Consider the isoparametric 5-node element shown in the figure for a stationary thermal conduction problem with material conductivity $\kappa = k\mathbf{I}$ (isotropic conductivity), $k = 2$. In order to capture the curved domain boundary geometry, the extra node 5 is inserted on the left boundary. For $L = 4$ the coordinates of the element in Cartesian system are:

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 & -2 & -2\sqrt{2} \\ -2 & -2 & 2 & 2 & 0 \end{bmatrix} \tag{1}$$

The questions for this element are:

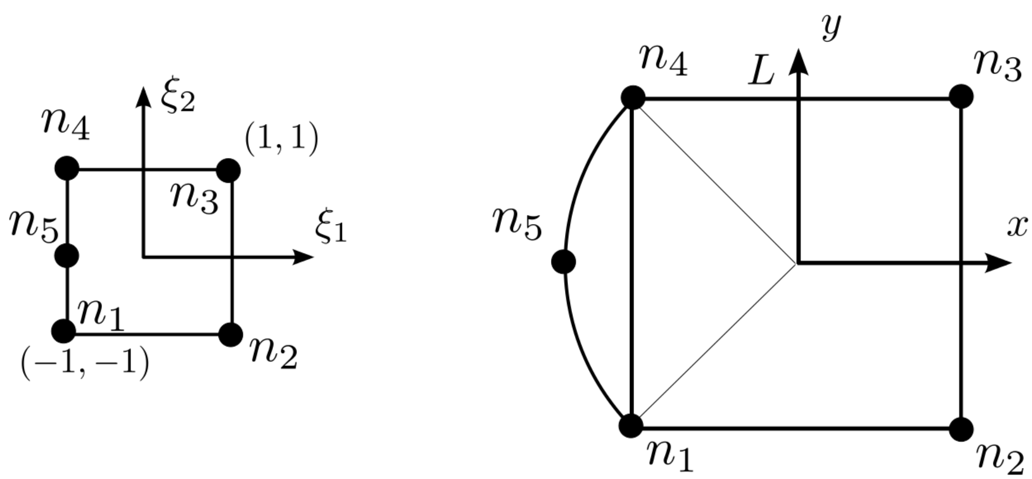

Figure 1: Parent element in (ξ_1, ξ_2) and the actual element.

1. $\boxed{\textbf{35 Points}}$ Starting from Q4 bilinear shape functions, show that the shape functions for the element are:

$$\begin{aligned} \mathbf{N} &= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(1-\xi_1)\xi_2(1-\xi_2)}{4} & \frac{(1+\xi_1)(1-\xi_2)}{4} & \frac{(1+\xi_1)(1+\xi_2)}{4} & \frac{(1-\xi_1)\xi_2(1+\xi_2)}{4} & \frac{(1-\xi_1)(1-\xi_2^2)}{2} \end{bmatrix} \end{aligned} \tag{2}$$

   and demonstrate that N_1 satisfy shape function δ property (1 at its dof; zero at other ones). **Starting from Q4 bilinear shape functions is compulsory.**

2. $\boxed{\textbf{35 Points}}$ Assuming that the element is isoparametric (*i.e.*, uses the same shape functions for geometry representation) demonstrate that:

$$x = (1-\xi_1)\left(-\sqrt{2} + (\sqrt{2}-1)\xi_2^2\right) + (1+\xi_1) \tag{3a}$$

$$y = 2\xi_2 \tag{3b}$$

3. $\boxed{\textbf{30 Points}}$ Given that,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} \end{bmatrix} \tag{4}$$

   express $\mathbf{J}, |J|, \mathbf{B}_\xi, \mathbf{B}$ in terms of ξ_1, ξ_2. **: No need to explicitly compute $\mathbf{J}^{-1}$ in your expressions.**

4. $\boxed{\textbf{30 Points}}$ For the point $(\xi_1, \xi_2) = (0, 0)$ compute 1) (x, y), 2) $\mathbf{N}$, 3) $\mathbf{B} = \nabla\mathbf{N}$. Assuming that the left boundary is on essential boundary condition with $\bar{T}(x, y) = y^2$ and the FEM solutions for nodes 2 and 3 given by $T_2 = 1, T_3 = 3$ compute 4) T and 5) $\mathbf{q} = -\kappa\nabla T$ at the same point. Note that $\kappa = 2\mathbf{I}$.

   Some values to verify your results: $B(1, 5) = -0.207, B(2, 4) = 0.125, T = 1, q_2 = -0.5$.

5. $\boxed{\textbf{40 Points}}$ Using $k^e = \int_e \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\mathrm{dA}$ and the values of $\mathbf{B}$ and dA show that:

$$k^e = k\int_{-1}^{1}\int_{-1}^{1}\begin{bmatrix} \frac{\xi_2(1-\xi_2)}{4} & \frac{(1-\xi_1)(2\xi_2-1)}{4} \\ \frac{1-\xi_2}{4} & -\frac{1+\xi_1}{4} \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\frac{\xi_2(1+\xi_2)}{4} & \frac{(1-\xi_1)(2\xi_2+1)}{4} \\ -\frac{1-\xi_2^2}{2} & -\xi_2(1-\xi_1) \end{bmatrix}\left\{\mathbf{J}^{-1}\mathbf{J}^{-\mathrm{T}}|\mathbf{J}|\right\}\begin{bmatrix} \frac{\xi_2(1-\xi_2)}{4} & \frac{1-\xi_2}{4} & \frac{1+\xi_2}{4} & -\frac{\xi_2(1+\xi_2)}{4} & -\frac{1-\xi_2^2}{2} \\ \frac{(1-\xi_1)(2\xi_2-1)}{4} & -\frac{1+\xi_1}{4} & \frac{1+\xi_1}{4} & \frac{(1-\xi_1)(2\xi_2+1)}{4} & -\xi_2(1-\xi_1) \end{bmatrix}\mathrm{d}\xi_1\mathrm{d}\xi_2 \tag{5}$$

   (a) This equation in fact holds for any 5 node thermal element with parent geometry shown before. Assuming that J is constant what are the maximum orders of integrand for ξ_1 ξ_2.

   (b) Based on the maximum orders of integrand for constant J list the 1) coordinates of quadrature points (ξ_1, ξ_2) 2) with their corresponding weight values; 3) schematically show these points in the parent geometry.

   (c) If we use Newton-Cotes to integrate k^e how many points are needed in ξ_1 and ξ_2 directions? Schematically, draw these quadrature points and compare them with Gauss quadrature points.

   (d) Can a 2×3 Gauss quadrature stencil integrate k^e exactly for the geometry shown in the figure? If not, can any order of Gauss quadrature integrate it exactly? What is a full integration order?

   (e) If the conductivity matrix is integrated exactly, what would it rank be? In other words, how many independent zero eigenvalues does the matrix possess? Comment on rank of conductivity matrix for a 1×1 Gauss integration scheme and its influence on FEM results.

6. $\boxed{\textbf{30 Points}}$ Prove that the FEM approximation of the left boundary is the parabola:

$$x = \frac{\sqrt{2}-1}{2}y^2 - 2\sqrt{2} \tag{6}$$

   Does this exactly match the actual boundary of the domain (90° circular arc)?

7. $\boxed{\textbf{50 Points}}$ **(Extra credit)** Compute the conductivity matrix k^e with a full Gauss integration scheme.