1. ($(20(a) + 30(b) + 30(b) =)$ **80 Points**) **Dispersion relation / phase and group velocity**: Consider the relaxed *advection-diffusion-reaction*(ADR) PDE,

$$\tau u_{,tt} - \nu u_{,xx} + u_{,t} + v u_{,x} = -ru \tag{1}$$

where $\tau, \nu, v, r \geq 0$ are the relaxation time, diffusion coefficient, advection velocity, and reaction coefficient ratio and have the units of $[T], [L]^2/[T], [L]/[T], 1/[T]$ with $[L]$ and $[T]$ being the length(space) and time scales, respectively.

   (a) Obtain the dispersion relation for (1); *cf.* §7.5.

   (b) Obtain the phase velocity from (593) and group velocity from (598) for the right-moving waves (*i.e.,* $\omega_R > 0$).

   (c) Write a finite difference stencil for (1) by using central difference stencils for $\ddot{u}_j^n, (u_{,xx})_j^n$, forward Euler for $\dot{u}_j^n$, and backward Euler for $v(u_{,x})_j^n$.
   Note the von Neumann analysis of this FD scheme provides the time step of this complex PDE and demonstrates that at different space/time scales time scale matches what the physical limits of the PDE dictated (*i.e.,* wave equation, diffusion equation, reaction equation, etc). The derivation of the time step and von Neumman analysis can be time consuming and only the expression of the FD scheme is requested.

2. ($(10(a)+10(b)+30(c)+10(d) =)$ **60 Points**) **Dispersion and dissipation error analysis**: Consider the advection equation,

$$u_{,t} + a u_{,x} = 0$$

solved by the Lax-Friedrich's scheme. By referring to (630b) for the values of g_R, g_I and following the solution scheme discussed in sections §7.7.5 and §7.7.6 for dispersion and dissipation analysis answer the following.

   (a) Show that numerical real and imaginary components of frequency for Lax-Friedrich's method ω^h are,

$$\omega_I^h = \frac{1}{2k}\log(g_R^2 + g_I^2) = \frac{1}{2k}\log\left(\cos^2\theta + \bar{k}^2\sin^2\theta\right) \tag{2a}$$

$$\omega_R^h = \frac{1}{k}\tan^{-1}\left(\frac{-g_I}{g_R}\right) = \frac{1}{k}\tan^{-1}\left(\bar{k}\tan\theta\right) \tag{2b}$$

   (b) For value of $\bar{k}$, ω_R^h, ω_I^h match the exact values $\omega_R = a\xi, \omega_I = 0$?

   (c) Show the asymptotic expressions, *i.e.,* $\theta \to 0$, for dissipation (and amplitude decay) and dispersion (and period elongation) for Lax Friedrich's method are,

$$\frac{\Delta\omega_I}{\omega_R} = \frac{1}{2\pi}\log(1 - A_d) \approx -\frac{1}{2\pi}A_d = \frac{1}{8}\left(\frac{\bar{k}^2 - 1}{\bar{k}}\right)\theta + \mathcal{O}\left(\theta^3\right) \tag{3a}$$

$$\frac{\Delta\omega_R}{\omega_R} = -\frac{\Delta T}{T} = \frac{1}{3}(1 - \bar{k}^2)\theta^2 + \mathcal{O}\left(\theta^4\right) \tag{3b}$$

   (d) The leading terms are zero for $\bar{k} = 1$. What do you expect the values of the higher order terms be for $\bar{k} = 1$ and why?

3. ($(20(a) + 20(b) + 20(c) =)$ **60 Points**) **Well-posedness, dynamic stability, and robustness**:

   (a) Show the following two equations are ill-posed.

$$u_{,tt} - u_{,x} = 0$$
$$u_{,ttt} - u_{,xx} = 0$$

   (b) Show that the following wave equation with negative reaction coefficient $r < 0$ is well-posed but not dynamically stable,

$$u_{,tt} - a^2 u_{,xx} = -ru \tag{4}$$

   (c) Show that Euler-Bernoulli equation (545),

$$u_{tt} + b^2 u_{,xxxx} = 0, \tag{5}$$

   while being well-posed, is not robust.
   Side Note (FYI): You can refer to §6.5 for the discussion of these concepts. Two of the examples above are directly discussed and solved in the course notes.