04/24/2014-2

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The content of this document is partially & with less detail covered in Bathe's book section 5.5.3 (pages 461-463)

Derivation of Gauss quadrature weights and points:
ω_i's ξ_i

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we noticed in class that the derivation of 2 point ($n=2$) Gauss quadrature weights (ω_i) and points (ξ_i) resulted in 4x4 nonlinear system:

$$\begin{cases} w_1 + w_2 = 2 \\ w_1 \xi_1 + w_2 \xi_2 = 0 \\ w_1 \xi_1^2 + w_2 \xi_2^2 = \frac{2}{3} \\ w_1 \xi_1^3 + w_2 \xi_2^3 = 0 \end{cases}$$

Herein, we present a methodology that simplifies the derivation of weights & points.

Basically, we derive ξ_i in 1st step (n unknowns) and then obtain w_i (the other n unknowns) as opposed to solving all n unknowns at once.

Given: number of Gauss points n

Goal: Gauss points ξ_i (n unknowns) and Gauss weights ω_i (n unknowns)

Since we have 2n unknowns we should be able to exactly integrate polynomials of orders $p \leq 2n-1$ as polynomial of order 2n-1 has 2n coefficients:

$$A(\xi) = a_0 + a_1\xi + \dots + a_{2n-1}\xi^{2n-1}$$

we should have:

$$\boxed{\int_{-1}^{1} A(\xi)\, d\xi = \sum_{i=1}^{n} w_i A(\xi_i)} \quad (1)$$

as before one can find w_i & ξ_i by direct evaluation of both sides of equation (1) and equating factors of $a_0, \dots, a_{2n-1}$:

$$\int_{-1}^{1} A\, d\xi = 2a_0 + 2\frac{a_2}{3} + \frac{2a_4}{5} + \dots + \frac{1+(-1)^j}{j+1} a_j + \dots + \frac{1}{2n-1} a_{2n-2}$$

$$= w_1\left(a_0 + a_1\xi_1 + \dots + a_{2n-1}\xi_1^{2n-1}\right) +$$
$$w_2\left(a_0 + a_1\xi_2 + \dots + a_{2n-1}\xi_2^{2n-1}\right) +$$
$$\dots$$
$$w_n\left(a_0 + a_1\xi_n + \dots + a_{2n-1}\xi_n^{2n-1}\right)$$

$$\Rightarrow \left\{ \begin{array}{l} w_1 + w_2 + \dots + w_n = 2 \\ w_1\xi_1 + w_2\xi_2 + \dots + w_n\xi_n = 0 \\ \vdots \\ w_1\xi_1^i + w_2\xi_2^i + \dots + w_n\xi_n^i = \frac{(-1)^i + 1}{i+1} \qquad 0 \leq i \leq 2n-1 \\ \vdots \\ \quad {}^{2n-1} \qquad {}^{2n-1} \qquad\qquad {}^{2n-1} \end{array} \right.$$

$$\begin{cases} \vdots \\ w_1\xi_1^{2n-1} + w_2\xi_2^{2n-1} + \cdots + w_n\xi_n^{2n-1} = 0 \end{cases}$$

Clearly the solution to this $2n \times 2n$ nonlinear system of equations is very challenging!

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in alternative approach we define

$$P(\xi) = (\xi - \xi_1) \cdots (\xi - \xi_n)$$ polynomial of order n

and divide $A(\xi)$ by $P(\xi)$ to obtain:

divisor   quotient   remainder

$$\boxed{A(\xi) = P(\xi)\,Q(\xi) + R(\xi)} \quad (2)$$

divident

order $2n-1$ order n

order of Q = order of A − order of P = $2n-1-n = n-1$

$$\boxed{\text{order of } R = \text{order of } A - 1 = n-1} \quad (3)$$

so we can express Q as

$$\boxed{Q(\xi) = q_0 + \cdots + q_{n-1}\xi^{n-1}} \quad (4)$$

Also for $R(\xi)$ we observe

for all $i \in \{1, \ldots, n\}$

$$A(\xi_i) = P(\xi_i)\,Q(\xi_i) + R(\xi_i)$$

but since

$$P(\xi) = (\xi-\xi_1)(\xi-\xi_2)\cdots(\xi-\xi_n) \Longrightarrow$$

$$P(\xi_i) = 0$$

and
$$\boxed{A(\xi_i) = R(\xi_i)} \quad (5)$$

we denote $A(\xi_i)$ by A_i

so functions A & R match at n ξ_i points;

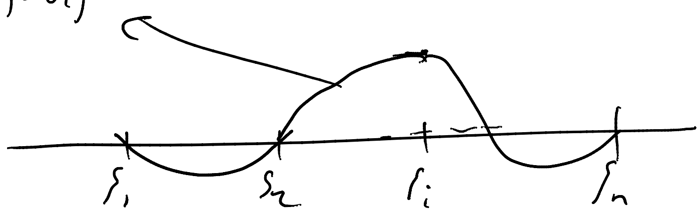
Noting that R is order $n-1$ we can use

Lagrange polynomials below to construct R

Lagrange polynomial $\longleftarrow$ $l_i(\xi_j) = \delta_{ij}$

is zero at all ξ_j $\quad j \neq i$

and 1 at ξ_i



l_i can easily be expressed as:

$$\boxed{l_i(\xi) = \frac{\overbrace{(\xi-\xi_1)\cdots(\xi-\xi_{(i-1)})(\xi-\xi_{(i+1)})\cdots(\xi-\xi_n)}^{\prod_{j=1,\,j\neq i}^{n}(\xi-\xi_i)}}{\underbrace{(\xi_i-\xi_1)\cdots(\xi_i-\xi_{(i-1)})(\xi-\xi_{(i+1)})\cdots(\xi_i-\xi_n)}_{\prod_{j=1,\,j\neq i}^{n}(\xi_i-\xi_j)}}} \quad (6)$$

it's easy to verify that $l_i(\xi_i)=1$ & $l_i(\xi_j)=0 \quad j\neq i$

and it's evident that l_i in expression above is of order $n-1$

So the function:

$$\boxed{\bar{R}(\xi) = \sum_{j=1}^{n} A_j \, l_j(\xi)} \quad (7)$$

satisfies these conditions

$$\bar{R}(\xi_i) = \sum_{j=1}^{n} A_j \, l_j(\xi_i) = \sum_{j=1}^{n} A_j \delta_{ij} = A_i$$

$\bar{R}$ is clearly an order $n-1$ polynomial (l_i's are all order $n-1$)

From (3) & (5) we also note that

R is order $n-1$ and

$R(\xi_i) = A_i$

So 2 polynomials of order $n-1$ $R, \bar{R}$ match @ n points ξ_i $\Longrightarrow$ $\boxed{R = \bar{R}}$ (8)

So from (2), (4), (6), (7), and (8) we have

$$A(\xi) = P(\xi)\, Q(\xi) + R(\xi) =$$

$$P(\xi)\left(q_0 + q_1\xi + \dots + q_{n-1}\xi^{n-1}\right) + \bar{R}(\xi) \Rightarrow$$

$$\boxed{A(\xi) = \sum_{i=0}^{n-1} q_i P(\xi)\, \xi^i + \sum_{i=1}^{n} A_i \, l_i(\xi)} \quad (9)$$

Now

$$\int^1 A(\xi) = \int^1 \left\{ \sum_{i=0}^{n-1} q_i \, P(\xi)\, \xi^i + \sum_{i=1}^{n} A_i \, l_i(\xi) \right\} d\xi \Rightarrow$$

$$\int_{-1}^{1} A(\xi) = \int_{-1}^{1} \left\{ \sum_{i=0}^{n-1} q_i P(\xi)\xi^i + \sum_{i=1} A_i l_i(\xi) \right\} d\xi \Longrightarrow$$

$$\boxed{\int_{-1}^{1} A(\xi) = \sum_{i=0}^{n-1} q_i \left(\int_{-1}^{1} P(\xi)\xi^i d\xi \right) + \sum_{i=1}^{n} A_i \int_{-1}^{1} l_i(\xi) d\xi} \quad (10)$$

At the same time we want the quadrature formula to hold that is

$$\boxed{\int_{-1}^{1} A(\xi) = \sum_{i=1}^{n} w_i A(\xi_i) = \sum_{i=1}^{n} A_i w_i} \quad (11)$$

By direct comparison of (10) & (11) we observe that

Step A: determination of $\xi_1, \ldots, \xi_n$

$\xi_1, \ldots, \xi_n$ satisfy

$$\boxed{\text{For } i=0,\ldots,n-1 \quad \int_{-1}^{1} \xi^i P(\xi) d\xi = 0} \quad (12)$$

n nonlinear equations yielding the values of ξ_i

where $P(\xi)$ was $P(\xi) = (\xi-\xi_1)\cdots(\xi-\xi_n)$

Step B: Now that ξ_i's are determined l_i's are fully determined from equation 6:

$$\boxed{l_i(\xi) = \frac{(\xi-\xi_1)\cdots(\xi-\xi_{(i-1)})(\xi-\xi_{(i+1)})\cdots(\xi-\xi_n)}{(\xi_i-\xi_1)\cdots(\xi_i-\xi_{(i-1)})(\xi-\xi_{(i+1)})\cdots(\xi_i-\xi_n)}} \quad (6)$$

comparison of second summations of (10) & (11) yields:

$$w_i = \int^{1} l_i(\xi) d\xi \quad (13)$$

$$\boxed{w_i = \int_{-1}^{1} \ell_i(\xi)\, d\xi} \quad (13)$$

Equations (12) & (13) provide values ξ_i & w_i in two steps

Example

Guess quadrature for $n=2$ $\Longrightarrow$

$$P(\xi) = (\xi-\xi_1)(\xi-\xi_2)$$

equation 12 yields (note that $n=2$)

$$i=0 \quad \int_{-1}^{1} (\xi-\xi_1)(\xi-\xi_2)\, d\xi = 0 \Rightarrow \int_{-1}^{1} \left(\xi^2 - (\xi_1+\xi_2)\xi + \xi_1\xi_2\right) d\xi = 0 \Rightarrow \frac{2}{3} + 2\xi_1\xi_2 = 0$$

$$i=1 \quad \int \xi(\xi-\xi_1)(\xi-\xi_2)\, d\xi = 0 \quad \int \left(\xi^3 - (\xi_1+\xi_2)\xi^2 + \xi_1\xi_2\xi\right) d\xi = 0 \Rightarrow \frac{2(\xi_1+\xi_2)}{3} = 0$$

$$\Longrightarrow$$

$$\xi_1\xi_2 = \frac{-1}{3}, \quad \xi_1+\xi_2 = 0 \Rightarrow \xi_1^2 = \frac{1}{3} \Rightarrow \boxed{\xi_1 = -\frac{1}{\sqrt{3}},\ \xi_2 = \frac{1}{\sqrt{3}}} \quad (14)$$

Step B derivation of weights

from (13)

$$w_i = \int^{1} \ell_i(\xi)\, d\xi$$

$w_i = \int_{-1} \cdots$

but

$$\ell_1(\xi) = \frac{(\xi-\xi_2)}{(\xi_1-\xi_2)} = \frac{(\xi-\frac{1}{\sqrt{3}})}{(-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}})} = \frac{\sqrt{3}\xi-1}{-2}$$

$$\Rightarrow \quad w_1 = \int_{-1}^{1}\left(\frac{\sqrt{3}\xi-1}{-2}\right)d\xi = \frac{-1}{2}\left(\frac{\sqrt{3}\xi^2}{2}-\xi\right)\Big|_{-1}^{1} = \frac{1}{2}\times 2 = 1$$

similarly $\ell_2(\xi) = \frac{(\xi-\xi_1)}{(\xi_2-\xi_1)} = \frac{(\xi+\frac{1}{\sqrt{3}})}{\frac{1}{\sqrt{3}}-(-\frac{1}{\sqrt{3}})} = \frac{\sqrt{3}\xi+1}{2}$

$$w_2 = \int_{-1}^{1}\left(\frac{\sqrt{3}\xi+1}{2}\right)d\xi = \frac{1}{2}\left(\frac{\sqrt{3}\xi^2}{2}+\xi\right)\Big|_{-1}^{1} = \frac{1}{2}\times 2 = 1$$

So $\boxed{\xi_1 = \frac{-1}{\sqrt{3}},\ \xi_2 = \frac{1}{\sqrt{3}},\ w_1 = 1,\ w_2 = 1}$

as we obtained these results before

Connection between Gauss points and the Roots of Legendre Polynomials

From equation (12) and definition of $P(\xi)$ we have:

$$\boxed{\begin{array}{l} \text{For } i = 0,\dots,n-1 \quad \int_{-1}^{1} \xi^i P(\xi)\, d\xi = 0 \\ P(\xi) = (\xi-\xi_1)(\xi-\xi_2)\cdots(\xi-\xi_n) \end{array}} \qquad (13)$$

this means polynomial $P(\xi)$ of order n is orthogonal to all polynomials of order 0 to order $n-1$ for the inner product $\langle p,q\rangle = \int_{-1}^{1} p(\xi)q(\xi)d\xi$ for polynomials p & q over $[-1,1]$

But we know that Legendre polynomials $P_i(\xi)$ are an orthonormal family of polynomials for $(p,q) = \int_{-1}^{1} p(\xi)q(\xi)d\xi$

$P_n(x)$
0
1
1
x
2
$\frac{1}{2}(3x^2-1)$
3
$\frac{1}{2}(5x^3-3x)$
4
$\frac{1}{8}(35x^4-30x^2+3)$
5
$\frac{1}{8}(63x^5-70x^3+15x)$
6
$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7
$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8
$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9
$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10
$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

Pasted from <http://en.wikipedia.org/wiki/Legendre_polynomials>

// So $P(\xi)$ in (13) is $P_n(\xi)$ (n'th order Legendre polynomial) except possibly ~~the~~ factor of ξ^n (in
that is $P(\xi) = \alpha\, P_n(\xi)$
$P(\xi)$ this factor is 1 were as in Legendre polynomials this may be different from 1)

Clearly in $P(\xi) = (\xi - \xi_1) \cdots (\xi - \xi_n)$

$\left.\begin{array}{l} \xi_1, \ldots, \xi_n \text{ are roots of } P \\ P = \alpha P_n \end{array}\right\} \Longrightarrow$

$$\boxed{\xi_1, \ldots, \xi_i, \ldots, \xi_n \text{ are roots of } P_n}$$

this can be easily seen in the table above. For example for $n = 2$

$$\frac{1}{2}(3\xi^2 - 1) = 0 \Rightarrow \xi_1 = -\frac{1}{\sqrt{3}} \quad \xi_2 = \frac{1}{\sqrt{3}}$$

and so forth.

Finally it can be shown that the weight are:

$w_i = \int_{-1}^{1} \ell_i(\xi)\, d\xi =$

P_n → Legander polynomial order n

$$w_i = \frac{2}{(1 - x_i^2)[P'_n(x_i)]^2}.$$

(Abramowitz & Stegun 1972, p. 887)

- Abramowitz, Milton; Stegun, Irene A., eds. (1965), "Chapter 25.4, Integration", *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover, ISBN 978-0486612720, MR 0167642.

Pasted from <http://en.wikipedia.org/wiki/Gaussian_quadrature>

Note: we have already observed the benefits of Legendre polynomials for example in Spectral methods and the extra credit problem in HW2