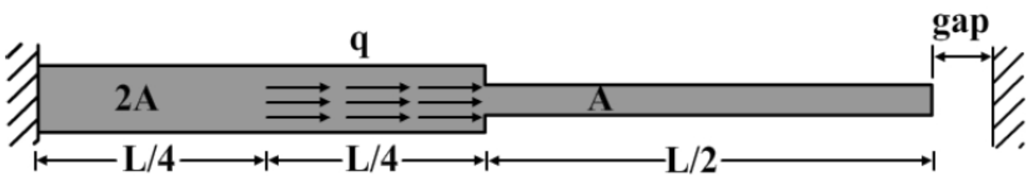

Figure 1: Bar problem

1. $\boxed{\textbf{50 Points}}$ Obtain an approximate solution of the following boundary value problem using spectral Galerkin method (that is trial functions are polynomials over the entire domain):

$$u''(x) + u(x) + x = 0 \tag{1a}$$
$$u(0) = 0 \tag{1b}$$
$$u(1) = 0 \tag{1c}$$

   (a) Derive the weak statement for the problem. (**20 Points**)

   (b) Using **one** appropriate quadratic polynomial trial function, find an approximate solution of the form $\phi(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ (*i.e.,* find appropriate α_i) such that $\phi(x) = \phi_1(x)$ satisfies all essential boundary conditions. Next, find an appropriate $\phi_p(x)$ that satisfies all homogeneous essential BCs and from which seek solution in the form $u^h(x) = \phi_p(x) + \phi(x)a$. Your weak statement solution involves only one unknown a. You should not directly plug $\phi(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ with α_i as unknowns in the weak statement.(**25 Points**)

   (c) What is the error between approximate solution and the exact solution $u(x) = -x + \frac{\sin(x)}{\sin(1.0)}$ at $x = 0.5$: $e(0.5) = u^h(0.5) - u(0.5)$? (**5 Points**)

2. $\boxed{\textbf{80 Points}}$ An axially loaded bar, fig. 1, carries a distributed load over one quarter of its length and has a small gap at one end as shown in the following figure. Divide the bar into appropriate number of elements and compute displacements and stresses in the bar. Note that if the load is large enough to close the gap then the gap can be treated as a known displacement boundary condition. Assume $L = 500$ mm, $A = 25$ mm^2, $E = 20,000$ N/mm^2, $q = 400$ N/mm, (a) gap = 1 mm, (b) gap = 20 mm.
Source: http://nptel.ac.in/downloads/105106051/

3. $\boxed{\textbf{60 Points}}$ Consider an 1D bar element with $E = 1$, $L = 1$, and $A(x) = 1 + x$.

   (a) Obtain the stiffness matrix for finite element method, using equation (373) in section 2.2 of course notes.(**10 Points**)

   (b) Obtain the stiffness matrix using the direct method using the methodology described in equations 307-308 of section 2.2 (equations 375-376). Include all the steps in equation 307 in your derivation. (**20 Points**)

   (c) Comparison of the two approaches: i) Compare the stiffness matrices; ii) Assume that $u_1 = 0, u_2 = 1$ (the end point displacements). Plot the displacement field, strain, and stress for the two methods; iii) (extra credit) compare internal energies $\mathcal{E} = \int_0^1 \frac{1}{2} E(x)A(x)\epsilon^2(x)\,\mathrm{d}x$ ($\epsilon = \frac{\mathrm{d}u}{\mathrm{d}x}$). (**30 Points**) + (**20 Points**) (extra credit)

4. **This problem is NOT counted in your grade, but I still keep it if anyone wants to work on it and return it for feedback.** For the truss shown in figure the prescribed dofs are: $\bar{U}_{\bar{1}} = \bar{U}_{\bar{2}} = \bar{U}_{\bar{4}} = \bar{U}_{\bar{5}} = 0; \bar{U}_{\bar{3}} = \frac{1}{10}$. For all four truss members $E = 1, A = 1$. Other information is provided in the figure 3.

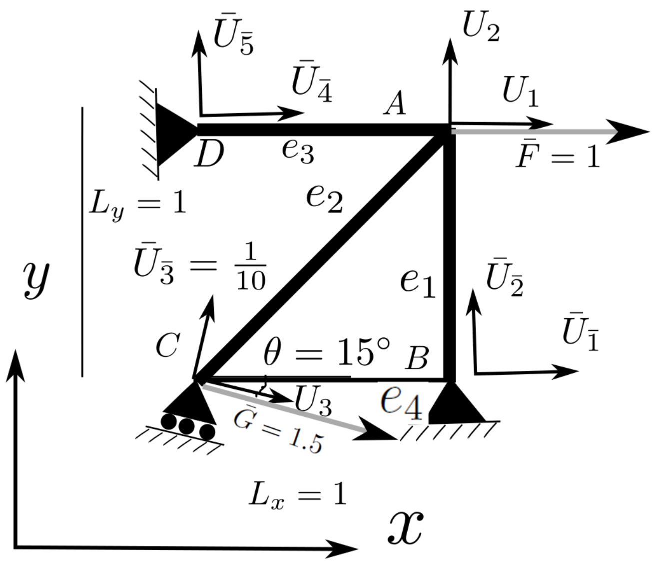

Figure 2: 3 dof truss with an angled support

(a) Form element force vectors and stiffness matrices. Note that for elements e_2 and e_4 two different coordinate systems are used at the two ends and an equation of the form (394) in section 2.3 should be employed. Also, the angle θ in the figure is the angle that the bar makes with x axis and is different from θ_1 and θ_2 in the figure used for equation (394).

(b) Form global stiffness matrix, total force and solve for global $\mathbf{U}_f$.

(c) Summarize element local displacements and obtain their axial force.

(d) Obtain support reactions at B (2), C (1), and D (2).

5. **80 Points** In figure 5 **Frame element** e_2 is hinged to **truss element** e_1. For the frame element, a concentrated moment $\bar{M} = 1$ is applied at $x = 0.75$ and a distributed load $q = x$ is applied over the length of the frame.

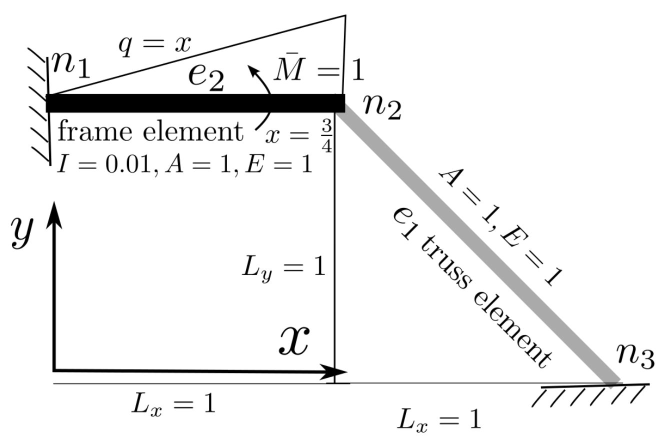

Figure 3: Frame and truss example.

(a) Number free and prescribed dofs. (**10 Points**)

(b) Form element force vectors and stiffness matrices. (**20 Points**)

(c) Form global stiffness matrix, total force and solve for global $\mathbf{U}_f$.(**20 Points**)

(d) Obtain displacement (y), rotation ($\theta = \frac{\mathrm{d}y}{\mathrm{d}x}$), and moment ($M = EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$) for the frame element at $x = 0.5$. Note that $y(\xi) = \sum_{i=1}^{4} N_i^e(\xi)a_i^e$. Also, since $\mathbf{B}^e = \frac{\mathrm{d}^2\mathbf{N}^e}{\mathrm{d}x^2} \Rightarrow M = EI\sum_{i=1}^{4} B_i^e(\xi)a_i^e$. (**30 Points**)