Before actual Finite element formulation, we focus on two things

1. representation of $\underline{\sigma}$, $\underline{\epsilon}$ tensors in vector form $\underline{s}, \underline{\gamma}$
2. Constitutive relation $\underline{\sigma} = \underline{\underline{C}}\, \underline{\epsilon}$ (alternatively $\underline{s} = \underline{\underline{D}}\, \underline{\gamma}$) for general elastostatic and its simplified isotropic case

FEM motivation: in weak statement below we should be able to
1. Evaluate $\epsilon(\hat{w}):\sigma$ which we show is equal to $\gamma(w)\cdot s$
2. Relate s & γ through constitutive relation $s = D\gamma$

Find $u \in \mathcal{V} = \{v \in H^1(\mathcal{D}) \mid \forall x \in \partial\mathcal{D}_u \; v(x) = \bar{u}(x)\}$, such that,
$\forall w \in \mathcal{W}\{v \in H^1(\mathcal{D}) \mid \forall x \in \partial\mathcal{D}_u \; v(x) = 0\}$

$$\int_{\mathcal{D}} \epsilon(w) : \sigma(u)\, dv = \int_{\mathcal{D}} w.\rho b\, dv + \int_{\partial\mathcal{D}_f} w.\bar{t}\, ds$$

(1)

we formulate the problem in 3D, then reduce it to 2D for numerical examples.

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

where $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are strains

for example $\epsilon_{11} = u_{1,1}$ & $\epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1})$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$ is the stress tensor.

Note that both σ & $\mathcal{E}$ are symmetric.

We use "^" to denote weight quantities;
That is, terms derived from ω:

$\hat{\mathcal{E}} = \mathcal{E}(\omega)$, $\hat{u} = \omega$, ...

$\sigma = f(\mathcal{E}, \dot{\mathcal{E}}, \ldots)$ σ is a function of strain

for hyperelastic material $\sigma = \dfrac{\partial U(\mathcal{E})}{\partial \mathcal{E}}$ strain energy density

for linear hyperelastic material $U = \frac{1}{2}\, \mathcal{E} : \mathbb{C} : \mathcal{E}$ 4th order elasticity tensor

2nd order sym.

$\hat{\varepsilon} : \mathbb{C} : \varepsilon$

4th order elasticity tensor

2nd order symmetric stress tensor

$\sigma = \mathbb{C} \hat{\varepsilon}$:

2nd order symm. strain

$\mathbb{C}$ fourth order elasticity tensor

In equation ①

$$\varepsilon(w) : \sigma(u) = \hat{\varepsilon} : \sigma = \hat{\varepsilon}_{11}\sigma_{11} + \hat{\varepsilon}_{12}\sigma_{12} + \hat{\varepsilon}_{13}\sigma_{13} + \hat{\varepsilon}_{21}\sigma_{21} + \hat{\varepsilon}_{22}\sigma_{22} + \hat{\varepsilon}_{23}\sigma_{23} + \hat{\varepsilon}_{31}\sigma_{31} + \hat{\varepsilon}_{32}\sigma_{32} + \hat{\varepsilon}_{33}\sigma_{33}$$

$\hat{\varepsilon} : \sigma = \hat{\varepsilon}_{ij}\, \sigma_{ij}$

corresponding to weight function

$\hat{\varepsilon} : \varepsilon(w)$

$= \frac{1}{2}(\nabla w + \nabla w^T)$

symmetry of σ, $\hat{\varepsilon}$ ⟹

$$\boxed{\hat{\varepsilon} : \sigma = \underbrace{(\hat{\varepsilon}_{11})}_{\gamma_1}\underbrace{\sigma_{11}}_{s_1} + \underbrace{(\hat{\varepsilon}_{22})}_{\gamma_2}\underbrace{\sigma_{22}}_{s_2} + \underbrace{(\hat{\varepsilon}_{33})}_{\gamma_3}\underbrace{\sigma_{33}}_{s_3} + \underbrace{(2\hat{\varepsilon}_{12})}_{\gamma_4}\underbrace{\sigma_{12}}_{s_4} + \underbrace{(2\hat{\varepsilon}_{23})}_{\gamma_5}\underbrace{\sigma_{23}}_{s_5} + \underbrace{(2\hat{\varepsilon}_{31})}_{\gamma_6}\underbrace{\sigma_{31}}_{s_6}}$$ ②

⟹ $\varepsilon : \sigma = \gamma \cdot S$ → Inner product

It's more convenient to represent stress & strain tensors in vector for for FEM formulation

Voigt notation

vector form stress $S = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$

Voight strain vector

$\gamma = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{bmatrix}$ from ②

shear strains

→ shear strains are multiplied by 2

→ engineering strain

$\sigma : \hat{\epsilon} = S \cdot \hat{\gamma}$

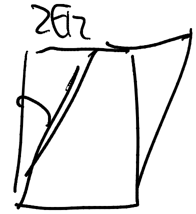
tensor 3x3 tensor 3x3

vector inner product

6x1 vectors

$2\epsilon_{12} = u_{1,2} + u_{2,1}$

eng. strain



③

γ is called engineering strain and the shear components $\gamma_{12}, \gamma_{23}, \gamma_{31}$ are twice ϵ_{12}, ϵ_{23}, and ϵ_{31}.

Constitutive equation

σ is a function of ϵ in elastostatics.

In linear case the linear relation can be written as

$$\underset{=}{\sigma} = \overset{\text{4th order elasticity tensor}}{\underset{\equiv}{C}} \, \underset{=}{\epsilon} \qquad \text{that is} \qquad \sigma_{ij} = C_{ijkl}\, \epsilon_{kl}$$

where $\underset{\equiv}{C}$ is the fourth order elasticity tensor.

— C is the generalization of elastic modulus E in 1D version of the equation $\sigma = E\varepsilon$

— C is a $3\times3\times3\times3$ fourth order tensor

— Instead of $\underset{\substack{3\times3 \\ \text{tensor}}}{\sigma} = \underset{\text{4th order tensor}}{C_{3\times3\times3\times3}} \; \underset{\substack{3\times3 \\ \text{tensor}}}{\epsilon}$

we seek a relation of the form

stiffness/ elasticity relating $\gamma \to S$ rather than $\varepsilon \to \sigma$

$$\underset{\substack{6\times1 \\ \text{6x1 stress vector}}}{S} = \underset{\substack{6\times6 \\ \text{2nd order tensor (matrix)}}}{D} \; \underset{\text{6x1 strain vector}}{\gamma}$$

Due to major & minor symmetries of C ($C_{ijkl} = C_{jikl}$, $C_{ijkl} = C_{ijlk}$ minor, $C_{ijkl} = C_{klij}$ major)

there are only 21 independent components in C.

— We want to form $D_{6\times6}$ from the components C_{ijkl}:

Examples below show how to derive components of D from C:

Examples below show how to derive components of D from C:

$$s_1 = \sigma_{11} = C_{11jk}\epsilon_{jk} = C_{1111}\epsilon_{11} + C_{1112}\epsilon_{12} + C_{1113}\epsilon_{13} + C_{1123}\epsilon_{23} + C_{1122}\epsilon_{22} + C_{1133}\epsilon_{33} + C_{1121}\epsilon_{21} + C_{1131}\epsilon_{31} + C_{1132}\epsilon_{32} \Longrightarrow$$

$\epsilon_{11} = \gamma_1$

$C_{1112} = C_{1121}, \epsilon_{12} = \epsilon_{21}$

$\epsilon_{22} = \gamma_2, \epsilon_{33} = \gamma_3$

$$= C_{1111}\gamma_1 + C_{1122}\gamma_2 + C_{1133}\gamma_3 + C_{1112}(\underbrace{2\epsilon_{12}}_{\gamma_4}) + C_{1123}(\underbrace{2\epsilon_{23}}_{\gamma_5}) + C_{1131}(\underbrace{2\epsilon_{31}}_{\gamma_6})$$

$$\Rightarrow \quad s_1 = \underbrace{C_{1111}}_{D_{11}}\gamma_1 + \underbrace{C_{1122}}_{D_{12}}\gamma_2 + \underbrace{C_{1133}}_{D_{13}}\gamma_3 + \underbrace{C_{1112}}_{D_{14}}\gamma_4 + \underbrace{C_{1123}}_{D_{5}}\gamma_5 + \underbrace{C_{1131}}_{D_6}\gamma_6$$

similar expression can be obtained for $s_2 = \sigma_{22}$ & $s_3 = \sigma_{33}$

Also for σ_{12} we have

$$s_4 = \sigma_{12} = C_{12jk}\epsilon_{jk} = C_{1211}\epsilon_{11} + C_{1212}\epsilon_{12} + C_{1213}\epsilon_{13} + C_{1222}\epsilon_{22} + C_{1223}\epsilon_{23} + C_{1233}\epsilon_{33} + C_{1221}\epsilon_{21} + C_{1231}\epsilon_{31} + C_{1232}\epsilon_{32}$$

(4)

following the same process

$$s_4 = C_{1211}\gamma_1 + C_{1222}\gamma_2 + C_{1233}\gamma_3 + \underbrace{C_{1212}}_{D_{44}}\gamma_4 + C_{1223}\gamma_5 + C_{1231}\gamma_6$$

$C_{41} = C_{1112} = C_{14}$ $\quad C_{42} = C_{2212} = C_{24}$ $\quad C_{43} = C_{34}$ $\quad C_{45} = C_{54}$ $\quad C_{46} = C_{64}$

The relation between σ & ϵ and s & γ can be summarized as,

$$\underline{\sigma} = \underset{3\times3\times3\times3}{\underline{\underline{C}}}\,\underline{\epsilon}, \text{ that is } \sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

$$\underline{s} = \underset{6\times6}{\underline{\underline{D}}}\,\underline{\gamma}, \quad s = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix}, \quad \gamma = \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{pmatrix}, \quad D = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array}\left[\begin{array}{cccccc} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\ & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} \\ & & C_{3333} & C_{3312} & C_{3323} & C_{3331} \\ & \text{sym.} & & C_{1212} & C_{1223} & C_{1231} \\ & & & & C_{2323} & C_{2331} \\ & & & & & C_{3131} \end{array}\right]$$

Voight elasticity matrix

21 independent quantities

Isotropic material

For isotropic materials 21 constants in D in (4) reduce to 2 as C is expressed in terms of two Lame' parameters:

http://en.wikipedia.org/wiki/Lam%C3%A9_parameters

Isotropic material

http://en.wikipedia.org/wiki/Lam%C3%A9_parameters

Isotropic material

$$C_{ijkl} = \delta_{ij}\delta_{kl}\lambda + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$D_{11} = D_{22} = D_{33} = C_{1111} = C_{2222} = C_{3333} = \lambda + 2\mu$

$D_{12} = D_{13} = D_{23} = C_{1122} = C_{1133} = C_{2233} = \lambda$

$D_{14} = D_{15} = D_{16} = D_{24} = D_{25} = D_{26} = \dots = C_{1112} = \dots = C_{2133} = 0 \rightarrow D =$

$$\begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ \text{sym} & & & & & \mu \end{bmatrix}$$

$D_{44} = D_{55} = D_{66} = C_{1212} = C_{1313} = C_{2323} = \mu$

$D_{45} = D_{46} = D_{56} = C_{1223} = C_{1231} = C_{2331} = 0$

Using the table below (wikipedia) D can alternatively represented in terms of E & ν

$E \downarrow$ Elastic modulus

$\nu \downarrow$ Poisson ratio

$(K,\ E)$	$(K,\ \lambda)$	$(K,\ G)$	$(K,\ \nu)$	$(E,\ G)$	$(E,\ \nu)$	$(\lambda,\ G)$	$(\lambda,\ \nu)$	$(G,\ \nu)$	$(G,\ M)$	
$K =$	K	K	K	K	$\frac{EG}{3(3G-E)}$	$\frac{E}{3(1-2\nu)}$	$\lambda + \frac{2G}{3}$	$\frac{\lambda(1+\nu)}{3\nu}$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$M - \frac{4G}{3}$
$E =$	E	$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{9KG}{3K+G}$	$3K(1-2\nu)$	E	E	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$2G(1+\nu)$	$\frac{G(3M-4G)}{M-G}$
$\lambda =$	$\frac{3K(3K-E)}{9K-E}$	λ	$K - \frac{2G}{3}$	$\frac{3K\nu}{1+\nu}$	$\frac{G(E-2G)}{3G-E}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	λ	λ	$\frac{2G\nu}{1-2\nu}$	$M - 2G$
$G =$	$\frac{3KE}{9K-E}$	$\frac{3(K-\lambda)}{2}$	G	$\frac{3K(1-2\nu)}{2(1+\nu)}$	G	$\frac{E}{2(1+\nu)}$	G	$\frac{\lambda(1-2\nu)}{2\nu}$	G	G
$\nu =$	$\frac{3K-E}{6K}$	$\frac{\lambda}{3K-\lambda}$	$\frac{3K-2G}{2(3K+G)}$	ν	$\frac{E}{2G} - 1$	ν	$\frac{\lambda}{2(\lambda+G)}$	ν	ν	$\frac{M-2G}{2M-2G}$
$M =$	$\frac{3K(3K+E)}{9K-E}$	$3K - 2\lambda$	$K + \frac{4G}{3}$	$\frac{3K(1-\nu)}{1+\nu}$	$\frac{G(4G-E)}{3G-E}$	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$\lambda + 2G$	$\frac{\lambda(1-\nu)}{\nu}$	$\frac{2G(1-\nu)}{1-2\nu}$	M

Pasted from <http://en.wikipedia.org/wiki/Lam%C3%A9_parameters>

$\Rightarrow$ For isotropic case equation (4) can be simplified to

Isotropic material

$\underset{3\times3}{\underline{\sigma}} = \underset{3\times3\times3\times3}{\underline{\underline{C}}} \; \underset{3\times3}{\underline{\epsilon}} \qquad C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$$\underline{s}_{6\times1} = \underline{D}_{6\times6}\,\underline{\epsilon}_{6\times1}, \quad D = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ \text{sym} & & & & & \mu \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{12} & 0 & 0 & 0 \\ & D_{11} & D_{12} & 0 & 0 & 0 \\ & & D_{11} & 0 & 0 & 0 \\ & & & D_{44} & 0 & 0 \\ & & & & D_{44} & 0 \\ \text{sym} & & & & & D_{44} \end{bmatrix} \quad (6)$$

where

$$D_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$D_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$D_{44} = \frac{E}{1+\nu}$$

The relation above is for 3D solid mechanics

In general in solid mechanics we have an equation of the form:

D Strain quantity

Force = D Strain quantity
(or C)

examples bar	$F =$	(EA)	ϵ
beam	$M =$	(EI)	$k \longrightarrow y''$
solid 3D	$\underline{S} =$	$\underline{\underline{D}}$	$\underline{\delta}$

$\vdots$

Bathe: Table 4.3 page 194 summarizes these relations :

TABLE 4.3 *Generalized stress-strain matrices for isotropic materials and the problems in Table 4.2*

Problem	Material matrix **C**
Bar	E
Beam	EI
Plane stress	$\dfrac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$
Plane strain	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ \dfrac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix}$
Axisymmetric	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 & \dfrac{\nu}{1-\nu} \\ \dfrac{\nu}{1-\nu} & 1 & 0 & \dfrac{\nu}{1-\nu} \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} & 0 \\ \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$
Three-dimensional	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & & & \\ \dfrac{\nu}{1-\nu} & 1 & \dfrac{\nu}{1-\nu} & & & \\ \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 1 & & & \\ & & & \dfrac{1-2\nu}{2(1-\nu)} & & \\ & \text{Elements not} & & & \dfrac{1-2\nu}{2(1-\nu)} & \\ & \text{shown are zeros} & & & & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix}$
Plate bending	$\dfrac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$

Notation: E = Young's modulus, ν = Poisson's ratio, h = thickness of plate, I = moment of inertia

$$S = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$