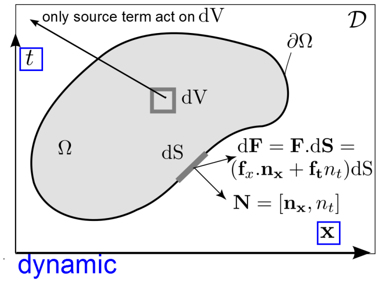
1 General solution schemes in space and time

1.1 Finite Element Method (FEM)

To derive the FEM formulation we follow the following steps:
Balance law ⇒ Strong form (+ BCs) ⇒ continuum weighted residual method ⇒ Continuum weak form ⇒ Discrete weak form ⇒ FEM method (by using shape functions)



1.1.1 Balance law (for FEM formulations)

- For solid mechanics the temporal flux, spatial flux, and source term are,

$$\mathbf{f}_t = \mathbf{p} = \rho\mathbf{v} \qquad p_i = \rho v_i \qquad \text{linear momentum density} \tag{1a}$$

$$\mathbf{f}_x = -\sigma = -\mathcal{C}\epsilon \qquad \sigma_{ij} = \mathcal{C}_{ijkl}\epsilon_{kl} \qquad \text{stress} \tag{1b}$$

$$\mathbf{r} = \rho\mathbf{b} \qquad \mathbf{b} = b_i\mathbf{e}_i \qquad \text{body force} \tag{1c}$$

  where $\mathcal{C}$ is the fourth order elasticity tensor and $\epsilon = \frac{1}{2}(\nabla\mathbf{u} + \nabla^{\mathrm{T}}\mathbf{u})$ is the strain tensor. Displacement is $\mathbf{u}$.

- Given that spacetime linear momentum density is,

$$\mathbf{F} = [\mathbf{f}_x|\mathbf{f}_t] = [-\sigma\ |\mathbf{p}]$$

- the balance of linear momentum for arbitrary domain $\mathit{\Omega}$ in spacetime is,

$$\forall\mathit{\Omega}\subset\mathcal{D}: \int_{\partial\mathit{\Omega}}\mathbf{F}.\mathrm{dS} - \int_{\mathit{\Omega}}\mathbf{r}\,\mathrm{dV} = \int_{\partial\mathit{\Omega}}(\mathbf{f}_x.\mathbf{n_x} + \mathbf{f_t}n_t)\mathrm{dS} - \int_{\mathit{\Omega}}\mathbf{r}\,\mathrm{dV} = \mathbf{0} \tag{2a}$$

$$\forall\mathit{\Omega}\subset\mathcal{D} = \int_{\partial\mathit{\Omega}}(-\sigma.\mathbf{n_x} + \mathbf{p}n_t)\mathrm{dS} - \int_{\mathit{\Omega}}\rho\mathbf{b}\,\mathrm{dV} = \mathbf{0} \tag{2b}$$

- Alternatively, in conventional expression of balance laws which are expressed for arbitrary domains ω in space rather than $\mathit{\Omega}$ in spacetime we have,

$$\forall\omega\subset\mathcal{D}\wedge\forall t: \int_{\omega}\mathbf{r}\,\mathrm{dv} - \int_{\partial\omega}\mathbf{f}_x.\mathrm{ds} = \int_{\omega}\mathbf{r}\,\mathrm{dv} - \int_{\partial\omega}(\mathbf{f}_x.\mathbf{n})\,\mathrm{ds} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega}\mathbf{f}_t\,\mathrm{dv} \quad\Rightarrow \tag{3a}$$

$$\forall\omega\subset\mathcal{D}\wedge\forall t: \int_{\omega}\rho\mathbf{b}\,\mathrm{dv} - \int_{\partial\omega}(-\sigma.\mathbf{n})\,\mathrm{ds} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega}\mathbf{p}\,\mathrm{dv} \quad\Rightarrow \tag{3b}$$

1.1.2 Derivation of Strong form from the balance law

- Strong form: In either case, by the application of divergence and localization theorems we obtain the strong form of the balance law,

$$\forall(\mathbf{x},t)\in\mathcal{D}: \nabla_{\mathrm{st}}\mathbf{F} - \mathbf{r} = \left(\dot{\mathbf{f}}_t + \nabla.\mathbf{f}_x\right) - \mathbf{r} = \mathbf{0} \quad \text{Strong Form} \tag{4}$$

  which for solid dynamics it reads as ($\mathbf{f}_t = \mathbf{p}$, $\mathbf{f}_x = -\sigma$),

$$\forall(\mathbf{x},t)\in\mathcal{D}: \dot{\mathbf{p}} - \nabla.\sigma - \rho\mathbf{b} = \mathbf{0} \quad \text{Solid mechanics Strong Form} \tag{5}$$

- Damping effects: In solid mechanics similar to fluids energy is dissipated by internal dissipative mechanisms. To motivate this, assume that the solid volume dv the opposing force to its motion is ($-\alpha\mathbf{v}$ dv) this force (in per volume form) is added to the body force

$$\rho\mathbf{b} \quad \rightarrow \quad \rho\mathbf{b} - \alpha\mathbf{v} \tag{6}$$

where α is the damping coefficient.

thus the final form of strong form for solid mechanics becomes,

$$\forall(\mathbf{x}, t) \in \mathcal{D} : \begin{cases} \dot{\mathbf{p}} - \nabla.\sigma + \alpha\mathbf{v} - \rho\mathbf{b} = \mathbf{0} & \Rightarrow \\ \rho\ddot{\mathbf{u}} + \alpha\dot{\mathbf{u}} - \nabla.\sigma = \rho\mathbf{b} & \text{that is} \\ \rho\ddot{u}_i + \alpha\dot{u}_i - \sigma_{ij,j} = \rho\ddot{u}_i + \alpha\dot{u}_i - (\mathcal{C}_{ijkl}u_{k,l})_j = \rho b_i \end{cases} \tag{7}$$

Solid mechanics Strong Form with damping

1.1.3 Continuum weighted residual statement (WRS) from strong form

- Continuum weighted residual statement (WRS)
- We define the following,
  - space domain $\mathcal{D}$.
  - time interval $\mathcal{I}^t := [t_{\min},\ t_{\max}]$ of the solution; IC is enforced at $t = t_{\min}$ and $t_{\max}$ is the terminal time.
  - spacetime domain $\mathcal{D}^t = \mathcal{D} \times \mathcal{I}^t$. Similarly we define,
    - spacetime domain boundary $\partial\mathcal{D}^t := \partial\mathcal{D} \times \mathcal{I}^t$.
    - spacetime essential BC domain $\partial\mathcal{D}_u^t = \partial\mathcal{D}_u \times \mathcal{I}^t$ ($\partial\mathcal{D}_u$ is spatial essential BC domain).
    - spacetime natural BC domain $\partial\mathcal{D}_f^t = \partial\mathcal{D}_f \times \mathcal{I}^t$ ($\partial\mathcal{D}_f$ is natural essential BC domain).

  The set of strong form equation for elastodynamics and boundary conditions are defined as,

PDE (balance law: strong form)
$$\dot{\mathbf{p}} + \alpha\mathbf{v} - \nabla.\sigma = \rho\mathbf{b} \quad \forall\mathbf{x} \in \mathcal{D}^t \tag{8a}$$

Boundary conditions (BCs))
$$\begin{cases} \mathbf{u} = \bar{\mathbf{u}} & \forall\mathbf{x} \in \partial\mathcal{D}_u^t \quad \text{Essential BC} \\ \mathbf{t} = \sigma.\mathbf{n} = \bar{\mathbf{t}} & \forall\mathbf{x} \in \partial\mathcal{D}_f^t \quad \text{Natural BC} \end{cases} \tag{8b}$$

- Residuals. Accordingly, we define the residuals,

$$\mathcal{R}_I(\mathbf{x}) = -\dot{\mathbf{p}} - \alpha\mathbf{v} + \nabla.\sigma + \rho\mathbf{b} \ @(\mathbf{x}) \qquad \forall\mathbf{x} \in \mathcal{D}^t \ \text{(interior residual)} \tag{9a}$$

$$\mathcal{R}_f(\mathbf{x}) = \bar{\mathbf{t}} - \mathbf{t} = \bar{\mathbf{t}} - \sigma.\mathbf{n} \ @(\mathbf{x}) \qquad \forall\mathbf{x} \in \partial\mathcal{D}_f^t \ \text{(Natural BC residual)} \tag{9b}$$

$$\mathcal{R}_u(\mathbf{x}) = \bar{\mathbf{u}} - \mathbf{u} \ @(\mathbf{x}) \qquad \forall\mathbf{x} \in \partial\mathcal{D}_u^t \ \text{(Essential BC residual)} \tag{9c}$$

- Weighted residual statement (WRS): For the exact solution all the residuals are zero and wise verse. In the weighted residual method (WRM) we have the option to satisfy (9a) and zero to two of BC residuals weakly by multiplying them with a weight function $\mathbf{w}$ and integrating it over their corresponding domains.

  As it is often done in the WRM, we will strongly (*i.e.*, "essentially") satisfy essential BC $\bar{\mathbf{u}} - \mathbf{u}$ on $\partial\mathcal{D}_u$. But $\mathcal{R}_I$ and $\mathcal{R}_f$ are satisfied weakly. The WRS is,

1.1.3.1 Weighted residual statement

$$\text{Find } \mathbf{u} \in \mathcal{V}^{\text{WRS}} = \{\mathbf{v} \in C^2(\mathcal{D}^t)\ |\forall\mathbf{x} \in \partial\mathcal{D}_u^t\ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}\},\ \text{such that,} \tag{10a}$$

$$\forall w \in \mathcal{W}^{\text{WRS}} = C^0(\mathcal{D}^t),\ \forall t \in \mathcal{I}^t \tag{10b}$$

no need to enforce the homogeneous essential BCs for WRS

$$\begin{aligned} 0 &= \int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_I \,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\mathcal{R}_f \,\mathrm{ds} \\ &= \int_{\mathcal{D}} \mathbf{w}.(-\rho\ddot{\mathbf{u}} - \alpha\dot{\mathbf{u}} + \underbrace{\nabla.\sigma}_{C_{ijkl}\,u_{k,lj}\mathbf{e}_i} + \rho\mathbf{b}) \,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.(\bar{\mathbf{t}} - \mathbf{t}) \,\mathrm{ds} \end{aligned} \tag{10c}$$

- equivalence of WRS (10) and strong form BVP (8): Clearly, if the strong form is satisfied, so is the WRS. The converse is also true. By the arbitrariness of **w** we can show that $\mathcal{R}_I$ and $\mathcal{R}_f$ are satisfies strongly (at every point), and $\mathcal{R}_u$ is already satisfied.

1.1.3.2 Implications on a discrete method

In a discrete method **u** is approximated in terms of n_f unknowns:

$$\mathbf{u}^h = \sum_{i=1}^{n_\mathrm{f}} a_i(t)\phi_i(\mathbf{x}) + \mathbf{u}^{ph}$$

where for spatial dimension d (*e.g.*, $d = 3$ in 3D),

- $\phi_i(\mathbf{x}) = [\phi_i^1(\mathbf{x})\ \phi_i^2(\mathbf{x})\ \phi_i^d(\mathbf{x})]$ is the i^{th} spatial test function vector. $\phi_i(\mathbf{x})$ satisfy homogeneous essential BC: $\forall\mathbf{x} \in \partial\mathcal{D}_u^t \phi_i(\mathbf{x}) = 0$.
- $\mathbf{u}^{ph}$ is a particular solution satisfying essential BC: $\forall\mathbf{x} \in \partial\mathcal{D}_u^t \mathbf{u}^{ph}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}, t)$.
- $\mathbf{a}(t) = [a_1(t)\ a_2(t)\ \cdots\ a_{n_\mathrm{f}}(t)]$ are system unknown coefficients (they are scalars depending on time t).

We further discuss this topic in §1.1.5.

Since, the number of unknowns are limited (n_f) so would be the number of weight functions that we can choose (which is equal to n_f. Thus PDE and natural BC cannot (may not) be satisfied strongly (at every point) while Essential BCs are satisfied strongly even in a discrete setting.

- In the WRM there are many choices for the weight functions. The most relevant one to us is the Galerkin method where the weight functions are equal to test functions ($\mathbf{w}_i = \phi_i, i \le n_\mathrm{f}$).
- (Continuous / conventional) Finite Element Method (FEM) is a spatial type of Galerkin method where the test functions are equal to shape functions $\phi_i = N_i$ that satisfy a delta property. That is, $N_i(x_j) = \delta_{ij}$ where x_j are the nodes of the mesh (discrete grid).
- Side note: In more general form, *e.g.*, 4th order beam problem, the shape functions have the delta property on the global degrees of freedom (dofs) rather than nodes.

1.1.4 Continuum weak statement (WK)

- Derivation of Continuum weak statement (WK) from weighted residual statement (WRS)

  The weight function vector **w** has zero derivatives and trial (solution) function **u** has two derivatives. As before, we want to transfer the derivative from **u** to **w** by using the divergence theorem. We want to show,

$$\int_{\mathcal{D}} \mathbf{w}.(\nabla.\sigma(\mathbf{u})) \,\mathrm{dv} = -\int_{\mathcal{D}} \epsilon(w) : \sigma(\mathbf{u}) \,\mathrm{dv} + \int_{\partial\mathcal{D}} \mathbf{w}.\mathbf{t}(\mathbf{u}) \,\mathrm{ds}$$

  where **t** is the traction vector and $\epsilon(w) = \frac{1}{2}\left(\nabla\mathbf{w} + (\nabla\mathbf{w})^{\mathrm{T}}\right)$ is the weight strain field.

  We observe,

$$\mathbf{w}.(\nabla.\sigma) = w_i\sigma_{ij,j} = \frac{\partial w_i\sigma_{ij}}{\partial x_j} - w_{i,j}\sigma_{ij} = \frac{\partial w_i\sigma_{ij}}{\partial x_j} - \frac{w_{i,j} + w_{j,i}}{2}\sigma_{ij} - \underbrace{\frac{w_{i,j} - w_{j,i}}{2}}_{=z_{ij}}\sigma_{ij} \tag{11}$$

  We note that $z_{ji} = \frac{w_{j,i} - w_{i,j}}{2} = -\frac{w_{i,j} - w_{j,i}}{2} = -z_{ij}$ and $\sigma_{ji} = \sigma_{ij}$ (symmetry of the stress tensor). Then we employ the asymmetry and symmetry of **z** and σ to show that:

$$z_{ji}w_{ji} \underbrace{=}_{\text{interchange of dummy indices}} z_{ji}w_{ji} = -(z_{ij})w_{ij} \ \Rightarrow\ 2z_{ij}w_{ij} = 0 \ \Rightarrow\ z_{ij}w_{ij} = 0 \tag{12}$$

also we note that,

$$\left(\frac{w_{i,j}+w_{j,i}}{2}\right)\sigma ij=\epsilon(\mathbf{w}):\sigma(\mathbf{u}) \tag{13}$$

According to (11), (12), and (13) we have:

$$\begin{aligned}
\mathbf{w}.(\nabla.\sigma(\mathbf{u})) &= \frac{\partial w_i\sigma_{ij}}{\partial x_j}-\epsilon(\mathbf{w}):\sigma \quad \Rightarrow\\
\int_{\mathcal{D}}\mathbf{w}.(\nabla.\sigma(\mathbf{u})) &= \int_{\mathcal{D}}\left(\frac{\partial w_i\sigma_{ij}}{\partial x_j}\right)\,\mathrm{dv}-\int_{\mathcal{D}}\epsilon(\mathbf{w}):\sigma(\mathbf{u})\,\mathrm{dv} \quad \Rightarrow \text{(Divergence theorem)}\\
\int_{\mathcal{D}}\mathbf{w}.(\nabla.\sigma(\mathbf{u})) &= \int_{\partial\mathcal{D}} w_i\underbrace{(\sigma_{ij}n_j)}_{t_i(\mathbf{u})}\,\mathrm{ds}-\int_{\mathcal{D}}\epsilon(\mathbf{w}):\sigma\,\mathrm{dv}\\
&= \int_{\partial\mathcal{D}}\mathbf{w}.\mathbf{t}(\mathbf{u})\,\mathrm{ds}-\int_{\mathcal{D}}\epsilon(\mathbf{w}):\sigma(\mathbf{u})\,\mathrm{dv}
\end{aligned}$$

which completes the proof

$$\int_{\mathcal{D}}\mathbf{w}.(\nabla.\sigma(\mathbf{u}))\,\mathrm{dv}=-\int_{\mathcal{D}}\epsilon(\mathbf{w}):\sigma(\mathbf{u})\,\mathrm{dv}+\int_{\partial\mathcal{D}}\mathbf{w}.\mathbf{t}(\mathbf{u})\,\mathrm{ds} \tag{14}$$

We plug (14) in (10c) to obtain:

$$\begin{aligned}
0 &= \int_{\mathcal{D}}\mathbf{w}.\left(-\rho\ddot{\mathbf{u}}-\alpha\dot{\mathbf{u}}+\nabla.\sigma(\mathbf{u})+\rho\mathbf{b}\right)\,\mathrm{dv}+\int_{\partial\mathcal{D}_f}\mathbf{w}.(\bar{\mathbf{t}}-\mathbf{t})\,\mathrm{ds} \quad \Rightarrow\\
0 &= \left\{-\int_{\mathcal{D}}\epsilon(\mathbf{w}):\sigma(\mathbf{u})\,\mathrm{dv}+\int_{\partial\mathcal{D}}\mathbf{w}.\mathbf{t}(\mathbf{u})\,\mathrm{ds}\right\}\\
&\quad -\int_{\mathcal{D}}\mathbf{w}.\left(\rho\ddot{\mathbf{u}}+\alpha\dot{\mathbf{u}}-\rho\mathbf{b}\right)\,\mathrm{dv}+\int_{\partial\mathcal{D}_f}\mathbf{w}.(\bar{\mathbf{t}}-\mathbf{t})\,\mathrm{ds} \;\Rightarrow\\
0 &= -\int_{\mathcal{D}}\epsilon(\mathbf{w}):\sigma(\mathbf{u})\,\mathrm{dv}-\int_{\mathcal{D}}\mathbf{w}.\left(\rho\ddot{\mathbf{u}}+\alpha\dot{\mathbf{u}}\right)\,\mathrm{dv}+\int_{\mathcal{D}}\mathbf{w}.\rho\mathbf{b}\,\mathrm{dv}\\
&\quad +\int_{\partial\mathcal{D}_f}\mathbf{w}.\mathbf{t}(\mathbf{u})\,\mathrm{ds}+\int_{\partial\mathcal{D}_f}\mathbf{w}.(\bar{\mathbf{t}}-\mathbf{t}(\mathbf{u}))\,\mathrm{ds}\\
&\quad +\int_{\partial\mathcal{D}_u}\mathbf{w}.\mathbf{t}(\mathbf{u})\,\mathrm{ds}
\end{aligned}$$

We used the decomposition $\partial\mathcal{D}=\partial\mathcal{D}_f\cup\partial\mathcal{D}_u,\ \partial\mathcal{D}_f\cap\partial\mathcal{D}_u=\emptyset$. Next We get,

$$\begin{aligned}
\int_{\mathcal{D}}\left[\mathbf{w}.\left(\rho\ddot{\mathbf{u}}+\alpha\dot{\mathbf{u}}\right)+\epsilon(\mathbf{w}):\sigma(\mathbf{u})\right]\,\mathrm{dv} &= \int_{\mathcal{D}}\mathbf{w}.\rho\mathbf{b}\,\mathrm{dv}+\int_{\partial\mathcal{D}_f}\mathbf{w}.\bar{\mathbf{t}}\,\mathrm{ds}\\
&\quad +\int_{\partial\mathcal{D}_u}\mathbf{w}.\mathbf{t}(\mathbf{u})\,\mathrm{ds}
\end{aligned} \tag{15}$$

As customary in FE formulations, we eliminate the last integration term on $\partial\mathcal{D}_u$ by enforcing homogeneous boundary conditions for $\mathbf{w}$ on $\partial\mathcal{D}_u$.

1.1.4.1 Continuum weak statement (WK)

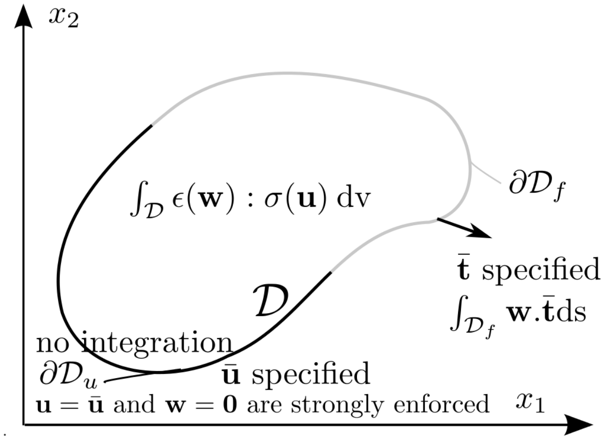
The weak statement for elastodynamics and the boundary conditions are:

$$\text{Find } \mathbf{u}\in\mathcal{V}=\{v\in C^1(\mathcal{D}^t)\mid\forall\mathbf{x}\in\partial\mathcal{D}_u^t\ \mathbf{v}(\mathbf{x})=\bar{\mathbf{u}}(\mathbf{x})\},\ \text{such that,} \tag{16a}$$

$$\forall\mathbf{w}\in\mathcal{W}=\{v\in C^1(\mathcal{D}^t)\mid\forall\mathbf{x}\in\partial\mathcal{D}_u^t\ \mathbf{v}(\mathbf{x})=0\},\ \forall t\in\mathcal{I}^t \tag{16b}$$

$$\int_{\mathcal{D}}\left[\rho\mathbf{w}.\ddot{\mathbf{u}}+\alpha\mathbf{w}.\dot{\mathbf{u}}+\epsilon(\mathbf{w}):\sigma(\mathbf{u})\right]\,\mathrm{dv}=\int_{\mathcal{D}}\mathbf{w}.\rho\mathbf{b}\,\mathrm{dv}+\int_{\partial\mathcal{D}_f}\mathbf{w}.\bar{\mathbf{t}}\,\mathrm{ds} \tag{16c}$$

- Both $\mathcal{V}$ and $\mathcal{W}$ have the same regularity ($C^m(\mathcal{D})$): $m = M/2$, $M = 2$ is the order of the differential equation.
- The less demanding regularity conditions for the solution compared to the weighted residual statement ($C^M(\mathcal{D}) \rightarrow C^m(\mathcal{D})$) takes us to the same function space needed for the balance law (highest derivative is for $\sigma(\mathbf{u}) = C_{ijkl}\, u_{k,l}$ is 1).
- Both $\mathcal{V}$ and $\mathcal{W}$ exactly enforce the essential boundary conditions, with the difference that $\mathcal{W}$ satisfies the homogeneous version.



1.1.5 Discrete solution & weight function space

- Discretization of weak form: We approximate continuum solution $\mathbf{u}$ by discrete solution $\mathbf{u}^h$ in terms of n_f unknowns,

$$\boxed{\mathbf{u}^h(\mathbf{x},t) = \sum_{i=1}^{n_\mathrm{f}} a_i^f(t)\phi_i(\mathbf{x}) + \mathbf{u}^{ph}(\mathbf{x},t)} \tag{17}$$

  This is a semi-discrete expansion of $\mathbf{u}^h$ which is appropriate for time marching schemes and exact integration in time approaches.

  The terms in (17) are,

  - n_f: number of free degrees of freedom (dof).
  - $a_i^f(t)$: Unknown coefficients that are scalar and function of time t.
  - $\phi_i(\mathbf{x})$: trial functions which are vectors $\phi_i(\mathbf{x}) = [\phi_i^1(\mathbf{x}),\ \phi_i^2(\mathbf{x}), \phi_i^3(\mathbf{x})]^\mathrm{T}$ (for $d = 3$) which are functions of space $\mathbf{x}$.
    - $\phi_i(\mathbf{x})$ satisfy homogeneous essential BC: $\forall \mathbf{x} \in \partial\mathcal{D}_u^t :\ \phi_i(\mathbf{x}) = 0.$ $\triangle$
    - ϕ_i is a vector because the unknown of the problem, displacement $\mathbf{u}$, is a vector. They will be a scalar for heat equation for temperature T, or any other order tensor depending on the tensor order of unknown of the problem.
  - $\mathbf{u}^{ph}(\mathbf{x},t)$ is a particular solution satisfying essential BC: $\forall \mathbf{x} \in \partial\mathcal{D}_u^t :\ \mathbf{u}^{ph}(\mathbf{x},t) = \bar{\mathbf{u}}(\mathbf{x},t).$ $\diamondsuit$
  - From $\triangle$ and $\diamondsuit$ $\forall \mathbf{x} \in \partial\mathcal{D}_u^t :\ \mathbf{u}^h(\mathbf{x},t) = \bar{\mathbf{u}}(\mathbf{x},t)$ that is $\mathbf{u}^h$ strongly satisfies the essential BC as we wanted.

- In a fully discrete spacetime scheme, we would have interpolated $\mathbf{u}^h$ as, $\mathbf{u}^h = \sum_{i=1}^{n_\mathrm{f}} a_i^f \phi_i(\mathbf{x},t) + \mathbf{u}^{ph}(\mathbf{x},t)$ that is the trial functions will be functions of space ($\mathbf{x}$) as well as time t. In that case the weak statement (16) should be expressed for $\mathcal{D}^t$ rather than $\mathcal{D}$ and being enforced for $t \in \mathcal{I}^t$.

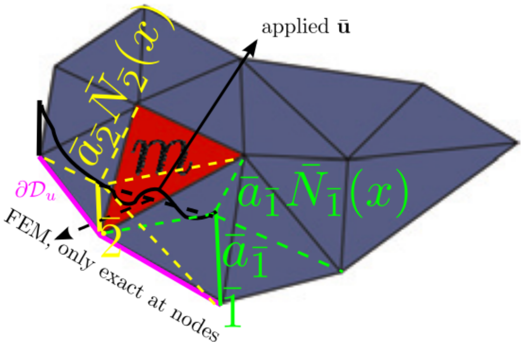
  For the moment, we focus on discretization (17).

FEM expression of trial function and shape functions.

- Shape functions: In FEM trial functions are called shape functions,

$$\begin{aligned} &\mathbf{N}_i^f(\mathbf{x}) = \phi_i(\mathbf{x}) \quad \text{that is} \\ &\mathbf{N}_i^f(\mathbf{x}) = [N_i^{f^1}(\mathbf{x}),\ N_i^{f^2}(\mathbf{x}),\ N_i^{f^3}(\mathbf{x})]^\mathrm{T} = [\phi_i^1(\mathbf{x}),\ \phi_i^2(\mathbf{x}),\ \phi_i^3(\mathbf{x})]. \end{aligned} \tag{18}$$

- Note that shape functions are vectors and $N_i^{f^1}(\mathbf{x})$ is component (direction) j of shape function $\mathbf{N}_i^f(\mathbf{x})$.
- dof: Dofs in FEM correspond to displacement components of nodes of the grid (and their derivatives for shells, plates and other higher order elements).
- FEM delta property: shape function $\mathbf{N}_i(\mathbf{x})$ takes the value of 1 at dof i and zero elsewhere.



- Particular solution $\mathbf{u}^{ph}(\mathbf{x}, t)$ is used to construct $\bar{\mathbf{u}}$ on $\partial\mathcal{D}_u^t$ for $\mathbf{u}^h$,

$$
\begin{aligned}
n_\mathrm{p} &= \text{ number of prescribed dofs} && \text{(19a)}\\
\mathbf{a}^p(t) &= \left[a_1^p(t), \ldots, a_{n_\mathrm{p}}^p(t)\right]^\mathrm{T}\\
&= \text{vector of prescribed values for prescribed dofs} && \text{(19b)}\\
\mathbf{N}^p(\mathbf{x}) &= \left[\mathbf{N}_1^p(\mathbf{x}), \ldots, \mathbf{N}_{n_\mathrm{p}}^p(\mathbf{x})\right]\\
&= \text{(row) vector of shape functions for prescribed dofs} && \text{(19c)}\\
\mathbf{u}^{ph}(\mathbf{x}, t) &= \mathbf{N}^p(\mathbf{x})\mathbf{a}^p(t) = \sum_{i=1}^{n_\mathrm{p}} a_i^p(t)\mathbf{N}_i^p(\mathbf{x}) && \text{(19d)}
\end{aligned}
$$

Size of these arrays are,

$\mathbf{a}^p(t)$	$1 \times n_\mathrm{p}$
$\mathbf{N}_i^p(\mathbf{x}) = [\bar{N}_i^{p^1}(\mathbf{x}),\ \bar{N}_i^{p^2}(\mathbf{x}), \bar{N}_i^{p^3}(\mathbf{x})]^\mathrm{T}$	3×1 vector prescribed shape function i
$\mathbf{N}^p(\mathbf{x})$	$3 \times n_\mathrm{p}$
$\mathbf{u}^{ph}(\mathbf{x}, t)$	3×1 particular displacement vector

The construction of particular solution for a scalar problem, *e.g.*, T in thermal equation, is shown in the figure.

Combining free and prescribed dofs,

- Discrete solution function can be written as (*cf.* (17), (18), (19)),

$$
\begin{aligned}
\mathbf{u}^h(\mathbf{x}, t) &= \mathbf{u}^{f^h} + \mathbf{u}^{ph}(\mathbf{x}, t) && \text{(20a)}\\
&= \sum_{i=1}^{n_\mathrm{f}} a_i^f(t)\mathbf{N}_i^f(\mathbf{x}) + \sum_{i=1}^{n_\mathrm{p}} a_i^p(t)\mathbf{N}_i^p(\mathbf{x}) && \text{(20b)}\\
&= \mathbf{N}.\mathbf{a} && \text{(20c)}
\end{aligned}
$$

where

$$
\mathbf{N} = \left[\mathbf{N}^f \quad \mathbf{N}^p\right] = \left[\mathbf{N}_1^f \quad \cdots \quad \mathbf{N}_{n_\mathrm{f}}^f \quad |\mathbf{N}_1^p \quad \cdots \quad \mathbf{N}_{n_\mathrm{p}}^p\right] \qquad 3 \times n \text{ array} \tag{21a}
$$

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}^f \\ \mathbf{a}^p \end{bmatrix} = \begin{bmatrix} a_1^f \\ \vdots \\ a_1^f \\ -- \\ a_1^p \\ \vdots \\ a_{n_\mathrm{p}}^p \end{bmatrix} \qquad n \times 1 \text{ array} \tag{21b}$$

$$\mathbf{u}^h(\mathbf{x}, t) \qquad 3 \times 1 \text{ array} \tag{21c}$$

$$n = n_\mathrm{f} + n_\mathrm{p} \quad \text{total number of dofs} \tag{21d}$$

Notes:

- Unknown quantities, $\mathbf{a}^f$ are shown in this color. There are n_f unknowns in $\mathbf{a}$.
- The coefficient vector $\mathbf{a}^p$ is known because the prescribed coefficients are given ("prescribed").
- $\mathbf{u}^{fh}$ satisfies the homogeneous essential BC.
- $\mathbf{u}^{ph}$ satisfies the essential BC.
- So $\mathbf{u}^h(\mathbf{x}, t)$ satisfies essential BC. strongly as we required.

Weight functions:

- We need two condition on weighted function set:
  - The number of weight functions should be equal to number of unknowns n_f.
  - Weight functions must satisfy essential boundary conditions.
  - If Galerkin method is used weight functions are equal to solution interpolant functions $\mathbf{w}_i = \phi_i$.
- The choice,

$$\mathbf{w}_i = a_i^f \quad i \leq n_\mathrm{f}$$

satisfies the first two conditions and based on item 3 correspond to a Galerkin method.

1.1.6 Voigt stress and strain notation / displacement to strain operator (tensor)

- Second order strain tensor is,

$$\epsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla^\mathrm{T}\mathbf{u}\right) \quad \text{that is}$$

$$\epsilon(\mathbf{u}) = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) & \frac{1}{2}(u_{1,3} + u_{3,1}) \\ \frac{1}{2}(u_{1,2} + u_{2,1}) & u_{2,2} & \frac{1}{2}(u_{2,3} + u_{3,2}) \\ \frac{1}{2}(u_{1,3} + u_{3,1}) & \frac{1}{2}(u_{2,3} + u_{3,2}) & \frac{1}{2}u_{3,3} \end{bmatrix}$$

- Voigt-notation strain vector is,

$$\gamma(\mathbf{u}) = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{bmatrix} = \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{1,2} + u_{2,1} \\ u_{2,3} + u_{3,2} \\ u_{3,1} + u_{1,3} \end{bmatrix} = L_m \mathbf{u} \quad \text{where} \tag{22a}$$

$$L_m = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \end{bmatrix} \quad \textbf{displacement to strain differential operator} \tag{22b}$$

- Note the dimensions are,
    - Displacement vector $\mathbf{u}$: 3×1.
    - strain tensor ϵ: 3×3.
    - Voigt strain vector γ: 6×1.
    - Displacement to strain differential operator L_m: 6×3.
- Computation of strain in FEM,

$$\left.\begin{array}{rcl} (\mathbf{u}(\mathbf{x},t))_{3\times 1} & = & (\mathbf{N}(\mathbf{x}))_{3\times n}.(\mathbf{a}(t))_{n\times 1} \\ (\gamma(\mathbf{x},t))_{6\times 1} & = & (L_m)_{6\times 3}(\mathbf{u}(\mathbf{x},t))_{3\times 1} = L_m\mathbf{N}(\mathbf{x}).(\mathbf{a}(t)) \end{array}\right\} \Rightarrow$$

$$\gamma(\mathbf{x},t) = \mathbf{B}(\mathbf{x})\mathbf{a}(t), \quad \text{where} \tag{23a}$$

$$(\mathbf{B}(\mathbf{x}))_{6\times n} = (L_m)_{6\times 3}(\mathbf{N}(\mathbf{x}))_{3\times n} \quad \text{displacement to strain array} \tag{23b}$$

- Second order stress tensor is,

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

- Voigt-notation stress vector is,

$$\mathbf{s} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} \tag{24a}$$

Constitutive equation:

- Constitutive equation for stress tensor (Voigt vector) can be obtained for complex models such as (nonlinear) hyperelasticity, hypoelasticity, plasticity, viscoplasticity, *etc.*.
- We focus on linear solid material where we have,

$$\sigma_{3\times 3} = \mathcal{C}_{3\times 3\times 3\times 3}\epsilon_{3\times 3} \quad \Rightarrow \quad \mathbf{s}_{6\times 1} = \bar{\bar{\mathbf{C}}}_{6\times 6}\gamma_{6\times 1} \tag{25}$$

where

$$\mathcal{C}_{3\times 3\times 3\times 3} = \mathcal{C}_{ijkl}\mathbf{e}_i \times \mathbf{e}_j \times \mathbf{e}_k \times \mathbf{e}_l \qquad \text{fourth order elasticity tensor} \tag{26a}$$

$$\mathcal{C}_{ijkl} = \mathcal{C}_{klij} = \mathcal{C}_{jikl} = \mathcal{C}_{ijlk} \tag{26b}$$

$$\bar{\bar{\mathbf{C}}}_{6\times 6} = \bar{\bar{C}}_{ij}\mathbf{e}_i \times \mathbf{e}_j (1 \le i,j \le 6) \qquad \text{second order Voigt elasticity array} \tag{26c}$$

$$\bar{\bar{C}}_{ij} = \bar{\bar{C}}_{ji} \tag{26d}$$

Stress stress term in weak form

- Continuum: The continuum weak statement (16c) has a term, $\epsilon(\mathbf{w}) : \sigma(\mathbf{u})$ which is virtual internal work from virtual displacement (weight function) $\mathbf{w}$ on solution stress $\sigma(\mathbf{u})$. For linear solid this term can be written as,

$$\epsilon(\mathbf{w}) : \sigma(\mathbf{u}) = \epsilon(\mathbf{w}) : \mathcal{C}\epsilon(\mathbf{u}) = \epsilon_{ij}(\mathbf{w})\mathcal{C}_{ijkl}\epsilon_{kl}(\mathbf{u}) \quad \text{or alternatively} \tag{27a}$$

$$\begin{aligned} \epsilon(\mathbf{w}) : \sigma(\mathbf{u}) &= \gamma(\mathbf{w}).\mathbf{s}(\mathbf{u}) = \gamma(\mathbf{w})^{\mathrm{T}}\bar{\bar{\mathbf{C}}}\gamma(\mathbf{u}) \\ &= \gamma_i(\mathbf{w})C_{ij}\gamma_j(\mathbf{u}) \quad (1 \le i,j \le 6) \quad \text{Voigt notation} \end{aligned} \tag{27b}$$

The factor of two for shear strains in (22a) are necessary for the following two properties we used,

- Symmetry of 6×6 Voigt elasticity $\bar{\bar{\mathbf{C}}}$ ($\bar{\bar{C}}_{ij} = \bar{\bar{C}}_{ji}$) in (25), (26d).
- Ability to express $\epsilon(\mathbf{w}) : \sigma(\mathbf{u})$ as $\epsilon(\mathbf{w}) : \mathcal{C}\epsilon(\mathbf{u})$ in (27).

- Expression of strain from displacement in Discrete setting:
  - In discrete version as we will see $\mathbf{w}_{3\times 1} = \mathbf{N}^f_{3\times n_\mathrm{f}}\hat{\mathbf{a}}$ where $\hat{\mathbf{a}}_{n_\mathrm{f}\times 1}$ is an arbitrary vector that enables spanning the n_f dimensional space of the weight functions. In addition, $\mathbf{u}^h = \mathbf{N}_{3\times n}.\mathbf{a}_{n\times 1}$.
  - From (27b), the above line, noting that $\mathbf{a}_{n\times 1}, \mathbf{N}_{3\times n}$ go with solution $\mathbf{u}^h$ and $\hat{\mathbf{a}}_{n_\mathrm{f}\times 1}, \mathbf{N}^f_{3\times n_\mathrm{f}}$ go with the weight $\mathbf{w}$, and $\gamma = \mathbf{B}\mathbf{a}$ we have,

$$\epsilon(\mathbf{w}) : \sigma(\mathbf{u}^h) = \gamma(\mathbf{w})^\mathrm{T}\bar{\bar{\mathbf{C}}}\gamma(\mathbf{u}^h) \qquad = (\mathbf{B}^p\hat{\mathbf{a}})^\mathrm{T}\bar{\bar{\mathbf{C}}}\mathbf{B}\mathbf{a} = \hat{\mathbf{a}}^\mathrm{T}\left(\mathbf{B}^{f\,\mathrm{T}}\bar{\bar{\mathbf{C}}}\mathbf{B}\right)\mathbf{a} \tag{28}$$

1.1.7 Discretization of weak form

- Trial solution and weight functions: Since $\mathbf{u}^{f\,h} = \sum_{i=1}^{n_\mathrm{f}} a_i^f(t)\mathbf{N}_i(\mathbf{x})$ satisfies the homogeneous essential BC and we use a Galerkin method $\mathbf{w}_i(\mathbf{x}) = \mathbf{N}_i(\mathbf{x})$. Clearly weight functions as required satisfy homogeneous essential BC. In addition, since $\mathbf{u}^{p\,h}(\mathbf{x},t)$ satisfies essential BC so does $\mathbf{u}^h(\mathbf{x},t)$ which again is required for the trial solution field. From this and (20), (21) we can write,

$$\begin{aligned} \mathbf{u}^h(\mathbf{x},t) &= \mathbf{N}(\mathbf{x}).\mathbf{a}(t) \qquad \Rightarrow \\ \gamma(\mathbf{u}(\mathbf{x},t)) &= \mathbf{B}(\mathbf{x}).\mathbf{a}(t) \qquad \dot{\mathbf{u}}^h(\mathbf{x},t) = \mathbf{N}(\mathbf{x}).\mathbf{a}(\dot{\mathbf{t}}) \qquad \ddot{\mathbf{u}}^h(\mathbf{x},t) = \mathbf{N}(\mathbf{x}).\mathbf{a}(\ddot{\mathbf{t}}) \qquad (29a) \\ \mathbf{w}(\mathbf{x},t) &= \mathbf{N}^f(\mathbf{x}).\hat{\mathbf{a}}(t) \qquad \Rightarrow \\ \gamma(\mathbf{w}(\mathbf{x},t)) &= \mathbf{B}^f(\mathbf{x}).\hat{\mathbf{a}}(t) \qquad (29b) \end{aligned}$$

- Discrete weak form: By plugging (29) into (16c), that is,

$$\int_\mathcal{D} [\rho\mathbf{w}.\ddot{\mathbf{u}} + \alpha\mathbf{w}.\dot{\mathbf{u}} + \epsilon(\mathbf{w}) : \sigma(\mathbf{u})]\ \mathrm{dv} = \int_\mathcal{D} \mathbf{w}.\rho\mathbf{b}\ \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{t}}\ \mathrm{ds}$$

and noting the expression for $\epsilon(\mathbf{w}) : \sigma(\mathbf{u})$ from (28) we obtain,

$$\forall\hat{\mathbf{a}}:\ \hat{\mathbf{a}}^\mathrm{T}\left\{\underbrace{\left[\int_\mathcal{D}\rho\mathbf{N}^{f\,\mathrm{T}}\mathbf{N}\ \mathrm{dv}\right]}_{\mathbf{M}}\ddot{\mathbf{a}} + \underbrace{\left[\int_\mathcal{D}\alpha\mathbf{N}^{f\,\mathrm{T}}\mathbf{N}\ \mathrm{dv}\right]}_{\mathbf{C}}\dot{\mathbf{a}} + \underbrace{\left[\int_\mathcal{D}\mathbf{B}^{f\,\mathrm{T}}\bar{\bar{\mathbf{C}}}\mathbf{B}\ \mathrm{dv}\right]}_{\mathbf{K}}\mathbf{a} \right.$$
$$\left. - \underbrace{\left[\int_\mathcal{D}\mathbf{N}^{f\,\mathrm{T}}\rho\mathbf{b}\ \mathrm{dv}\right]}_{\mathbf{F}_r} - \underbrace{\left[\int_{\partial\mathcal{D}_f}\mathbf{N}^{f\,\mathrm{T}}\bar{\mathbf{t}}\ \mathrm{ds}\right]}_{\mathbf{F}_N}\right\} = 0$$

Since $\hat{\mathbf{a}}$ is arbitrary (to span the n_f dimensional space of the weight functions), we conclude that,

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{F}_r + \mathbf{F}_N \tag{30}$$

where,

$$\mathbf{M} = \int_\mathcal{D}\rho\mathbf{N}^{f\,\mathrm{T}}\mathbf{N}\ \mathrm{dv} \qquad \text{Mass matrix}$$
$$\mathbf{C} = \int_\mathcal{D}\alpha\mathbf{N}^{f\,\mathrm{T}}\mathbf{N}\ \mathrm{dv} \qquad \text{Damping matrix}$$
$$\mathbf{K} = \int_\mathcal{D}\mathbf{B}^{f\,\mathrm{T}}\bar{\bar{\mathbf{C}}}\mathbf{B}\ \mathrm{dv} \qquad \text{Stiffness matrix}$$
$$\mathbf{F}_r = \int_\mathcal{D}\mathbf{N}^{f\,\mathrm{T}}\rho\mathbf{b}\ \mathrm{dv} \qquad \text{Source term (body force) force vector}$$

$$\mathbf{F}_N = \int_{\partial\mathcal{D}_f} {\mathbf{N}^f}^{\mathrm{T}} \bar{\mathbf{t}} \, \mathrm{ds} \qquad \text{Natural BC force vector}$$

Since all matrices $\mathbf{M}, \mathbf{C}, \mathbf{K}$ have dimensions $n_{\mathrm{f}} \times n$ $(n = n_{\mathrm{f}} + n_{\mathrm{p}})$ and $\mathbf{a}$ has dimension of n rather than the dimension of unknowns n_{f} we separate the matrices and $\mathbf{a}$ to the part that corresponding to $\mathbf{a}^f$ and $\mathbf{a}^p$. For example,

$$\begin{aligned}\mathbf{M}\ddot{\mathbf{a}} &= \left\{\int_{\mathcal{D}} \rho {\mathbf{N}^f}^{\mathrm{T}} \mathbf{N} \, \mathrm{dv}\right\} \ddot{\mathbf{a}} = \left\{\int_{\mathcal{D}} \rho {\mathbf{N}^f}^{\mathrm{T}} \begin{bmatrix}\mathbf{N}^f & \mathbf{N}^p\end{bmatrix} \mathrm{dv}\right\} \begin{bmatrix}\ddot{\mathbf{a}}^f \\ \ddot{\mathbf{a}}^p\end{bmatrix} \\ &= \underbrace{\left\{\int_{\mathcal{D}} \rho {\mathbf{N}^f}^{\mathrm{T}} \mathbf{N}^f \, \mathrm{dv}\right\}}_{\mathbf{M}^{ff}} \ddot{\mathbf{a}}^f + \underbrace{\left\{\int_{\mathcal{D}} \rho {\mathbf{N}^f}^{\mathrm{T}} \mathbf{N}^p \, \mathrm{dv}\right\}}_{\mathbf{M}^{fp}} \ddot{\mathbf{a}}^p\end{aligned}$$

Similar process can be applied to $\mathbf{K}$ and $\mathbf{C}$ terms. Finally, the system can be summarized as follows,

$$\mathbf{M}^{ff}\ddot{\mathbf{a}}^f + \mathbf{C}^{ff}\dot{\mathbf{a}}^f + \mathbf{K}^{ff}\mathbf{a}^f = \mathbf{F}_r + \mathbf{F}_N - \mathbf{F}_D \quad \text{ODE} \quad \text{where} \tag{31a}$$

$$\ddot{\mathbf{a}}^f(t=0) = \ddot{\mathbf{a}}_0^f, \dot{\mathbf{a}}^f(t=0) = \dot{\mathbf{a}}_0^f \qquad \text{Initial condition (IC)} \tag{31b}$$

$$\mathbf{F}_r = \int_{\mathcal{D}} {\mathbf{N}^f}^{\mathrm{T}} \rho \mathbf{b} \, \mathrm{dv} \qquad \text{Source term (body force) force vector} \tag{31c}$$

$$\mathbf{F}_N = \int_{\partial\mathcal{D}_f} {\mathbf{N}^f}^{\mathrm{T}} \bar{\mathbf{t}} \, \mathrm{ds} \qquad \text{Natural (Neumann) BC force vector} \tag{31d}$$

$$\mathbf{F}_D = \mathbf{M}^{fp}\ddot{\mathbf{a}}^p + \mathbf{C}^{fp}\dot{\mathbf{a}}^p + \mathbf{K}^{fp}\mathbf{a}^p \qquad \text{Essential (Dirichlet) BC force vector} \tag{31e}$$

$$\mathbf{M}^{ff} = \int_{\mathcal{D}} \rho {\mathbf{N}^f}^{\mathrm{T}} \mathbf{N}^f \, \mathrm{dv}, \quad \mathbf{M}^{fp} = \int_{\mathcal{D}} \rho {\mathbf{N}^f}^{\mathrm{T}} \mathbf{N}^p \, \mathrm{dv} \qquad \text{Mass matrices} \tag{31f}$$

$$\mathbf{C}^{ff} = \int_{\mathcal{D}} \alpha {\mathbf{N}^f}^{\mathrm{T}} \mathbf{N}^f \, \mathrm{dv}, \quad \mathbf{C}^{fp} = \int_{\mathcal{D}} \alpha {\mathbf{N}^f}^{\mathrm{T}} \mathbf{N}^p \, \mathrm{dv} \qquad \text{Damping matrices} \tag{31g}$$

$$\mathbf{K}^{ff} = \int_{\mathcal{D}} {\mathbf{B}^f}^{\mathrm{T}} \bar{\bar{\mathbf{C}}} \mathbf{B}^f \, \mathrm{dv}, \quad \mathbf{K}^{fp} = \int_{\mathcal{D}} {\mathbf{B}^f}^{\mathrm{T}} \bar{\bar{\mathbf{C}}} \mathbf{B}^p \, \mathrm{dv} \qquad \text{Stiffness matrices} \tag{31h}$$

Often , (31b) are written in the short form,

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{F}_r + \mathbf{F}_N - \mathbf{F}_D \qquad \text{ODE} \quad \text{where} \tag{32a}$$

$$\ddot{\mathbf{a}}(t=0) = \ddot{\mathbf{a}}_0, \quad \dot{\mathbf{a}}(t=0) = \dot{\mathbf{a}}_0 \qquad \text{Initial condition (IC)} \tag{32b}$$

where for short the superscripts for free dofs is dropped knowing that free and prescribed dofs are handles according to (31).

- As we will see through the following example, we actually DO NOT form $\mathbf{M}^{fp}$, $\mathbf{C}^{fp}$, $\mathbf{K}^{fp}$ directly, rather computing their corresponding values from elements and assemble their effects to global force $\mathbf{F}_D$. Local versions of $\mathbf{F}_D$ is

$$\mathbf{f}_D^e = \mathbf{M}^e\ddot{\mathbf{a}}^e + \mathbf{C}^e\dot{\mathbf{a}}^e + \mathbf{k}^e\mathbf{a}^e \tag{33}$$

where $\mathbf{a}^e$ is the local displacement vector of element formed by

1. Having zero values for free dofs.
2. Having prescribed values for prescribed dofs.

1.1.8 Types of damping matrix

- Rayleigh damping matrix, generalizes the formula for $\mathbf{C}$ from (31g) by basically adding a coefficient of stiffness matrix. That is,

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{34}$$

  - Justification for α is as before by modeling the equation of motion as in (6), that is $\rho\mathbf{b} \rightarrow \rho\mathbf{b} - \alpha\mathbf{v}$ and getting (7) which is,

$$\dot{\mathbf{p}} - \nabla.\sigma + \alpha\mathbf{v} - \rho\mathbf{b} = \mathbf{0}$$

  - Justification for β is modifying equation of motion in the form,

$$\sigma = \mathcal{C}(\epsilon + \beta\dot{\epsilon}) \tag{35}$$

1.1.9 Stiffness and mass matrices for 1D elastostatics

- Shape functions are given by,

$$\begin{aligned} &N^e = [N_1^e \; N_2^e] \quad \text{where} \\ &N_1^e = \frac{x_{i+1} - x}{L}, \quad N_2^e = \frac{x - x_i}{L}, \quad L = x_{i+1} - x_i \end{aligned}$$

- In 1D displacement to strain relation is $\epsilon = \frac{\partial}{\partial x}u$ so $L_m = \frac{\partial}{\partial x}$, *cf.* (22b) for 3D version of L_m, and B^e is given by,

$$B^e = L_m[N_1^e \; N_2^e] = \frac{\partial}{\partial x}[\frac{x_{i+1} - x}{L} \;\; \frac{x - x_i}{L}] = \frac{1}{L}[-1 \; 1]$$

- Stiffness matrix From (31h), but for an element level where all dofs would be free, we have,

$$\mathbf{k}^e = \int_{\mathcal{D}} B^{e\mathrm{T}} \bar{\bar{\mathbf{C}}} B^e \,\mathrm{dv} = \int_{x_i}^{x_{i+1}} \frac{1}{L}\begin{bmatrix}-1\\1\end{bmatrix} E \begin{bmatrix}-1 & 1\end{bmatrix} (A\,\mathrm{d}x) = \frac{AE}{L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \tag{36}$$

  where A is area section, E elastic modulus, and L length of the bar. A, E are assumed to be constant.

- (Consistent) mass matrix is obtained from (31f), again element level; all dofs being free,

$$\begin{aligned} \mathbf{M}^e &= \int_{\mathcal{D}} N^{e\mathrm{T}} \rho N^e \,\mathrm{dv} = \int_{x_i}^{x_{i+1}} \begin{bmatrix}\frac{x_{i+1}-x}{L}\\ \frac{x-x_i}{L}\end{bmatrix} \rho \begin{bmatrix}\frac{x_{i+1}-x}{L} & \frac{x-x_i}{L}\end{bmatrix} (A\,\mathrm{d}x) \\ &= \frac{AL\rho}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix} = \frac{m^e}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix} \end{aligned} \tag{37}$$

  where ρ is mass density and $m^e = AL\rho$ is the mass of element. Note that,

  - The sum of components of $\mathbf{M}^e$ is equal to m^e.
  - The mass matrix is NOT diagonal.
  - As we will see this results in NON-DIAGONAL global level mass matrix.

1.1.10 Lumped mass matrix

To obtain a diagonal mass matrix, that would result in system level mass matrix:

- Use quadrature rules whose points match finite element nodes.

- We have,

$$M_{ij}^e = \int_{\mathcal{D}} N_i^e(x)\rho N_j^e(x)\,\mathrm{dv} \tag{38}$$

- The numerical quadrature of M^e_{ij} is then,

$$M^e_{ij} \approx M^{e_q}_{ij}, \qquad \text{where numerical quadrature of mass matrix } M^{e_q}_{ij} \text{ is } \quad M^{e_q}_{ij} = \sum_{k=1}^{n_q} w_k N^e_i(x_k) N^e_j(x_k) \rho(x_k) J(x_k) \tag{39}$$

where w_k are the weight values of the quadrature, n_q is the number of quadrature points, and x_k are the quadrature points. J is the Jacobian function for the transformation from the element coordinate to the quadrature parent element.

- Now if we use a quadrature scheme whose quadrature points match element nodal points by delta property of FEM shape functions (that is shape function i takes that value of 1 at node i and zero at other nodes) and the coincidence of nodal positions and quadrature points we have,

$$N_i(x_j) = \delta_{ij} \tag{40}$$

- Using in (39) it is clear that the mass matrix becomes diagonal and diagonal values are

$$M^{e_q}_{ii} = \sum_{k=1}^{n_q} w_k N^e_i(x_k) N^e_i(x_k) \rho(x_k) J(x_k) \qquad \Rightarrow \boxed{M^{e_q}_{ii} = w_i \rho(x_i) J(x_i) \qquad (\text{no summation on } i)} \tag{41}$$

- For a constant density and cross section (1D) or thickness (2D) we simply have,

$$\boxed{M^{e_q}_{ii} = w_i m^e} \quad m^e = \text{ element mass}, \qquad (\text{Uniform mass density and section (1D)/ thickness (2D)}) \tag{42}$$

again there is no summation on i.

- As an example consider the lumped mass matrix for the first order 1D bar element.
- The quadrature scheme for a line of length L is:

$$\text{Quadrature}(\int_0^L f(x)\,\mathrm{d}x) = \frac{L}{2} f(0) + \frac{L}{2} f(L) \tag{43}$$

- That is this quadrature scheme uses the two end points of a line segment, which will be the end points of an element in computing the lumped mass of the element.

$$\mathbf{M}^e \approx \mathbf{M}^{e_q} = \text{Quadrature}\left(\int_{\mathcal{D}} N^{e\mathrm{T}} \rho(x) N^e \,\mathrm{dv}\right) = \frac{L}{2}\begin{bmatrix} N_1(0) \\ N_2(0)\end{bmatrix} \rho(0)A(0) \begin{bmatrix} N_1(0) & N_2(0)\end{bmatrix} + \frac{L}{2}\begin{bmatrix} N_1(L) \\ N_2(L)\end{bmatrix} \rho(L)A(L) \begin{bmatrix} N_1(L) & N_2(L)\end{bmatrix}$$

$$\mathbf{M}^{e_q} = \frac{L}{2}\begin{bmatrix} 1 \\ 0\end{bmatrix} \rho(0)A(0)\begin{bmatrix} 1 & 0\end{bmatrix} + \frac{L}{2}\begin{bmatrix} 0 \\ 1\end{bmatrix} \rho(L)A(L)\begin{bmatrix} 0 & 1\end{bmatrix} = \begin{bmatrix} \rho(0)A(0)\frac{L}{2} & 0 \\ 0 & \rho(L)A(L)\frac{L}{2}\end{bmatrix} \tag{44}$$

where $\mathrm{dv} = A\,\mathrm{d}x$.

- If further we assume that ρ and A are uniform we have,

$$\mathbf{M}^e \approx \mathbf{M}^{e_q} = \frac{m^e}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} \qquad \text{diagonal mass matrix for constant } \rho \text{ and } A \tag{45}$$

This time we note,

- The sum of components of $\mathbf{M}^e$ is again equal to m^e.
- The mass matrix IS diagonal.
- As we will see this will results in DIAGONAL global level mass matrix.

- For the second order bar element we use the Simpson rule,

$$\text{Quadrature}(\int_0^L f(x)\,\mathrm{d}x) = \frac{L}{6}f(0) + \frac{4L}{6}f(L/2) + \frac{L}{6}f(L) \tag{46}$$

  which results in

$$\mathbf{M}^e = \frac{m^e}{6}\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \text{for second order 1D element} \tag{47}$$

  note that,

  - The lumped mass matrix is again diagonal.
  - The diagonal values may NOT necessarily be equal.

- For even higher order (p) bar elements, unfortunately using uniform distant internal element nodes at L/p positions and quadrature points corresponding to those points limits the order of accuracy in which the mass matrix is integrated.

- This in turn affects the FEM solutions convergence rates for the nodal solution $\mathbf{U}$ and other solution features such as modal quantities; *cf.* §**??**.

- To not sacrifice the order in which the element mass matrix is integrated we do two things:

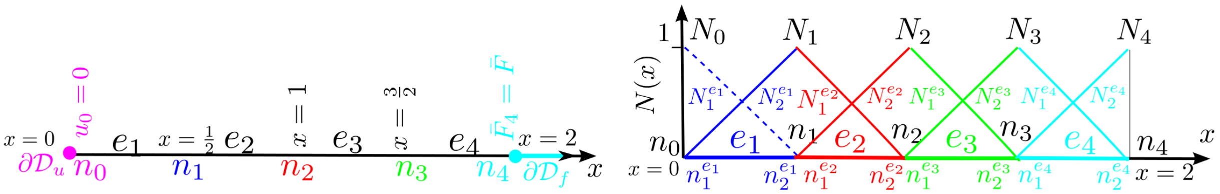
  1. Choose the two end point of the element as two of the quadrature points because we have no freedom in changing the position of the element end nodes.
  2. Similar to Gauss quadrature formulation we optimize the position of high order element nodal (*i.e.*, quadrature) positions.

- As a result the corresponding quadrature scheme with these optimized points will have sufficient order of accuracy and the FEM solution convergence rates are not affected.

- The optimized scheme of quadrature points that include the end points is called Lobatto quadrature

- For more information refer to Section 7.3.2 [**?**].

1.1.11 Example for the assembly of global matrix systems



- We recall that the element matrices for first order 1D bar elements were

$$\mathbf{k}^e = \frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \quad \text{Stiffness matrix} \tag{48a}$$

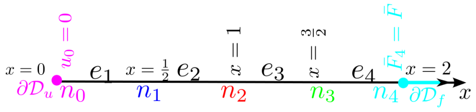
$$\mathbf{M}^e = \frac{m^e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \quad \text{consistent mass matrix} \tag{48b}$$

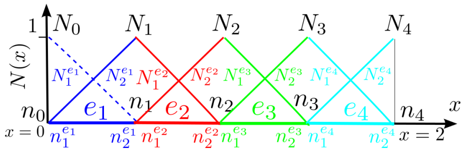
$$\mathbf{M}^e = \frac{m^e}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \text{lumped mass matrix} \tag{48c}$$

- Local stiffness matrix: Since, $E = 1, A = 1, L = \frac{1}{2}$ for all elements, $\mathbf{k}^e$ is given by (48a)

$$\mathbf{k}^e = \frac{(1).(1)}{1/2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$$

1D elastostatics example: Assembly of **M**, **C**, & **K**





- Local mass matrix: Since, $m^e = 1(A=1, L=\frac{1}{2}, \rho=2)$ for all elements, $\mathbf{M}^e$ is given by (48b) and (48c)

$$\mathbf{M}^e = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{consistent mass matrix} \tag{49a}$$

$$\mathbf{M}^e = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{lumped mass matrix} \tag{49b}$$

e	e_1	e_2	e_3	e_4
$\mathbf{k}^e$	$\begin{array}{cc} & \begin{matrix}\bar{1} & 1\end{matrix} \\ \begin{matrix}\bar{1} \\ 1\end{matrix} & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\end{array}$	$\begin{array}{cc} & \begin{matrix}1 & 1\end{matrix} \\ \begin{matrix}1 \\ 1\end{matrix} & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\end{array}$	$\begin{array}{cc} & \begin{matrix}2 & 3\end{matrix} \\ \begin{matrix}2 \\ 3\end{matrix} & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\end{array}$	$\begin{array}{cc} & \begin{matrix}3 & 4\end{matrix} \\ \begin{matrix}3 \\ 4\end{matrix} & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\end{array}$
$\mathbf{M}^e$	$\begin{array}{cc} & \begin{matrix}\bar{1} & 1\end{matrix} \\ \begin{matrix}\bar{1} \\ 1\end{matrix} & \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}\end{array}$	$\begin{array}{cc} & \begin{matrix}1 & 1\end{matrix} \\ \begin{matrix}1 \\ 1\end{matrix} & \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}\end{array}$	$\begin{array}{cc} & \begin{matrix}2 & 3\end{matrix} \\ \begin{matrix}2 \\ 3\end{matrix} & \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}\end{array}$	$\begin{array}{cc} & \begin{matrix}3 & 4\end{matrix} \\ \begin{matrix}3 \\ 4\end{matrix} & \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}\end{array}$

e	e_1	e_2	e_3	e_4
$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e + \mathbf{M}f^{e_1}\ddot{\mathbf{a}}_1^e = \begin{matrix}\bar{1} \\ 1\end{matrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e + \mathbf{M}f^{e_2}\ddot{\mathbf{a}}_2^e = \begin{matrix}1 \\ 2\end{matrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{matrix}2 \\ 3\end{matrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{matrix}3 \\ 4\end{matrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \mathbf{f}_r^{e_1} + \mathbf{f}_N^{e_1} - \mathbf{f}_D^{e_1} = \begin{matrix}\bar{1} \\ 1\end{matrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \mathbf{f}_N^{e_2} - \mathbf{f}_D^{e_2} = \begin{matrix}1 \\ 2\end{matrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{f}_e^{e_3} = \begin{matrix}2 \\ 3\end{matrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{f}_e^{e_4} = \begin{matrix}3 \\ 4\end{matrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\mathbf{K} = \begin{bmatrix} 2+2 & -2 & 0 & 0 \\ -2 & 2+2 & -2 & 0 \\ 0 & -2 & 2+2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \frac{1}{3}+\frac{1}{3} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{1}{3}+\frac{1}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{1}{3}+\frac{1}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix}, \quad \mathbf{F} = \mathbf{F}_N + \mathbf{F}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{F}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{F}(t) \end{bmatrix} \Rightarrow$$

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{F} \quad \text{that is} \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \\ \ddot{a}_3 \\ \ddot{a}_4 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{F}(t) \end{bmatrix} \quad \begin{array}{l} \underline{\text{consistent}} \\ \text{mass} \\ \text{matrix} \end{array} \tag{50}$$

with lumped mass matrix (49b) we would have got

1D elastostatics example: Assembly of **M**, **C**, & **K**

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{F} \quad \text{that is} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \\ \ddot{a}_3 \\ \ddot{a}_4 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{F(t)} \end{bmatrix} \quad \begin{array}{l} \underline{\text{lumped}} \\ \text{mass} \\ \text{matrix} \end{array} \tag{51}$$