

We want to solve the advection equation,

$$u_{,t} + a(x, t)u_{,x} = 0 \quad \text{PDE} \quad (1a)$$

$$u(x, t = 0) = u_0(x) \quad \text{IC} \quad (1b)$$

The specific of the problem are,

1. $a = 0.1$.
2. Computational domain is $[-10, 10]$.
3. Initial condition (IC): Two different cases are considered,

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & 0 \leq x \end{cases} \quad \text{nonsmooth IC option (not even a } \mathcal{C}^0 \text{ function)} \quad (2a)$$

$$u_0(x) = \begin{cases} 1 & x < -1 \\ -x^3(6x^2 + 15x + 10) & -1 \leq x < 0 \\ 0 & 0 \leq x \end{cases} \quad \text{smooth IC option (} \mathcal{C}^2 \text{ function)} \quad (2b)$$

1. ((90 + 20 + 10 =) **120 Points**) Numerical solution of (1) with given computational domain size and $a = 0.1$:
 - Solve **ONLY the nonsmooth IC**.
 - Consider FD spatial grid size $h = 0.1$.
 - Consider **two** different normalized time steps,

$$\bar{k} = 0.8, 1.1, \quad \bar{k} = \frac{ka}{h}$$

- Consider 3 target times of $T_t = 3, 6, 20$. Depending on the value of \bar{k} you need to choose a time step value just above target values. For example, $\bar{k} = 0.8 \Rightarrow 0.8 = \bar{k} = \frac{ka}{h} = \frac{0.1k}{0.1} \Rightarrow k = 0.8 \Rightarrow$ for $T_t = 3$ number of time steps = ceiling(T_t/k) = ceiling($3/0.8$) = ceiling(3.75) = 4 \Rightarrow 4 time steps are needed and the *computed final time* instead of 3 would be $4k = 4 \times 0.8 = 3.2$. Different \bar{k} given slightly different final times than 3, 6, 20. Below, is the list of these values for your reference (T refers to the time that samples will be taken for visualization):

$$\begin{array}{lll} T = 3.2, 6.4, 20 & \text{for } T_t & = 3, 6, 20, \text{ and } \bar{k} = 0.8 \\ T = 3.3, 6.6, 20.9 & \text{for } T_t & = 3, 6, 20, \text{ and } \bar{k} = 1.1 \end{array}$$

You can interpolate values between different time steps to exactly match T_t but the values T above suffice for this study.

- Solve for the schemes **BTFS, BTBS, FTFS, FTCS, FTBS**.
- submit all the plots in one or several file(s) AND the code that you have used to generate the results by email or by using dropbox.
- Each plot shows numerical results u^h versus x along with the exact solution for the given time.
- Since some solutions blow up, use the following limits for your x and $u(y)$ axes:

$$\begin{array}{ll} x \text{ axis} & [-1.5 \ 4] \\ y \text{ axis} & [-4 \ 4] \end{array}$$

- For boundary conditions (BC) use $u = 1$ at $x = -10$ and $u = 0$ for $x = 10$ (or beyond as may be needed for FTFS scheme).
- The three explicit schemes (FTFS, FTCS, FTBS) can be solved for one point at a time u_m^n . The two implicit schemes can also be solved starting from the left side going to the right side as demonstrated in the class notes (part 2) for this particular problem.

Material that should be submitted are:

- Present your results in 12 plots $\{2 \bar{k} (0.8, 1.1)\} \times \{3 T_t (3, 6, 20)\} \times \{2 (\text{Implicit schemes BTFS, BTBS versus explicit schemes FTFS, FTBS, FTCS})\}$ names as **kn(\bar{k} value), $T=(T)$, (Explicit or Implicit)** where terms in () are replaced by the values displayed in the plots. If it is difficult to group explicit and implicit results together, you can submit the plot for each scheme separately.
 - For each of the 5 schemes (BTFS, BTBS, FTFS, FTCS, FTBS) write the stencil equation you use for the solution of a new value in terms of \bar{k} parameter. For example, from the course notes for this problem the equation for BTFS is $(1 - \bar{k})u_m^{n+1} + \bar{k}u_{m+1}^{n+1} = u_m^n \Rightarrow u_{m+1}^{n+1} = \frac{1}{\bar{k}}u_m^n - \frac{1-\bar{k}}{\bar{k}}u_m^{n+1}$. Right the update equations for all the schemes in your solution sheets.
 - In the solution sheet, write the exact solution to this problem for the nonsmooth initial condition.
2. ($(5 \times 6 \times 3 =)$ **90 Points**) Continuing with the same problem and using your solutions, class notes, and convergence plots provided to you in the folder “HW02/P1,2/ConvergencePlots”, answer the following questions for EACH of the schemes BTFS, BTCS, BTBS, FTFS, FTCS, FTBS:
- Classify the method as i) Unconditionally unstable (UU), ii) Conditionally stable (CS), iii) Unconditionally stable (US).
 - List the set $S_{\bar{k}}$ of stability in terms of \bar{k} . For example, for an unconditionally unstable scheme $S_{\bar{k}} = \emptyset$, conditional stable with CFL number 0.5 it is $S_{\bar{k}} = [0, 0.5]$.
 - For the schemes that are conditionally stable or unconditionally unstable comment on how from your solutions instabilities develop and grow in time T_t from 3 to 6 to 20 for either $\bar{k} = 0.8$ or $\bar{k} = 1.1$ (which one(s) that apply). Also, comment whether there are regions where the solution does not blow up, but does not converge to the correct solution.
 - Referring to convergence plots (in HW2.FD_convergencePlots folder) comment on the rate of convergence of the error versus h for the ranges of \bar{k} (kn in the plots) where the method is stable. Specifically i) by visual inspection of convergence plots provide approximate values for both smooth and nonsmooth IC cases; ii) comment whether nonsmooth and smooth IC problems have different convergence rates or not; iii) specify if the method has the highest spatial convergence rate among those considered; iv) if the method is unconditionally stable comment whether for any k the convergence plots may erroneously suggest a convergence of error to zero as $h \rightarrow 0$.

Background: The L2 error norm is defined as,

$$L_2(u^h - u) = \sqrt{\int_{-10}^{10} (u^h(x) - u(x))^2 dx} \quad (3)$$

In many numerical methods, *e.g.*, FV, FD, FEM, the convergence rate of an error (*e.g.*, the L2 error norm above) is expressed as,

$$e \propto C_h h^p + C_k k^s \quad (4)$$

where p and s are spatial and temporal convergence rates and C 's are constants depending on the details of the method, grid, *etc.*. If the exact solution is NOT smooth enough the convergence rates decrease depending of the level of regularity of the exact solution (*e.g.*, if it is a $\mathcal{C}^0, \mathcal{C}^1, \dots, \mathcal{C}^r$ function). For a static problem the temporal part drops. If the error is entirely from the spatial part (or is dominated by it), we can write

$$e \approx C_h h^p \quad \Rightarrow \log(e) \approx \log(C_h) + p \log(h) \quad (5)$$

So, the convergence rate is the slope at which $\log(e)$ decreases as h decreases in log-log plots (or obtained from mathematical analysis). Note that to compute convergence rate, the ranges of small h should be considered where the asymptotic convergence rates are achieved. At large h the convergence rates do not hold.

- (e) Comment on the role of normalized time step \bar{k} on the convergence plots. Is there a value that beyond of \bar{k} beyond which the temporal discretization error is ignorable compared to spatial discretization error? Does taking smaller \bar{k} improve the convergence rate. Specifically for BTCS method discuss why the convergence rate changes as $\bar{k} \rightarrow 0$.

- List your solutions as BTFS(a), BTFS(b), \dots , BTFS(e), BTCS(a), \dots , BTCS(e), \dots .
- Be brief in your answers.

3. ((6 × 15 =) **90 Points**) Consider the 1D elastodynamic problem,

$$u_{,tt} - c^2 u_{,xx} = 0 \quad (6)$$

for zero body $b = 0$.

- The problem can be solved as a **1-field** problem where u (*i.e.*, U_m^n being the unknowns) is directly discretized or as a **3-field** problem where u, v, s (*i.e.*, U_m^n, V_m^n, S_m^n are unknowns) are discretized. We use F1 and F3 to refer to these two approaches.
- For each F1 or F3 we can use average fluxes which recovers a normal FD scheme or use Riemann solutions which in the following discussion we label it as FV. So, FD means we have used average fluxes, FV Riemann fluxes.
- In the class notes (part 2) we demonstrated that the FV update for the stress, velocity (and added displacement here) in a 3F scheme is,

$$\frac{S_m^{n+1} - S_m^n}{k} - E \frac{V_{m+1}^n - V_{m-1}^n}{2h} - \frac{hc}{2} \frac{S_{m+1}^n + S_{m-1}^n - 2S_m^n}{h^2} = 0 \quad (7a)$$

$$\frac{V_m^{n+1} - V_m^n}{k} - \frac{1}{\rho} \frac{S_{m+1}^n - S_{m-1}^n}{2h} - \frac{hc}{2} \frac{V_{m+1}^n + V_{m-1}^n - 2V_m^n}{h^2} = 0 \quad (7b)$$

$$\frac{U_m^{n+1} - U_m^n}{k} = V_m^n \quad (7c)$$

the terms in the first two equations after $\frac{hc}{2}$ (terms in red) are added terms by using Riemann fluxes (labeled as FV scheme herein) compared to average fluxes (labeled as FD scheme).

- By eliminating S and V from (7b) we obtain the 1F update:

$$\frac{U_m^{n+1} - 2U_m^n + U_m^{n-1}}{k^2} - c^2 \frac{U_{m+2}^n - 2U_m^n + U_{m-2}^n}{4h^2} - D_h \frac{1}{k} \left\{ \left[\frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{h^2} \right] - \left[\frac{U_{m+1}^{n-1} - 2U_m^{n-1} + U_{m-1}^{n-1}}{h^2} \right] \right\} = 0 \quad (8)$$

the terms in red (after D) are added when Riemann fluxes are used. That is, with them we solve FV1F and without them FD1F.

- The solution to all four combinations FD3F, FV3F, FD1F, FV1F options are provided in the folder P3/F3 and P3/F1 folders with subfolders FD and FV.
- Solutions are given for three different grid spatial resolutions $h = 0.001, 0.01, 0.1$ to represent refined, medium, and coarse resolutions.
- To numerically observe the stability limits of these methods, solutions are given for $\bar{k} = 0.1, 0.49, 0.5, 0.51, 0.99, 1.01, 1.99, 2.01$.
- The initial condition consists of a hat function (with peak 1) spanning $[0.4, 0.6]$ in the computational domain $[0, 1]$ with period boundary conditions.
- Material properties are $E = 0.04, \rho = 1 \Rightarrow c = Z = 0.2$.
- Results are presented for the target time $T_t = 1.75$ (slightly different values are realized based on the closest time step).
- Within 1F and 3F folders sample Matlab files are provided that would generate the results for problem 3. They can help you in writing compute code for the first problem, although in problem 1 many aspects (e.g., boundary conditions) are different.

Answer the following questions using the provided numerical results:

- (a) Classify FD3F, FV3F, FD1F, FV1F schemes as i) Unconditionally unstable (UU), ii) Conditionally stable (CS), iii) Unconditionally stable (US).
- (b) If one of the schemes is unconditionally unstable, compare the instability with a corresponding FD stencil instability when only one first order PDE is solved.
- (c) For each scheme list the set $S_{\bar{k}}$ of stability in terms of \bar{k} .
- (d) Given that results are compared for a fixed target time (slightly different times are realized for different \bar{k}), discuss which one of small or large grid sizes h expose instabilities better and justify your answer.
- (e) Between FD and FV schemes describe which one is more dissipative, *i.e.*, dampens the sharp peaks. Explain the cause for this, if one scheme is more dissipative.
- (f) Based on your observations briefly (less than 3 sentences) describe what scheme (FV vs. FD) you would choose for 1F and 3F choices.