

Consider the isoparametric 5-node element shown in the figure for a stationary thermal conduction problem with material conductivity $\kappa = k\mathbf{I}$ (isotropic conductivity), $k = 2$. In order to capture the curved domain boundary geometry, the extra node 5 is inserted on the left boundary. For $L = 4$ the coordinates of the element in Cartesian system are:

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 & -2 & -2\sqrt{2} \\ -2 & -2 & 2 & 2 & 0 \end{bmatrix} \tag{1}$$

The questions for this element are:

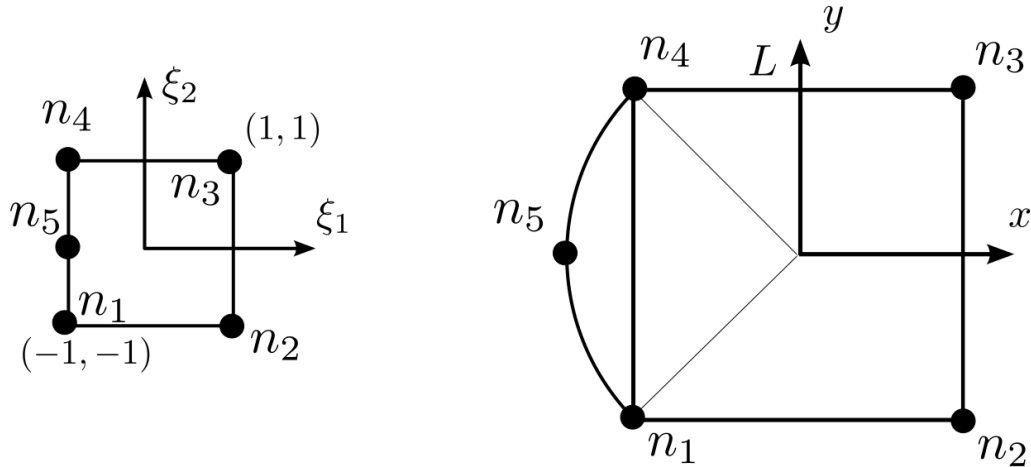


Figure 1: Parent element in (ξ_1, ξ_2) and the actual element.

1. **35 Points** Starting from Q4 bilinear shape functions, show that the shape functions for the element are:

$$\mathbf{N} = [N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5] = \left[-\frac{(1-\xi_1)\xi_2(1-\xi_2)}{4} \quad \frac{(1+\xi_1)(1-\xi_2)}{4} \quad \frac{(1+\xi_1)(1+\xi_2)}{4} \quad \frac{(1-\xi_1)\xi_2(1+\xi_2)}{4} \quad \frac{(1-\xi_1)(1-\xi_2^2)}{2} \right] \tag{2}$$

and demonstrate that N_1 satisfy shape function δ property (1 at its dof; zero at other ones).

Starting from Q4 bilinear shape functions is compulsory.

2. **35 Points** Assuming that the element is isoparametric (*i.e.*, uses the same shape functions for geometry representation) demonstrate that:

$$x = (1 - \xi_1) \left(-\sqrt{2} + (\sqrt{2} - 1)\xi_2^2 \right) + (1 + \xi_1) \tag{3a}$$

$$y = 2\xi_2 \tag{3b}$$

3. **30 Points** Given that,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial y}{\partial \xi_1} \\ \frac{\partial x}{\partial \xi_2} & \frac{\partial y}{\partial \xi_2} \end{bmatrix} \tag{4}$$

express $\mathbf{J}, |J|, \mathbf{B}_\xi, \mathbf{B}$ in terms of ξ_1, ξ_2 . : **No need to explicitly compute \mathbf{J}^{-1} in your expressions.**

4. **30 Points** For the point $(\xi_1, \xi_2) = (0, 0)$ compute 1) (x, y) , 2) \mathbf{N} , 3) $\mathbf{B} = \nabla \mathbf{N}$. Assuming that the left boundary is on essential boundary condition with $\bar{T}(x, y) = y^2$ and the FEM solutions for nodes 2 and 3 given by $T_2 = 1, T_3 = 3$ compute 4) T and 5) $\mathbf{q} = -\kappa \nabla T$ at the same point. Note that $\kappa = 2\mathbf{I}$.

Some values to verify your results: $B(1, 5) = -0.207, B(2, 4) = 0.125, T = 1, q_2 = -0.5$.

5. **40 Points** Using $k^e = \int_e \mathbf{B}^T \mathbf{D} \mathbf{B} \, dA$ and the values of \mathbf{B} and dA show that:

$$k^e = k \int_{-1}^1 \int_{-1}^1 \begin{bmatrix} \frac{\xi_2(1-\xi_2)}{4} & \frac{(1-\xi_1)(2\xi_2-1)}{4} \\ \frac{1-\xi_2}{4} & -\frac{1+\xi_1}{4} \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\frac{\xi_2(1+\xi_2)}{4} & \frac{(1-\xi_1)(2\xi_2+1)}{4} \\ -\frac{1-\xi_2^2}{2} & -\xi_2(1-\xi_1) \end{bmatrix} \{ \mathbf{J}^{-T} \mathbf{J}^{-1} |\mathbf{J}| \} \begin{bmatrix} \frac{\xi_2(1-\xi_2)}{4} & \frac{1-\xi_2}{4} & \frac{1+\xi_2}{4} & -\frac{\xi_2(1+\xi_2)}{4} & -\frac{1-\xi_2^2}{2} \\ \frac{(1-\xi_1)(2\xi_2-1)}{4} & -\frac{1+\xi_1}{4} & \frac{1+\xi_1}{4} & \frac{(1-\xi_1)(2\xi_2+1)}{4} & -\xi_2(1-\xi_1) \end{bmatrix} d\xi_1 d\xi_2 \quad (5)$$

- (a) This equation in fact holds for any 5 node thermal element with parent geometry shown before. Assuming that J is constant what are the maximum orders of integrand for ξ_1 ξ_2 .
- (b) Based on the maximum orders of integrand for constant J list the 1) coordinates of quadrature points (ξ_1, ξ_2) 2) with their corresponding weight values; 3) schematically show these points in the parent geometry.
- (c) If we use Newton-Cotes to integrate k^e how many points are needed in ξ_1 and ξ_2 directions? Schematically, draw these quadrature points and compare them with Gauss quadrature points.
- (d) Can a 2×3 Gauss quadrature stencil integrate k^e exactly for the geometry shown in the figure? If not, can any order of Gauss quadrature integrate it exactly? What is a full integration order?
- (e) If the conductivity matrix is integrated exactly, what would it rank be? In other words, how many independent zero eigenvalues does the matrix possess? Comment on rank of conductivity matrix for a 1×1 Gauss integration scheme and its influence on FEM results.
6. **30 Points** Prove that the FEM approximation of the left boundary is the parabola:

$$x = \frac{\sqrt{2}-1}{2} y^2 - 2\sqrt{2} \quad (6)$$

Does this exactly match the actual boundary of the domain (90° circular arc)?

7. **50 Points (Extra credit)** Compute the conductivity matrix k^e with a full Gauss integration scheme.