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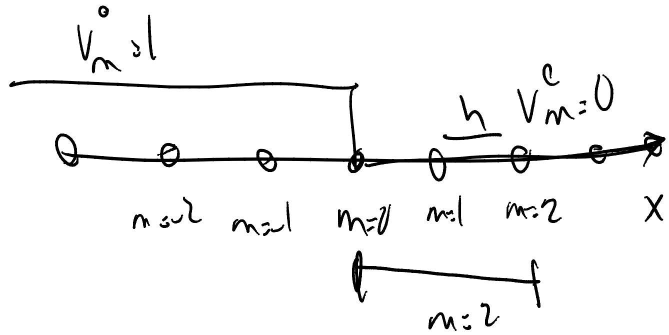
Simplified von Neumann analysis

$$v_m^0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{v}^0(\xi) d\xi$$



$$\frac{1}{\sqrt{2\pi}} e^{i(mh)\xi} \hat{v}^0(\xi)$$

$$= e^{ix\xi} \hat{v}^0(\xi)$$

$mh = x$

$$v_m^0 = v^0(x = mh) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \frac{1}{\sqrt{2\pi}} e^{ix\xi} \hat{v}^0(\xi) d\xi$$

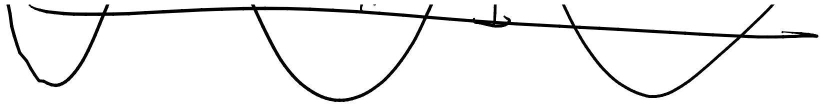
Solution is the summation of simple harmonics

$$\underbrace{\sum_{\xi=-\frac{\pi}{h}}^{\xi=\frac{\pi}{h}} \left[\frac{1}{\sqrt{2\pi}} \hat{v}^0(\xi)\right]}_{\text{amplitude}} \underbrace{e^{ix\xi}}_{\text{harmonic solution}}$$

$$e^{ix\xi} = \cos x\xi + i \sin x\xi$$

$$\xi = 0 \qquad \frac{1}{\sqrt{2\pi}} \hat{v}^0(0)_{\xi=0}$$

$$+ \xi = \frac{\pi}{h}$$



any $-\frac{\pi}{h} \le \xi \le \frac{\pi}{h}$

$$v^0(x = mh) = \int_{-\pi/h}^{\pi/h} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \hat{v}^0(\xi)\right]}_{\text{amplitude}} \underbrace{e^{i\xi x}}_{\text{harmonic functional}} \, d\xi$$

what frequencies? $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h})$

How do we get given wavenumber ξ amplitude?

$\hat{v}^0(\xi) = h \sum_{m=-\infty}^{\infty} e^{-imh\xi} v_m^0.$

For the solution at time step n

$$v_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{v}^n(\xi) \mathrm{d}\xi$$

Assume that IC is simply

$v^0_{\xi_m} = v^0_\xi(mh, 0) = e^{ix\xi} \hat{v}^0(\xi)$

$$\boxed{v^n_{\xi m} = e^{ix\xi} \hat{v}^n(\xi)}$$

plug this in FD stencil

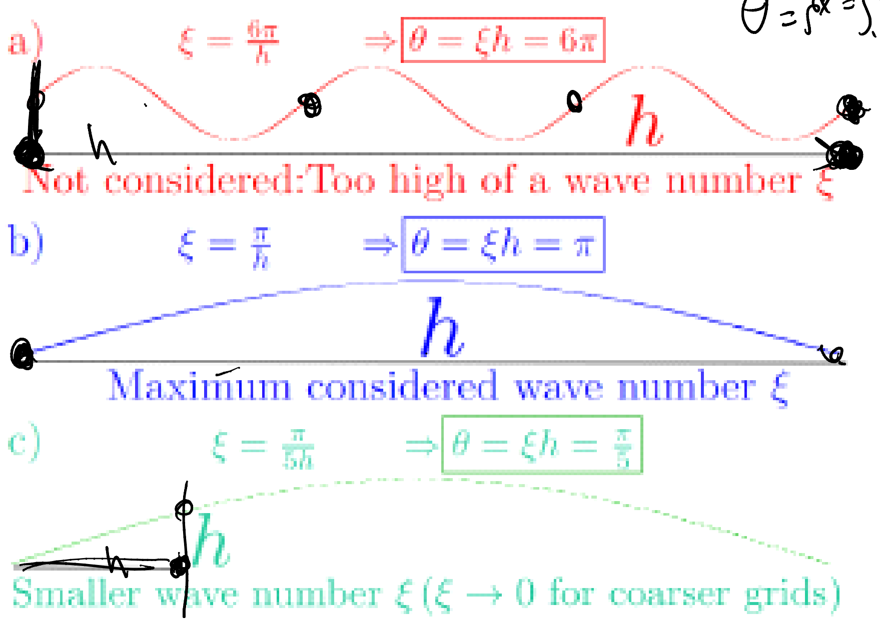
$-\frac{\pi}{h} \le \xi \le \frac{\pi}{h}$

$-\frac{\pi}{h} \leq \xi \leq \frac{\pi}{h}$

$-\pi \leq h\xi \leq \pi$

$\theta = \xi x = \xi h = 6\pi$

$e^{ix\xi}$



For von Neumann analysis plug this in the FD stencil

$$\boxed{v^n_\xi(x) = e^{ix\xi}\,\hat{v}^n(\xi)}$$

Fourier function for time step n

$$-\frac{\pi}{h} \leq \xi \leq \frac{\pi}{h}$$

$$V^n_m = V^n(x = mh) = e^{i(mh)\xi}\,\hat{V}^n$$

$$= e^{im(\underbrace{h\xi}_{\theta})}\,\hat{V}^n$$

$$\boxed{V_m^n = e^{im\theta}\,\hat{V}^n}$$

$$\boxed{\begin{aligned} V_{m+a}^{n+b} &= e^{i(m+a)\theta}\,\hat{V}^{n+b} \\ &= e^{ia\theta}\,e^{im\theta}\,\hat{V}^{n+b} = e^{ia\theta}\,V_m^{n+b} \end{aligned}}$$

If we have a one step method

$$\hat{V}_m^{n+1} = g\,\hat{V}_m^n$$

amplification factor

$$V_{m+a}^{n+b} = e^{ia\theta}\,V_m^{n+b} = e^{ia\theta}\,g^b\,V_m^n$$

$$\boxed{V_{m+a}^{n+b} = e^{ia\theta}\,g^b\,V_m^n}$$

Example:

6.3.7 Sample stability analyses with von Neumann method

- In §6.3.3 we obtained the amplification factor of FTBS scheme $g(h\xi) := \left[(1-\bar{k}) + \bar{k}e^{-\imath h\xi}\right]$. As discussed for $a > 0$ the stability was achieved if $\bar{k} = a\frac{k}{h} \leq 1$.
- This resulted in a conditional stability condition in the form $k \leq h/a$.
- We provides a more straightforward computation of g by directly plugging a solution of the form (444) in the FD stencil.
- For FTBS scheme the update equation from (3) is,

$$\boxed{v_m^{n+1} = (1-\bar{k})v_m^n + \bar{k}v_{m-1}^n}$$

Update equation

$n+b$ $\quad$ $ia\theta$ $\quad$ b $\quad$ n

upwind equation

$$V^{n+b}_{m+a} = e^{ia\theta} g^b V^n_m \Rightarrow$$

$$v^{n+1}_m = g\, v^n_m \qquad g v^n_m = (1-\bar{k}) v^n_m + \bar{k} e^{-i\theta} v^n_m$$

$$v^n_{m-1} = e^{-i\theta} v^n_m \quad \Rightarrow \quad \boxed{g = (1-\bar{k}) + \bar{k} e^{-i\theta}}$$

$$\bar{k} = \frac{ak}{h}$$

Does g depend on k? Yes

" " explicitly depend on k? No

then $|g| \leq 1 \iff$ stability

$$\forall\theta \quad |g| \leq 1 \quad \equiv \quad \bar{k} \leq 1 \qquad k \leq \frac{h}{a}$$

Example 4 *Stability of the Lax-Friedrichs scheme (source [Strikwerda, 2004] Example 2.2.4),*

- Consider the Lax-Friedrichs FD equation for the advection equation $u_{,t} + au_{,x} = 0$ from (27d),

$$\frac{v^{n+1}_m - \frac{1}{2}\left(v^n_{m-1} + v^n_{m+1}\right)}{k} + a\frac{v^n_{m+1} - v^n_{m-1}}{2h} = 0$$

$$v^{n+1}_m = \overbrace{\frac{ak}{2h}}^{\bar{k}}\left(v^n_{m-1} - v^n_{m+1}\right) + \frac{1}{2}\left(v^n_{m-1} + v^n_{m+1}\right)$$

$$v^{n+1}_m = \frac{\bar{k}}{2}\left(v^n_{m-1} - v^n_{m+1}\right) + \frac{1}{2}\left(v^n_{m-1} + v^n_{m+1}\right)$$

$$v^n(x) = e^{ix\xi}\,\hat{v}^n(\xi)$$

$$v^n_m = v^n(x=mh) = e^{im(h\xi)}\,\hat{v}^n(\xi) = e^{im\theta}\,\hat{v}^m(\xi)$$

$$\left.\begin{aligned} V_m^{n+1} &= \frac{\bar{k}}{2}\left(V_{m-1}^n - V_{m+1}^n\right) + \frac{1}{2}\left(V_{m-1}^n + V_{m+1}^n\right) \\ V_m^{n+1} &= e^{im\theta}\,\hat{V}^{n+1} \\ V_{m-1}^n &= e^{i(m-1)\theta}\,\hat{V}^n \\ V_{m+1}^n &= e^{i(m+1)\theta}\,\hat{V}^n \end{aligned}\right\} \Rightarrow$$

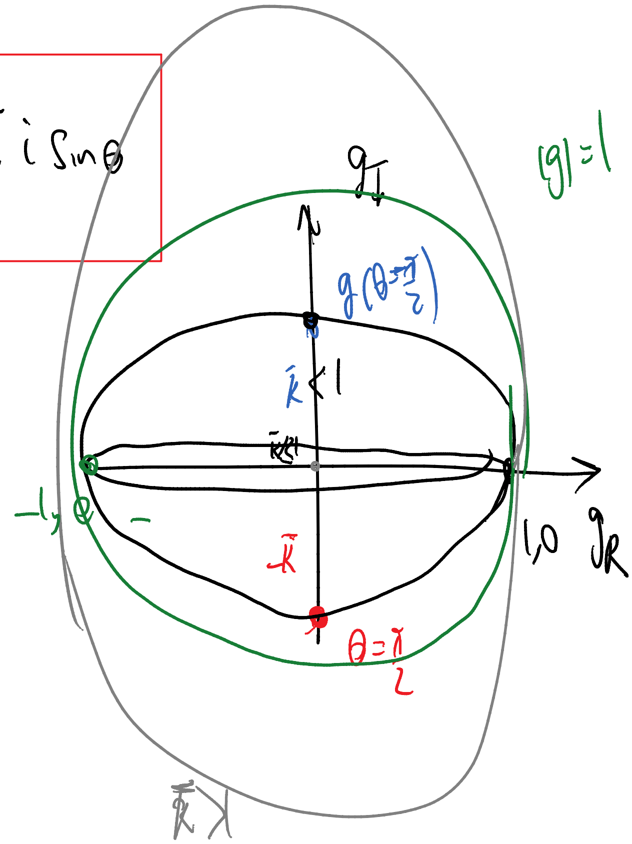
$$e^{im\theta}\,\hat{V}^{n+1} = \frac{\bar{k}}{2} e^{im\theta}\left(e^{-i\theta} - e^{i\theta}\right)\hat{V}^n + \frac{1}{2} e^{im\theta}\left(e^{-i\theta} + e^{i\theta}\right)\hat{V}^n$$

$$\hat{V}^{n+1} = \left[-\bar{k} i \underbrace{\left(\frac{e^{+i\theta} - e^{i\theta}}{2i}\right)}_{\sin\theta} + \underbrace{\left[\frac{e^{i\theta} + e^{-i\theta}}{2}\right]}_{\cos\theta}\right]\hat{V}^n$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\hat{V}^{n+1} = \underbrace{\left[-\bar{k} i \sin\theta + \cos\theta\right]}_{g(\xi)}\hat{V}^n \qquad \theta = \xi h$$

$$\boxed{g(\xi) = \cos\theta - \bar{k}\, i \sin\theta}$$



$$g(\theta) = \cos\theta - \imath\bar{k}\sin\theta$$

$$g(\theta=0) = 1 + 0i$$

$$g(\theta=\tfrac{\pi}{2}) = 0 - i\bar{k}$$

$$g(\theta=\pi=-\pi) = -1$$

$\bar{k} \leq 1$ for the method to be stable

$$k \leq \frac{h}{a}$$

Example 5 ***Numerical Stability*** *of the Lax-Friedrichs scheme applied to a dynamically unstable problem (source [Strikwerda, 2004] Example 2.2.3),*

- Consider the following problem,

$$u_{,t} + au_{,x} - u = 0 \tag{456}$$

$e^{i\xi x + \omega t}$

ξ: wavenumber, ω: frequency

$$(\omega + ai\xi)\, e^{i\xi x + \omega t} = 1\, e^{i\xi x + \omega t}$$

$$\omega = 1 - ai\xi$$

$$\boxed{u(x,t) = e^{i\xi(x-at)}\, e^{t}}$$

$$\lim_{t\to\infty} u(x,t) = \infty$$ Dynamically unstable

6.4.1 von Neumann analysis for leapfrog scheme

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \quad \Rightarrow \tag{464a}$$

$$v_m^{n+1} = \bar{k}(v_{m-1}^n - v_{m+1}^n) + v_m^{n-1} \tag{464b}$$

- This as shown results in (447) $v_{m+a}^{n+b} = e^{\imath(m+a)\theta}\hat{v}^{n+b}$.
- By using this equation in (464b) we obtain,

$$e^{\imath m\theta}\hat{v}^{n+1} = \bar{k}\left(e^{\imath(m-1)\theta}\hat{v}^n - e^{\imath(m+1)\theta}\hat{v}^n\right) + e^{\imath m\theta}\hat{v}^{n-1} \qquad \Rightarrow \qquad e^{\imath m\theta}\hat{v}^{n+1} = e^{\imath m\theta}\left\{-2\bar{k}\hat{v}^n\frac{e^{\imath\theta} - e^{-\imath\theta}}{2} + \hat{v}^{n-1}\right\}$$

$$\boxed{\hat{v}^{n+1} = -2\imath\bar{k}\sin\theta\hat{v}^n + \hat{v}^{n-1}}$$

$$\hat{v}^{n+1} = g\,\hat{v}^{n} = g^2\,\hat{v}^{n-1}$$

$$g^2 = -2i\bar{k}\sin\theta\, g + 1$$

$$\boxed{g^2 + 2i\bar{k}\sin\theta\, g - 1 = 0}$$

$$g_+ = -\imath\bar{k}\sin\theta + \sqrt{1-\bar{k}^2\sin^2\theta}$$

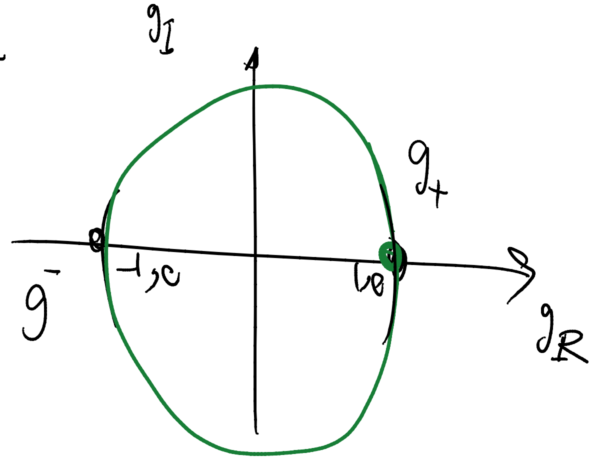
$$g_- = -\imath\bar{k}\sin\theta - \sqrt{1-\bar{k}^2\sin^2\theta}$$

if $\bar{k} \leq 1$ real

$$g_+ = \overbrace{\sqrt{1-\bar{k}^2\sin^2\theta}}^{g_R} + -i\bar{k}\sin\theta$$

$$|g_+|^2 = g_R^2 + g_I^2 = 1 - \bar{k}^2\sin^2\theta + \bar{k}^2\sin^2\theta = 1$$

Similarly $|g_-|^2 = 1$



$\bar{k} \leq 1$ $|g_+| = |g_-| = 1$

stable

What if $\bar{k} > 1$

$$g_-(\pi) = -\imath\bar{k}\sin(\pi/2) - \overbrace{\sqrt{1-\bar{k}^2\sin^2(\pi/2)}}^{\text{imaginary}} = \left|-\imath\left(\bar{k} + \sqrt{\bar{k}^2-1}\right)\right| > 1 \qquad (\text{since } \bar{k} > 1)$$

$\sqrt{1-k^2}$ $\quad \bar{k} > 1$

$g_-(\pi)$

$\bar{k} < 1$ stable

$\bar{k} > 1$ unstable

$\bar{k} = 1$ unstable ! refer to course notes on why

$\bar{k} = 1$ results in instability

basically for $\theta = \pi$ & $\bar{k} = 1$ we will have repeated roots of $+1$ for g_+ & -1 for g_- and

$$v^n = A(\pm 1)^n + Bn(\pm 1)^n$$

weak instability