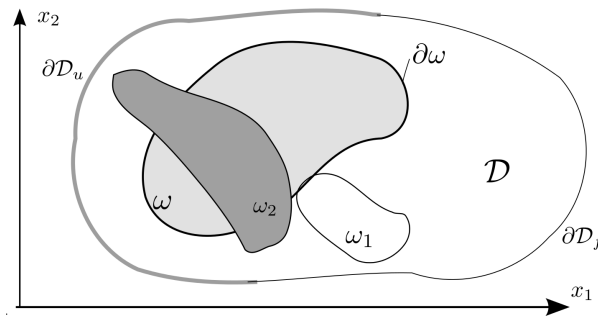


Consider the 2D spatial domain  $\mathcal{D}$  with essential (Dirichlet) boundary  $\partial\mathcal{D}_u$  and natural (Neumann) boundary  $\partial\mathcal{D}_f$  ( $\overline{\partial\mathcal{D}_u} \cup \overline{\partial\mathcal{D}_f} = \overline{\mathcal{D}}$ ,  $\partial\mathcal{D}_u \cap \partial\mathcal{D}_f = \emptyset$ ) for steady state heat equation. The purpose of this problem is to derive the weak form from two different approaches: balance law approach in problems 1-3 and functional approach in problem 4.



1. **Balance law (30 Points):** The spatial flux  $\mathbf{f}$  and source term  $s$  for this problem are: *heat flux vector*  $\mathbf{f} = \mathbf{q}$  and *heat source (scalar)*  $s = Q$ . This balance law corresponds to steady state balance of energy when only temperature effects are considered.  
Write the balance law statement for **arbitrary**  $\omega \subset \mathcal{D}$ .
2. **Strong form (100 Points):** In this step we derive the partial differential equation (PDE) and close the system with constitutive/compatibility equations and boundary conditions.
  - (a) PDE: From the balance law derive the PDE. In less than 2-3 lines briefly write the intermediate equations and theorems used to derive the PDE. Why the space of acceptable solutions for the PDE is more limited than for the balance law.
  - (b) BC: Write the expressions for the boundary conditions in terms of primary field temperature  $T$  and heat flux vector  $\mathbf{q}$  on  $\partial\mathcal{D}_u$  and  $\partial\mathcal{D}_f$ . The applied boundary values are  $\bar{T}$  and  $\bar{q}$ . Note that  $\bar{q}$  is scalar and is the outward net flux and pay attention to the fact that  $\mathbf{q}$  is a vector when writing natural BC. Can we solve the system of these boundary conditions and the PDE?
  - (c) Constitutive equation: Write the constitutive equation based on Fourier law (relating  $\mathbf{q}$  and  $T$ ). Does Fourier heat flux equation apply to all material types? How do you compare this with a balance law statement?
  - (d) Boundary value problem (BVP): Combine all the steps above to obtain the BVP in terms of  $T$ .
3. **Weak form (100 Points):** We want to obtain weak statements (integral forms) whose solution is the same as the solution to the BVP above.
  - (a) **Weighted Residual (WR) Statement**: As it is the common practice, for the candidate solution  $T$  we strongly satisfy the essential boundary condition residual. That is,  $\forall \mathbf{x} \in \partial\mathcal{D}_u$  :  $\mathcal{R}_u(\mathbf{x}) = \bar{T}(\mathbf{x}) - T(\mathbf{x}) = 0$ .
    - i. Write the interior residual  $\mathcal{R}_I$  for  $\mathbf{x} \in \mathcal{D}$  and natural BC residual for  $\mathbf{x} \in \partial\mathcal{D}_f$ .
    - ii. Write the Weighted Residual statement.
    - iii. What space should the candidate solution function  $T(\mathbf{x})$  belong to? Noting this set by  $\mathcal{V}_w$ , that is  $T \in \mathcal{V}_w$ , write  $\mathcal{V}_w$  in the form  $\mathcal{V}_w = \{u \in C^n(\mathcal{D}) \mid \text{required boundary conditions (if any)}\}$ . That is, specify i) number of derivatives ( $m$ ) and boundary conditions needed for  $T$ .
    - iv. What space does the weight function belong to ( $w \in \mathcal{W}_w$ )? Again write it in the form  $\mathcal{W}_w = \{u \in C^n(\mathcal{D}) \mid \text{required boundary conditions (if any)}\}$ .

(b) **Weak Statement:** Noting that  $w(\nabla \cdot (\kappa \nabla T)) = \nabla \cdot (w \kappa \nabla T) - \nabla w \cdot (\kappa \nabla T)$ :

i. Use the Gauss (divergence) theorem to transform the weighted residual statement to the *weak statement*.

Hints: 1. Make sure in the WR statement  $\mathcal{R}_f$  is added with the right sign so that boundary terms generated by  $(w \kappa \nabla T)$  term above cancel some of  $\mathcal{R}_f$  terms; 2. After the application of Gauss theorem some boundary terms are generated on  $\partial \mathcal{D}_u$ . Make judicious choice for the spaces of the functions  $T$  or  $w$  so that those terms would disappear.

ii. What space should the candidate solution function  $T(\mathbf{x})$  belong to? Noting this set by  $\mathcal{V}$  express it in the form  $\mathcal{V} = \{u \in C^n(\mathcal{D}) \mid \text{required boundary conditions (if any)}\}$ . That is,  $m$  and BCs should be specified.

iii. What space does the weight function belong to ( $w \in \mathcal{W}$ )? Again write it in the form  $\mathcal{W} = \{u \in C^n(\mathcal{D}) \mid \text{required boundary conditions (if any)}\}$ . Note that due to the relation between  $\mathcal{W}$  and  $\mathcal{V}$ ,  $\mathcal{W}$  is often denoted by  $\mathcal{V}_0$ .

iv. Briefly (less than 2-3 lines) discuss how the space of the solution for Weak statement compares to those from BVP and WR as well as that from the balance law (+BCs and constitutive equation).

4. **Functional approach (70 Points)**<sup>1</sup> The solution to the heat equation can alternatively be obtained by minimization of the following functional (no convective BCs):

$$\Pi(T, \nabla T) = \int_{\mathcal{D}} \left\{ \frac{1}{2} \nabla T \cdot (\kappa \nabla T) - TQ \right\} dV - \int_{\partial \mathcal{D}_f} T(-\bar{q}) dS \quad (1)$$

where  $T$  strongly satisfies the essential BC:

$$\forall \mathbf{x} \in \partial \mathcal{D}_u : T(\mathbf{x}) = \bar{T}(\mathbf{x}) \quad (2)$$

(a) By using the extremum condition  $\delta \Pi(T, \nabla T) = 0$  and letting  $w := \delta T$  derive the weak statement obtained by balance law approach in 3.b.

(b) Why the weight function strongly satisfy the *homogeneous essential boundary condition*: ( $\forall \mathbf{x} \in \partial \mathcal{D}_u : w(\mathbf{x}) = 0$ )? Hint:  $w = \delta T$  and use equation (2).

(c) Briefly less than 2-3 lines compare the functional approach with balance law approach in obtaining the weak statement.

<sup>1</sup>In this case the functional does not correspond to an “energy”.