2016/04/18

Monday, April 18, 2016

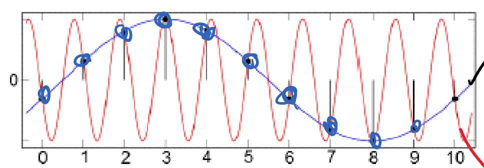
11:38 AM

While the proof of the relation

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} h \sum_{m=-\infty}^{\infty} e^{-\imath m h \xi} v_m, \quad \text{for} \quad \xi \in [-\frac{\pi}{h}, \frac{\pi}{h}] \qquad \Leftrightarrow$$

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{\imath m h \xi} \hat{v}(\xi) \mathrm{d}\xi$$

Is sufficient what is the meaning of the limit of integral being from $-\frac{\pi}{h}$ to $\frac{\pi}{h}$ instead of $-\infty$ to $+\infty$



Source: Wikipedia

ξ = corresponds to $-\frac{\pi}{h} \leq \xi \leq \frac{\pi}{h}$

high wave number $\quad \xi_{red} = \xi + q\frac{2\pi}{h}$ addition

ξ_{red} at position $x = mh$

$$e^{\imath \overbrace{mh}^{x} \overbrace{(\xi + q(2\pi/h))}^{\xi_{red}}} = \underbrace{e^{imh\xi}}_{\text{blue solution}} \, e^{im\cancel{h}q\frac{2\pi}{\cancel{h}}} = e^{imh\xi} e^{imq2\pi}$$

integer for position v_m

$e^{i\alpha \xi_{red}}$

blue solution: $e^{i\alpha \xi_{blue}}$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$v_m^n = v(x_m, t_n) = v(\underbrace{hm}_{x}, nk)$$

$$e^{imh(\xi + q(2\pi/h))} = e^{imh\xi}\left(\underbrace{\cos mq2\pi}_{1} + \underbrace{i\sin 2\pi q m}_{0}\right)$$

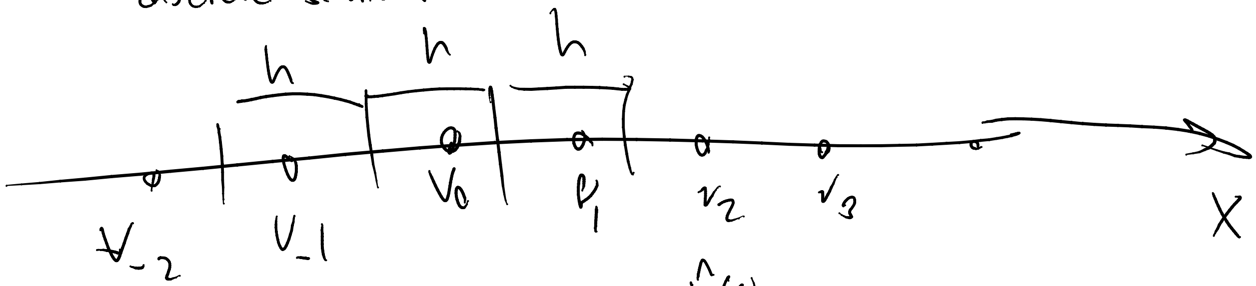
$$= e^{imh\xi}$$

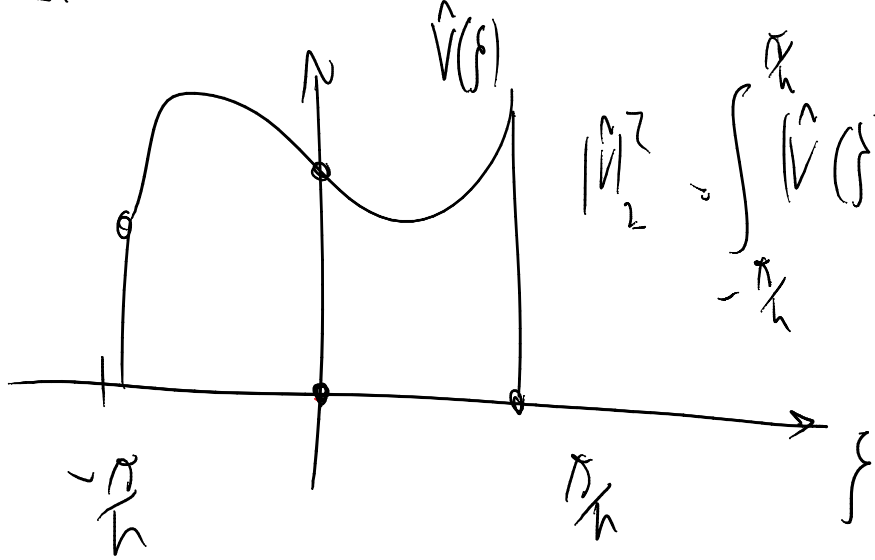
Parseval's inequality for Fourier series:

Parseval's inequality for Fourier series:

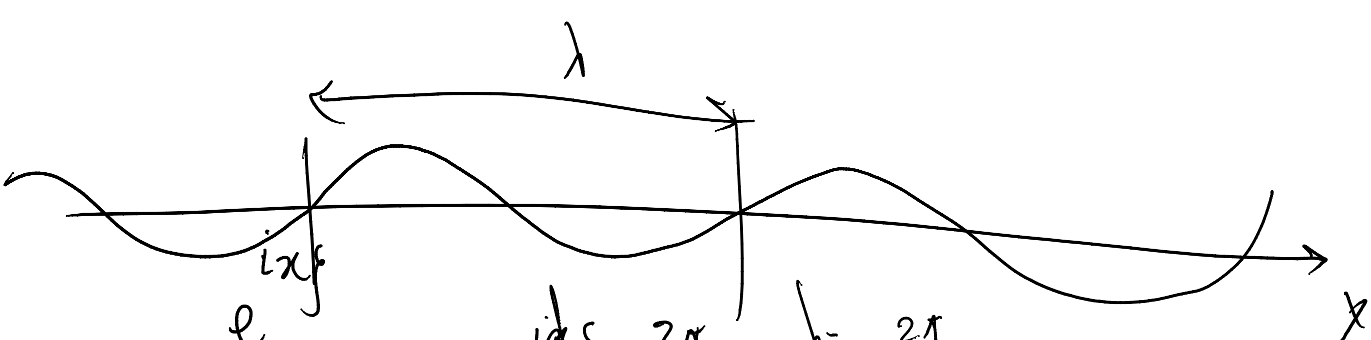
$$h\sum_{m=-\infty}^{\infty}|v_m|^2=\int_{-\pi/h}^{\pi/h}|\hat{v}(\xi)|^2 d\xi \Rightarrow \boxed{\|v\|_h=\|\hat{v}\|_h}$$

$$\underbrace{\|v\|_h^2}_{\text{the norm of discrete solution}} = \|\hat{v}\|_2^2$$





$$\|\hat{v}\|_2^2 = \int_{-\pi/h}^{\pi/h}|\hat{v}(\xi)|^2 d\xi$$



$e^{ix\xi}$

$i\lambda\xi = 2\pi$ $\quad \lambda = \frac{2\pi}{\xi}$

wave length

counter of period in time

λ = spatial period

ξ = wave number = $\frac{2\pi}{\lambda}$ = spatial frequency

What is the use of Fourier analysis?

Stability condition

Definition 4 *Stability of temporally first order PDEs: A finite difference scheme $P_{h,k}v_m^n = 0$ for a temporally first-order PDE is stable in the stability region Λ if there an integer J such that for any positive time T, there is a constant C_T such that,*

solution norm² at time step n

$$h||v^n||_h^2 \leq C_T h \sum_{j=0}^{J} ||v^j||_h^2 \qquad \text{for } 0 \leq nk \leq T \quad \text{with} \quad (h,k) \in \Lambda. \tag{406}$$

bounded by solution norm² @ first time step

example 1step method for temporally first order PDE

Initial condition

$$||v^n||_h^2 = h \sum_{m=-\infty}^{\infty} |v_m^n|^2 \leq C_T ||v^0||^2 = C_T \left(h \sum_{m=-\infty}^{\infty} |v_m^0|^2 \right)$$

time stepping

$$||v^n||_h^2 = ||\hat{v}^n||_h^2 \qquad ||v^0||_h^2 = ||\hat{v}^0||_h^2$$

x space at time n

ξ (Fourier space) again at time n

So stability for a one step method is equivalent to

$$\boxed{\|\hat{v}^n\|_h^2 \leq C_T \|\hat{v}^0\|_h^2}$$

Stability in wavenumber (Fourier space)

$j > 1$

$$\|\hat{v}^n\|_h^2 \leq C_T \sum_{j=0}^{J} \|\hat{v}^j\|_h^2$$

Stability of FTBS method for advection equation:

6.3.3 Analysis in frequency domain: Amplification factor

$\dot{u} + a u_{,x} = 0$

- Consider FTBS scheme ($\boxed{27b}$),

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = 0$$

$$v_m^{n+1} = -\frac{ak}{h}\left(v_m^n - v_{m-1}^n\right) + v_m^n = -\bar{k}\left(v_m^n - v_{m-1}^n\right) + v_m^n$$

$\bar{k}$ = normalized time step

$$\boxed{v_m^{n+1} = (1 - \bar{k})\, v_m^n + \bar{k}\; v_{m-1}^n}$$

$$\boxed{V_m = (1-\bar{k})V_m + \bar{k}\, V_{m-1}^{n}}$$

$$V_m^n = \underline{V^n(x_m = mh)} = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i x \xi}\, \hat{V}^n(\xi)\, d\xi \qquad x = mh$$

$$\boxed{V_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi}\, \hat{V}^n(\xi)\, d\xi}$$

$$V_m^{n+1} = (1-\bar{k})V_m^n + \bar{k}\, V_{m-1}^n$$

$$V_{m-1}^n = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i(m-1)h\xi}\, \hat{V}^n(\xi)\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}} e^{-ih\xi} e^{imh\xi}\, \hat{V}^n(\xi)\, d\xi \Rightarrow$$

$$V_m^{n+1} = (1-\bar{k})V_m^n + \bar{k}\, V_{m-1}^n = \frac{(1-\bar{k})}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi}\, \hat{V}^n(\xi)\, d\xi +$$

$$-\frac{\cdot\cdot}{h}$$

$$\frac{\bar{k}}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{-ih\xi}\, e^{imh\xi}\, \hat{V}^n(\xi)\, d\xi$$

$$\boxed{\begin{aligned} V_m^{n+1} &= \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} \underbrace{\left((1-\bar{k})+\bar{k}e^{-ih\xi}\right)\hat{V}^n(\xi)}\, e^{imh\xi}\, d\xi \\ V_m^{n+1} &= \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} \overbrace{\hat{V}^{n+1}(\xi)}\, e^{imh\xi}\, d\xi \end{aligned}}$$

$$\boxed{\hat{V}^{n+1}(\xi) = \overbrace{\left((1-\bar{k})+\bar{k}\, e^{-ih\xi}\right)}^{g(\theta)}\hat{V}^n(\xi)}$$

$$\underset{\text{wavenumber}}{\xi} \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right] \qquad h\xi \in [-\pi, \pi]$$

$$h\xi = \theta \in [-\pi, \pi]$$

$$h = 0 \rightarrow \theta = 0$$

$$h = \frac{\pi}{\xi} \rightarrow \theta = \pi$$

$$\hat{V}^{n+1}(\xi) \quad \ldots \quad \hat{V}^n(\xi)$$

$$\cancel{k^{n+1} = a k^n}$$

$$\boxed{\hat{v}^{n+1}(\xi) = g(\theta)\,\hat{v}^{n}(\xi) \qquad g(\theta) = (1-\bar{k}) + \bar{k}e^{-i\theta} \qquad \theta \in [-\pi, \pi]}$$

Amplification factor

$$v_m^{n+1} = g\,v_m^{n}$$

Impossible

$$\hat{v}^{n+1}(\xi) = g(\theta)\,\hat{v}^{n}(\xi) = g(\theta)\left(g(\theta)\,\hat{v}^{n-1}(\xi)\right)$$

$$= g^2(\theta)\,\hat{v}^{n-1}(\xi)$$

$$\hat{v}^{n+1}(\xi) = g^{n+1}(\theta)\,\hat{v}^{0}(\xi)$$

$$\boxed{\hat{v}^{n}(\xi) = g^{n}(\theta)\,\hat{v}^{0}(\xi)}$$

what happens if $|g(\bar{k},\theta)| \leq 1$ for all $\theta \in [-\pi,\pi]$

$$\|\hat{v}^n\|_h^2 = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\hat{v}^n(\xi)|^2 d\xi = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\hat{v}^0(\xi)\, g^n(\acute{k},\theta)|^2 d\xi$$

$$= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\hat{v}^0(\xi)|^2 \, |g(\acute{k},\theta)|^{2n} d\xi$$

$\acute{k}$ is such that

$$\underbrace{\le \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\hat{v}^0(\xi)|^2 d\xi = \|\hat{v}^0\|_h^2}$$

$|g(\acute{k},\theta)| \le 1$

for $\theta \in [-\pi, \pi]$

i.e. $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$

if $|g(\acute{k},\theta)| \le 1$ for all θ

$$\|\hat{v}^n\|_h^2 \le \|\hat{v}^0\|_h^2$$

Parseval's equality
$$\begin{cases} \|\hat{v}^n\|_h^2 = \|v^n\|_h^2 \\ \|\hat{v}^0\|_h^2 = \|v^0\|_h^2 \end{cases} \Rightarrow \|v^n\|_h^2 \le \|v^0\|_h^2 \quad C_T = 1$$

so the scheme is stable

$e^{i\phi} = \cos\phi + i\sin\phi$

$g(\theta) := (1-\bar{k}) + \bar{k}e^{-\imath\theta} \quad \Rightarrow$

$g(\theta) = g_R + \imath g_I,$ where $g_R = (1-\bar{k}) + \bar{k}\cos\theta, \quad g_I = -\bar{k}\sin\theta \quad$ (428a)

$\bar{k} = k\frac{a}{h}$

$\theta \in [-\pi, \pi]$

$\theta = 0 \quad g_R = 1 \quad g_I = 0$

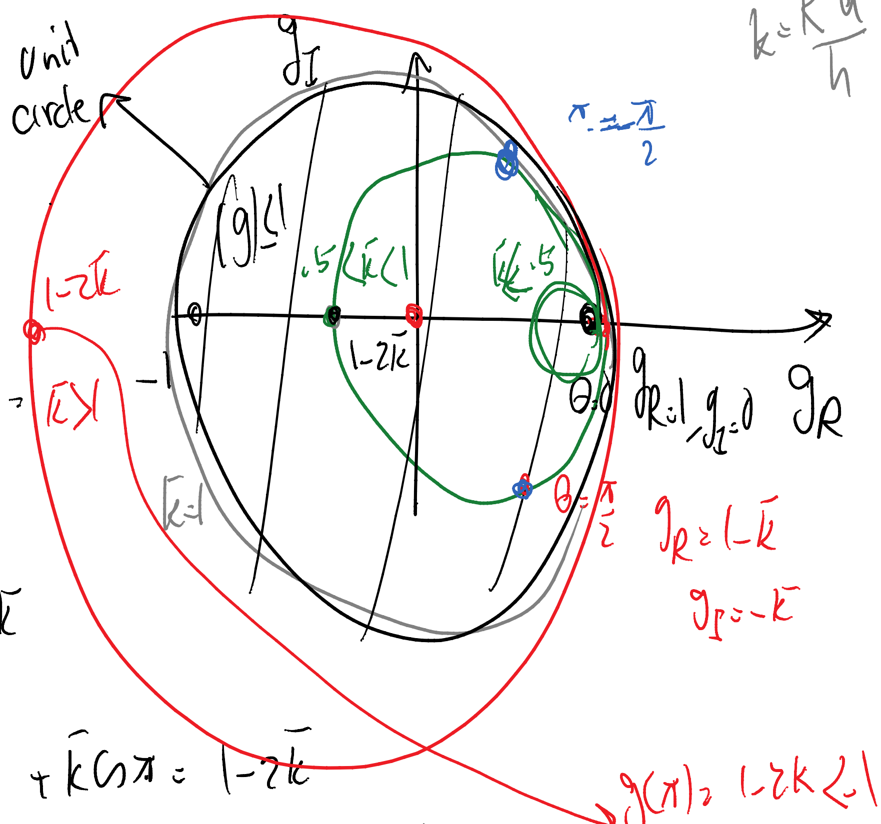
$\theta = \frac{\pi}{2} \quad g_R = (1-\bar{k}) + \bar{k}\cos\frac{\pi}{2} = 1-\bar{k}$

$g_I = -\bar{k}\sin\theta$

$= -\bar{k}\sin\frac{\pi}{2} = -\bar{k}$

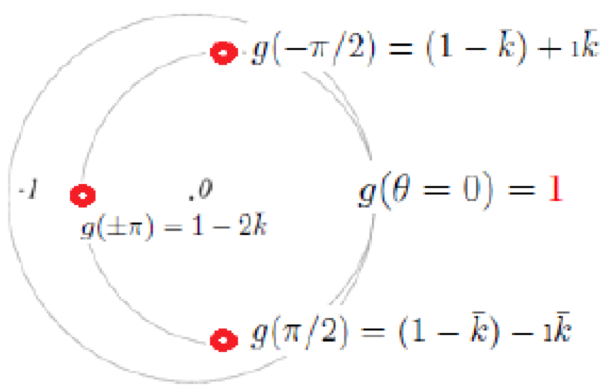
$\theta = \pi \quad g_R = (1-k) + \bar{k}\cos\pi = 1-2\bar{k}$

$g_I = -\bar{k}\sin\theta = 0$



assume $.5 < \bar{k} < 1$

$|g| > 1$



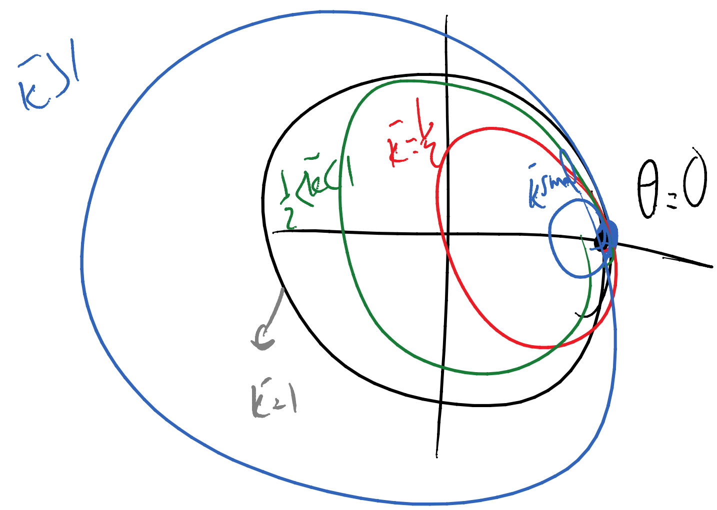
The image of $g(\theta)$ for the forward-time backward-space scheme.

$\bar{k} \leq 1$ for the method to be stable

$\bar{k} = \frac{ka}{h}$ $\qquad k \leq \frac{h}{a}$

Why all g loci are attached to

$g_R = 1 \quad g_I = 0 \quad (g = 1)$ at $\theta = 0$?



$$\hat{V}^{n+1}(\xi) = g(\theta)\hat{V}^{n}(\xi)$$

$$\theta = \xi h$$

$$\theta = 0 \Rightarrow h = 0$$

$$h \to 0 \qquad \hat{V}^{n+1}(\xi) \Rightarrow \hat{V}^{n}(\xi) \Rightarrow V_m^{n+1} \to V_m^{n} \quad \forall m$$

$$\bar{k} = \frac{ak}{h} \qquad k = \frac{\bar{k}h}{a}$$