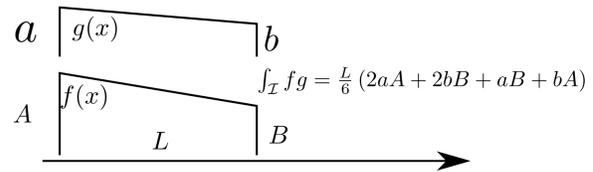


(a) Schematic of the 1D bar example.



(b) Integration of product of linear by linear functions.

- HW3 is graded out of 275 (problems 1-5). So, you **DO NOT** need to do any of the 3 extra credit problems.
- From extra credit problems 1-3 a maximum of 75 of the total 85 points can be added to your grade.

Consider the 1D bar example shown with the following boundary value problem:

$$\begin{cases} \frac{d\sigma(u(x))}{dx} + q(x) = \frac{d}{dx} \left(EA \frac{du(x)}{dx} \right) + q(x) = 0 & \text{Strong form} \\ u(0) = \bar{u} & \text{Essential BC on } \partial\mathcal{D}_u = \{0\} \\ F(2) = EA \frac{du(x)}{dx} = \bar{F} & \text{Natural BC on } \partial\mathcal{D}_f = \{2\} \end{cases} \quad (1)$$

Material and load properties are,

$$q(x) = \begin{cases} 2 - 2x & x < 1 \\ 0 & 1 \leq x \leq 2 \end{cases} \quad E(x) = 1, \quad A(x) = 1, \quad \bar{u} = 1, \quad \bar{F} = 1$$

More details on this problem can be found in course notes equations (177) to (179).

1. **(10 Points)** Starting with the interior residual and residual on natural boundary derive weighted residual statement (equation (180) in course notes).

$$\begin{aligned} \text{Find } u \in \mathcal{V} = \{v \mid v \in C^2([0, 2]), v(0) = \bar{u} = 1\} : \forall w \in \mathcal{W} = C^0([0, 2]) \\ \int_0^2 w \left(\frac{d^2 u}{dx^2} + q(x) \right) dx + w^f \left(1 - \frac{du}{dx}(2) \right) = 0 \quad (\text{no restrictions for } w \text{ on } \partial\mathcal{D}_u) \end{aligned} \quad (2)$$

2. **(10 Points)** Apply integration by parts to (2) to obtain the weak statement for $w^f = \mathbf{w}$ (equation (187) in course notes). In the process determine how certain boundary terms are eliminated.

$$\begin{aligned} \text{Find } u \in \mathcal{V} = \{v \mid v \in C^1, v(0) = \bar{u} = 1\} : \forall w \in \mathcal{V}_0 = \{v \mid v \in C^1, v(0) = \bar{u} = 0\} \\ \int_0^2 \frac{dw}{dx} \frac{du}{dx} dx = \int_0^1 w(x) \cdot q(x) dx + w(2) \end{aligned} \quad (3)$$

3. **(155 Points)** For 4 unknowns ($n = 4$):

- (a) Solve the problem for 1) Subdomain method (S); 2) Collocation method (C), 3) Finite Difference (FD), 4) Least Square (L), 5) Galerkin method with $\phi = [x \ x^2 \ x^3 \ x^4]$ (G), and 6) Galerkin method with Finite Element hat functions (FE) shown in figure 3a . Trial functions for 1, 2 and 4 is also $\phi = [x \ x^2 \ x^3 \ x^4]$. For your solutions use $\phi_p = 1$. For each case clearly indicate \mathbf{K} , \mathbf{F} and the solution \mathbf{a} (for finite difference method the linear matrix equation corresponds to unknown values $u_1^{FD} = u^{FD}(0.5)$, $u_2^{FD} = u^{FD}(1)$, $u_3^{FD} = u^{FD}(1.5)$, and

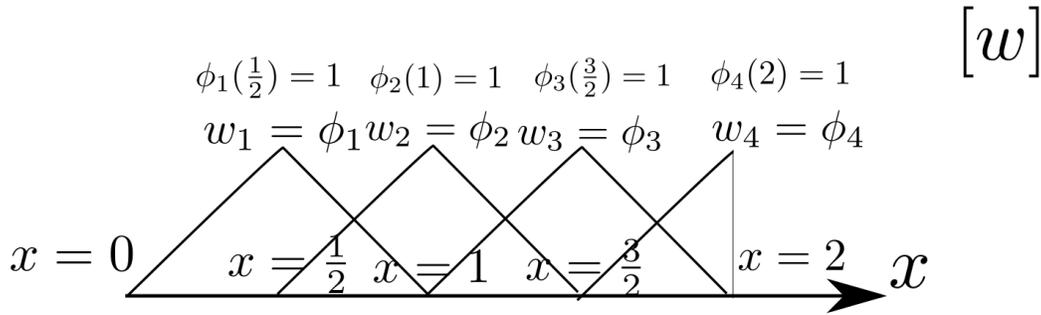


Figure 1: Trial and Weight functions for $n = 1$ 1D Finite Element Method.

Method	ϕ	\mathbf{w}	Can WR (2) be used and why?	Can WK (3) be used and why?
S	$[x \ x^2 \ x^3 \ x^4]$			
C	$[x \ x^2 \ x^3 \ x^4]$			
L	$[x \ x^2 \ x^3 \ x^4]$			
G	$[x \ x^2 \ x^3 \ x^4]$			
FE	figure 3a			

$u_4^{FD} = u^{FD}(2)$. Use weighted residual method (2) for **S, C, L** and weak statement (3) for **G, FE** ($6 \times 25 =$ **150 Points**)).

(Hint: For FE case there are many integrands that are products of two linear functions. The formula in figure (b) can be used to evaluate such integrals.)

- (b) Summarize the unknowns based on the given solution orders above in the matrix U (**5 Points**):

$$U = \begin{bmatrix} a_1^S & a_2^S & a_3^S & a_4^S \\ a_1^C & a_2^C & a_3^C & a_4^C \\ u_1^{FD} & u_2^{FD} & u_3^{FD} & u_4^{FD} \\ a_1^L & a_2^L & a_3^L & a_4^L \\ a_1^G & a_2^G & a_3^G & a_4^G \\ a_1^{FE} & a_2^{FE} & a_3^{FE} & a_4^{FE} \end{bmatrix} \tag{4}$$

- 4. (**50 Points**) Fill out table (4). If extra space needed the three questions about \mathbf{w} and applicability of WR and WK can be directly addressed in text. For L (least square) also list \mathbf{w}^f (weight functions on $\partial\mathcal{D}_f = \{2\}$).

Special attention should be paid for the applicability of WK (3) to S and C (if not explain) and WR (2) to FE.

- 5. (**50 Points**) Matlab data processing: Run the provided Matlab function WRM1D.m from the course web page ¹ with the input argument U in (4) to generate various plots for $n = 1$ to $n = 4$ for all the method discussed above.

- (a) File “WRM-error-method6-FE.png” displays the error between Finite Element solution and exact solution ($\Delta u_{FE} = u_{FE} - u$) versus spatial position ($x = 0$ to $x = 2$) for $n = 1$ to $n = 4$). Discuss the special property of the error at finite element nodes (e.g., $x = 2$ for $n = 1$, $x = 1, 2$ for $n = 2$, $x = \frac{2}{3}, \frac{4}{3}, 2$ for $n = 3$, and $x = \frac{1}{2}, 1, \frac{3}{2}, 2$ for $n = 4$) (**10 Points**).

- (b) In collocation and finite difference methods we enforce the equations at discrete points. Why the solutions at these points do not match the exact solution?

¹<http://rezaabedi.com/wp-content/uploads/Courses/FEM/WRM1D.m>

Files “WRM-solutions(n)Unknown.png” ($n = 1$ to $n = 4$) display the solutions. **(10 Points)**

- (c) In Least square method, we minimize R^2 for the entire admissible function space $\{x^1, \dots, x^n\}$ ($n = 1$ to $n = 4$). We however observe that other methods (for example Galerkin with the same trial functions (G)) is a more accurate solution. This can be observed in “WRM-solutions(n)Unknown.png” “WRM-errors(n)Unknown.png” file. Explain why this is not a contradiction? **(10 Points)**
- (d) For Finite element methods errors between discrete and continuum solutions (here denoted by e) can be expressed as,

$$e = Ch^\alpha \quad \Rightarrow \quad \log(e) = \log(C) + \alpha \log(h)$$

That is in a log-log plot, $\log(e)$ should be approximately linear versus $\log(h)$ with the slope of α . Refer to file “WRM-all-errors-log-h.png” and verify if the data set for Finite Element is linear and from the plot provide an approximate value for α . **(10 Points)**

- (e) From the same file “WRM-all-errors-log-h.png” discuss which method has the fastest decrease of the error versus h ? Does the rate of the error convergence (slope of $\log(e)$ versus $\log(h)$) increases as h decreases for this method? **(10 Points)**

Extra credit problems: A maximum of **75 points** from extra credit problems can be added to your grade. You do not need to solve these problems.

- (10 Points)** In Weighted Residual statement (2) it is stated that the weight functions (in \mathcal{W}) should be continuous (C^0). Reviewing the weight functions in (4) and your solutions in problem 3. In which approach (S, C, L) the weight functions are not C^0 . Is this a problem?
- (10 Points)** In Weak statement (3), it is stated that the solution, hence the trial functions ϕ_i , should have continuous derivative **everywhere** (C^1). Are the trial functions for Finite Element method (*cf.* figure 3a) C^1 ? If not, does this cause a problem in evaluating the weak statement?
- (65 Points)** In Finite Element method (and many other numerical methods) the solution of algebraic equation $\mathbf{K}\mathbf{a} = \mathbf{F}$ is major source of computational costs. If the matrix \mathbf{K} is diagonal, the solution is trivial ($a_i = \frac{F_i}{K_{ii}}$, no summation on i). In particular, for identity \mathbf{K} :

$$\mathbf{K} = \mathbf{1} \wedge \mathbf{K}\mathbf{a} = \mathbf{F} \quad \Rightarrow \quad \delta_{ij}a_j = F_i \quad \Rightarrow \quad \boxed{\mathbf{a} = \mathbf{F}} \quad (5)$$

Thus, for identity \mathbf{K} we circumvent the solution of a linear system. From (3) we observe that for weak statement,

$$K_{ij} := \mathcal{A}(\phi_i, \phi_j) = \int_0^2 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx \quad (6)$$

where the bilinear operator $\mathcal{A}(\phi_i, \phi_j)$ has the properties of an inner product and for any given basis for $\{\phi_1, \phi_2, \dots, \phi_n\}$ we can form an orthonormal basis using Gram Schmidt method. That is, for the new basis system \mathbf{K} is equal to an identity matrix.

There are many orthogonal function basis in mathematics. For example, Legendre polynomials satisfy the following orthogonality condition:

$$\int_{-1}^1 P_m(\xi)P_n(\xi) d\xi = \frac{2}{2n+1} \delta_{mn} \quad \text{no sum on } n \quad (7)$$

These functions can be obtained from the recursive equation:

$$P_0(\xi) = 1 \quad (8a)$$

$$P_1(\xi) = \xi \quad (8b)$$

$$(n+1)P_{n+1}(\xi) = (2n+1)\xi P_n(\xi) - nP_{n-1}(\xi) \quad (8c)$$

For our purpose, the first four functions are given by,

$$\begin{array}{ll}
 n & P_n(\xi) \\
 0 & 1 \\
 1 & \xi \\
 2 & \frac{1}{2}(3\xi^2 - 1) \\
 3 & \frac{1}{2}(5\xi^3 - 3\xi)
 \end{array} \tag{9}$$

To use these orthogonal basis for our bar example, we use the coordinate transformation,

$$\xi = x - 1 \quad \text{for } x \in [0 \ 2] \tag{10}$$

to map x to $\xi \in [-1 \ 1]$ used in definition of Legendre polynomials.

(a) Show that the following trial functions:

$$\boxed{\phi_{i+1}(\xi) = \sqrt{i + \frac{1}{2}} \int_{-1}^{\xi} P_i(\eta) \, d\eta} \tag{11}$$

- are admissible trial functions for weak statement (3). That is, $\phi_i(x=0) = \phi_i(\xi=-1) = 0$. **(5 Points)**
- are orthonormal with respect to inner product in (6): **(10 Points)**

$$K_{ij} := \mathcal{A}(\phi_i, \phi_j) = \int_0^2 \frac{d\phi_i}{dx}(x) \frac{d\phi_j}{dx}(x) \, dx = \int_{-1}^1 \frac{d\phi_i}{d\xi}(\xi) \frac{d\phi_j}{d\xi}(\xi) \frac{d\xi}{dx} \, d\xi = \delta_{ij}$$

(Hint: use (7) and (11))

(b) Using (11) and (9) show that, **(10 Points)**

$$\begin{array}{ll}
 i & \phi_i(\xi) \\
 1 & \sqrt{\frac{1}{2}}(\xi + 1) \\
 2 & \sqrt{\frac{3}{8}}(\xi^2 - 1) \\
 3 & \sqrt{\frac{5}{8}}(\xi^3 - \xi) \\
 4 & \frac{1}{8}\sqrt{\frac{7}{2}}(5\xi^4 - 6\xi^2 + 1)
 \end{array} \tag{12}$$

(c) Using (3) we have,

$$F_i = \int_0^1 \phi_i(x) \cdot q(x) \, dx + \phi_i(x=2) = \int_{-1}^0 \phi_i(\xi) \cdot q(\xi) \frac{dx}{d\xi} \, d\xi + \phi_i(\xi=1) \quad \text{for}$$

$$q(\xi) = \begin{cases} -2\xi & \xi < 0 \\ 0 & 0 \leq \xi \leq 1 \end{cases}$$

Find $\mathbf{F} = \mathbf{a}$ **(20 Points)**

(d) From the values \mathbf{a} above, (12), $u^h = a_i \phi_i + \phi_p$, $\phi_p = 1$, obtain u^h in terms of x and compare it with the Galerkin solution in question 3. (Hint: it is easier to obtain each $a_i \phi_i$ first as a function of ξ then change $\xi \rightarrow x$ before plugging into $u^h = a_i \phi_i + \phi_p$.)

(20 Points)