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Overview of von Neumann analysis of leapfrog method

6.4.1 von Neumann analysis for leapfrog scheme

- Consider the leapfrog scheme (27e) for the advection equation $u_{,t} + au_{,x} = 0$,

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \quad \Rightarrow \tag{464a}$$

$$v_m^{n+1} = \bar{k}(v_{m-1}^n - v_{m+1}^n) + v_m^{n-1} \tag{464b}$$

$u(x, t_n) = e^{i\xi x}\hat{v}^n$

v_m^{n+1}

v_{m-1}^n v_{m+1}^n

v_m^{n-1}

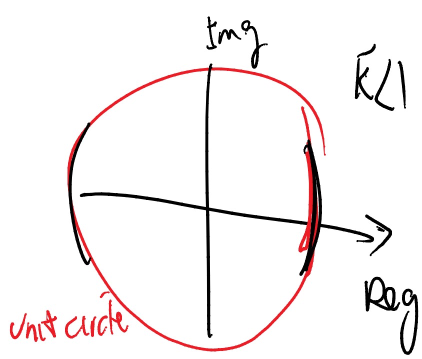
$) \; v_{m+a}^{n+b} = e^{\imath(m+a)\theta}\hat{v}^{n+b}.$

$$\boxed{\hat{v}^{n+1} = -2\imath\bar{k}\sin\theta\,\hat{v}^n + \hat{v}^{n-1}}$$

$\hat{v}^{n+1} = g\,\hat{v}^n = g^2\,\hat{v}^{n-1}$

$$g^{n+1} + (2\imath\bar{k}\sin\theta)g^n - g^{n-1} = 0 \quad \Rightarrow \quad \boxed{g^2 + (2\imath\bar{k}\sin\theta)g - 1 = 0}$$

$$g_+ = -\imath\bar{k}\sin\theta + \sqrt{1 - \bar{k}^2\sin^2\theta}$$
$$g_- = -\imath\bar{k}\sin\theta - \sqrt{1 - \bar{k}^2\sin^2\theta}$$



General solution looks like this:

$$\hat{v}^n = A_+(\xi)g_+^n(\theta) + A_-(\xi)g_-^n(\theta)$$

what if $g_+ = g$

Is it possible that g+ = g-

Yes,

Let $\bar{k}=1$

$$g^2 + 2i\sin\theta\, g - 1 = 0$$

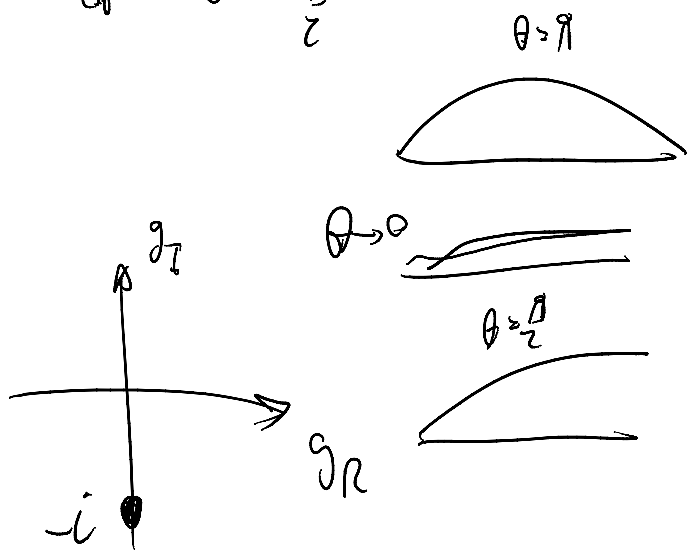
$$g_\pm = -\imath\sin\theta \pm \sqrt{1-\sin^2\theta} \qquad \text{for } \bar{k}=1$$

$\sin\theta = 1$ $\qquad \theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$

$\bar{k}=1$ & $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$

$\theta = 90° = \frac{\pi}{2}$

$g_\pm := g = -\imath \qquad \text{for } \bar{k}=1 \text{ and } \theta = \pm\pi/2$



How is this equation modified?

$$\hat{v}^n = A_+(\xi)g_+^n(\theta) + A_-(\xi)g_-^n(\theta)$$

if $g_- = g_+$

$$\hat{v}^n = A\, g^n(\theta) + n\, B\, g^n(\theta)$$

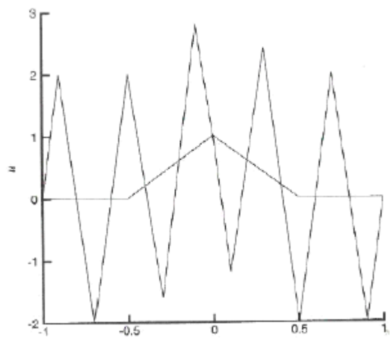
It repeated once

Here $g^- = g^+ = i$

$$V^n = A\,(i)^n + n B\,(i)^n$$

It grows

algebraic growth



Leapfrog weak (algebraic) instability for $\bar{k} = 1$.

Weak instability for leapfrog method

6.4.2 von Neumann analysis for a temporally 2$^{\text{nd}}$PDE

- For example, consider the wave equation (56a) ($u_{,tt} - c^2 u_{,xx} = r$) but without the source term,

$$u_{,tt} - a^2 u_{,xx} = 0 \tag{479}$$

$$u_{,tt} - a^2 u_{,xx} = 0$$

- Now consider a central-space central-time FD scheme being applied to the solution of (479). The FD equation will be,

$$\frac{v_m^{n+1} - 2v_m^n + v_m^{n-1}}{k^2} - a^2 \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} = 0 \tag{481}$$

$$\ddot{u} - a^2 u_{,xx} = 0$$

$u(x) = 0$

$\dot{u}(x) = 1$

$u(x,t) = t$

growing linearly

that's why in this case repeated $g_+ = g_- = g$, $|g| = 1$ is acceptable

Von Neumann analysis for the CSCT for wave equation results in the following second order equation of g:

$$g^2 - 2A_1 g + A_2 = 0, \qquad \text{for } A_1 = 1 - 2\bar{k}^2 \sin^2\frac{\theta}{2}, \quad A_2 = 1$$

$$g_\pm = \left[1 - 2\bar{k}^2 \sin^2\frac{\theta}{2}\right] \pm \left[2\bar{k}\sin\frac{\theta}{2}\sqrt{\bar{k}^2 \sin^2\frac{\theta}{2} - 1}\right]$$

$g_1 + g_2 = A_1$ $\quad g_1 g_2 = A_2$

$$g^2 - 2A_1 g + A_2 = 0,$$

A_1 & A_2 real

$|g_1|$ & $|g_2| \leq 1$ for $g_1 \neq g_2$ or $g_1 = g_2$ but $|g_1| = |g_2|$

$|A_2| \leq 1$

$|A_1| \leq \frac{A_2+1}{2}$

$\frac{1+A_2}{2} = 1 \quad A_2 = 1$

- Recall that the necessary and sufficient conditions for the roots of (483) to satisfy $|g| \leq 1$ is (362) was $-1 \leq A_2 \leq 1$, $-\frac{A_2+1}{2} \leq A_1 \leq \frac{A_2+1}{2}$ except the point with repeated roots of -1 or $+1$. That is,

$$-1 \leq |A_2| = |1| \leq 1 \qquad \text{satisfied} \tag{485a}$$

$$-1 \leq 1 - 2\bar{k}^2 \sin^2 \frac{\theta}{2} \leq 1 \qquad \bar{k}^2 \sin^2 \frac{\theta}{2} \leq 1 \tag{485b}$$

satisfied for all $\theta \in [-\pi, \pi]$ if

$\theta = \frac{\pi}{2}$

$\bar{k} \leq 1$

How about if $\bar{k} = 1$

$$g^2 - 2(1 - 2\sin^2 \frac{\theta}{2})g + 1 = 0 \quad \Rightarrow \quad g^2 - 2(\cos\theta)g + 1 = 0, \quad \text{for } \bar{k} = 1 \tag{486}$$

$g = \cos\theta \pm \sqrt{\cos^2\theta - 1}$

$\cos\theta = 0$ $\quad \theta = \cancel{0}, \pi$

$g = \cancel{\pm} 1$

$\underline{g(\pi) = -1}$

for $\theta = \pi$

$$\hat{v}^n = A(\xi)g^n + B(\xi)ng^n = A(\xi)(\cancel{\pm}1)^n + B(\xi)n(\cancel{\pm}1)^n$$

this is fine because the wave equation permits solutions like $u(x,t) = t$

equations,

$u_{,t} + a u_{,x} = 0$	1D advection equation
$u_{,tt} - a^2 u_{,xx} = 0$	1D wave equation
$u_{,t} - D u_{,xx} = 0$	1D diffusion equation
$\tau u_{,tt} + u_{,t} - D u_{,xx} = 0$	1D relaxed diffusion equation
$\rho \mathbf{u}_{,tt} - \nabla.(\mathcal{C}\nabla \mathbf{u}) = \mathbf{0}$	elastodynamic problem

The idea is if we can propagate a wave with speed c?

$$u = f(x - ct)$$

scalar function

speed

$\mathbf{u} = f(\mathbf{x}.\mathbf{n} - ct)\Phi$ shape

Φ = mode vector

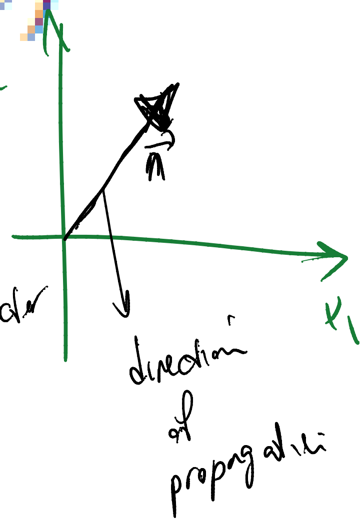
Solid Mechanics

f = planar wave function

c = wave speed

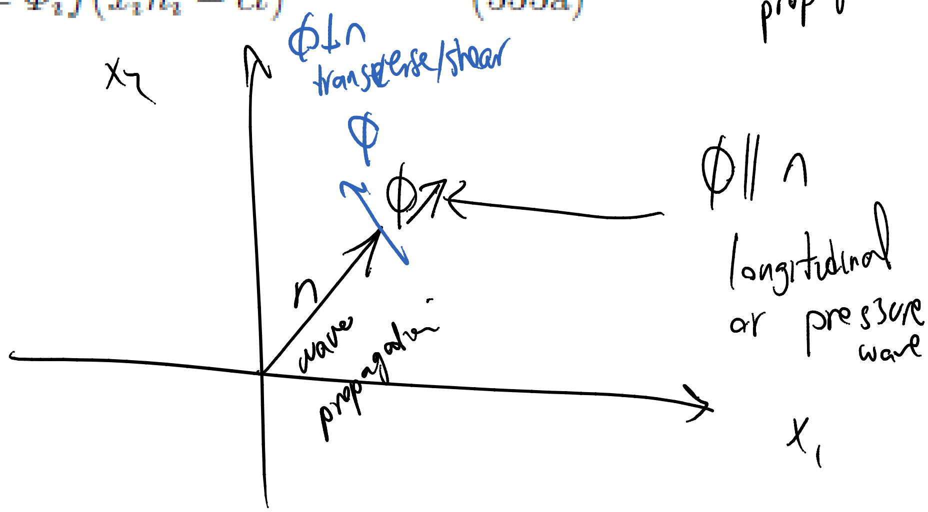
u is a vector $\Rightarrow$ ϕ a vector

$\mathbf{n}$ = direction of wave propagation (a unit vector)



that is $u_i = \Phi_i f(x_i n_i - ct)$ (555a)

$u_i \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$



The ability to propagate planar waves

Advection equation $u_{,t} + a u_{,x} = 0$

$$u = f(x - ct)$$

$$-c f'(x-ct) + a f'(x-ct) = 0 \Rightarrow (a-c) f'(x-ct) = 0$$

$$\boxed{c = a}$$

Wave equation $u_{,tt} - a^2 u_{,xx} = 0$

$$(-c)^2 f'' - a^2 f'' = 0 \qquad (c^2 - a^2) f'' = 0$$

$$\boxed{c = \pm a}$$

Diffusion equation

$$u_{,t} - D u_{,xx} = 0$$

$$u = f(x - ct)$$

$$(-c) f'(x-ct) - D f''(x-ct) = 0$$

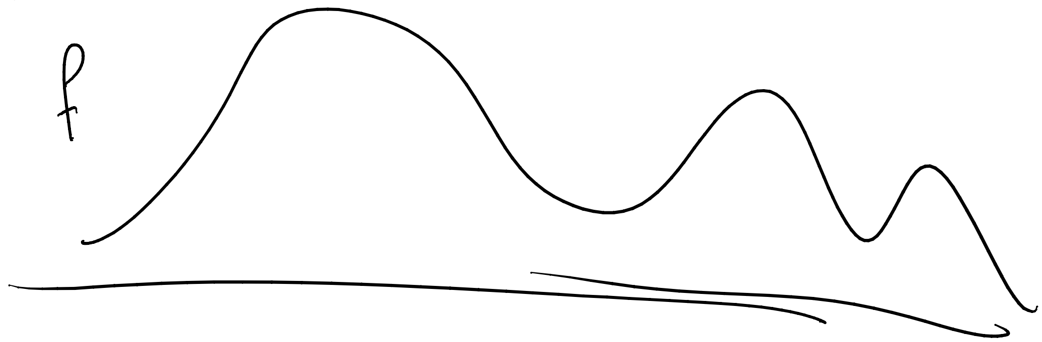
in general no solution

We observed advection eqn $u_{,t} + a u_{,x} = 0$

wave eqn $u_{,tt} - a^2 u_{,xx} = 0$

Elastodynamics eqn $\rho \ddot{u} - \nabla \sigma = 0$

Can propagate arbitrary planar waves



But Diffusion eqn couldn't

$$u_{,t} - D u_{,xx} = 0$$

What is the next level in ability of a PDE to propagate a shape

$$u(x,t) = e^{i(x\xi - \omega t)}$$

$$\cos(x\xi - \omega t) + i \sin(x\xi - \omega t)$$

Scalar in 1D

$$u(x,t) = e^{\imath(\xi x - \bar{\omega} t)} = e^{\imath(\xi x - \bar{\omega}_R t)} e^{\bar{\omega}_I t}$$

scalar in 2D & 3D

$$u(\mathbf{x},t) = e^{\imath(\boldsymbol{\xi} \cdot \mathbf{x} - \bar{\omega} t)} = e^{\imath(\boldsymbol{\xi} \cdot \mathbf{x} - \bar{\omega}_R t)} e^{\bar{\omega}_I t}$$

vector in 2D, 3D

$$\mathbf{u}(\mathbf{x},t) = \boldsymbol{\Phi} e^{\imath(\boldsymbol{\xi} \cdot \mathbf{x} - \bar{\omega} t)}$$

shape

wave number vector

scalar (1D)

$$u(x,t) = e^{i(\xi x - \omega t)} =$$

$$e^{i(\xi x - \omega_R t - i\omega_I t)}$$

$$\boxed{u(x,t) = e^{i(\xi x - \omega_R t)} \, e^{\omega_I t}}$$

propagating mode

$\omega_I \leq 0$ for decaying solution

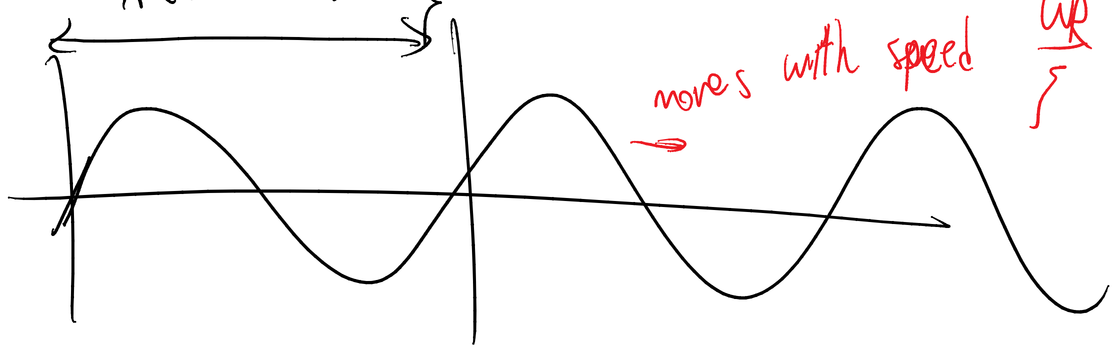
$$u(x,t) = e^{i\xi(x - \underbrace{\frac{\omega_R}{\xi}}_{c} t)} \, e^{\omega_I t}$$

c → wave speed

decays as $e^{\omega_I t}$ in time

$e^{i\xi x}$

λ = wave length = $\frac{2\pi}{\xi}$



moves with speed $\frac{\omega_R}{\xi}$

$$u = e^{i(\xi x - \omega t)} \qquad \underline{u_{,t}} + a u_{,x} = b u$$

$$(-i\omega) e^{i(\xi x - \omega t)} + a(i\xi) e^{i(\xi x - \omega t)} = b e^{i(\xi x - \omega t)}$$

$$-i\omega + a i \xi = b$$

$$\omega = a\xi - \frac{b}{i}$$

$$\boxed{\omega = a\xi + ib}$$

$$e^{i\xi(x - at)} e^{bt}$$

growing

$$u_{,tt} + \omega_0 u_{,t} - a^2 u_{,xx} = 0, \qquad a = \sqrt{\frac{D}{\tau}}, \omega_0 = \frac{1}{\tau}$$