2015/02/09

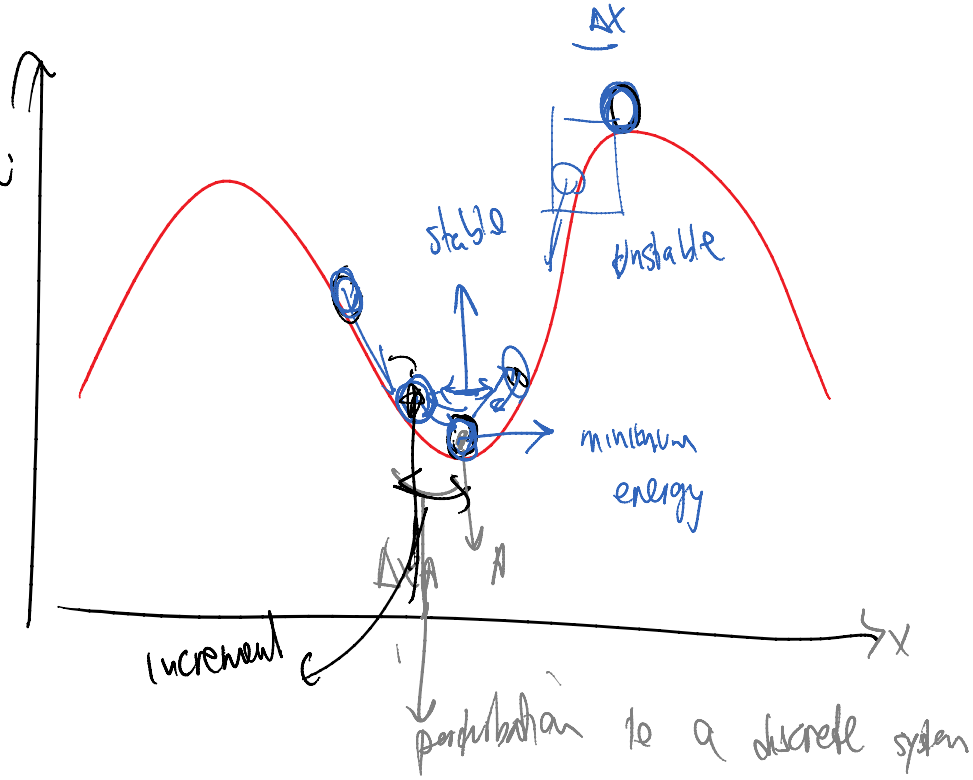
Monday, February 09, 2015
11:37 AM

y = mountain elevation

α

Π

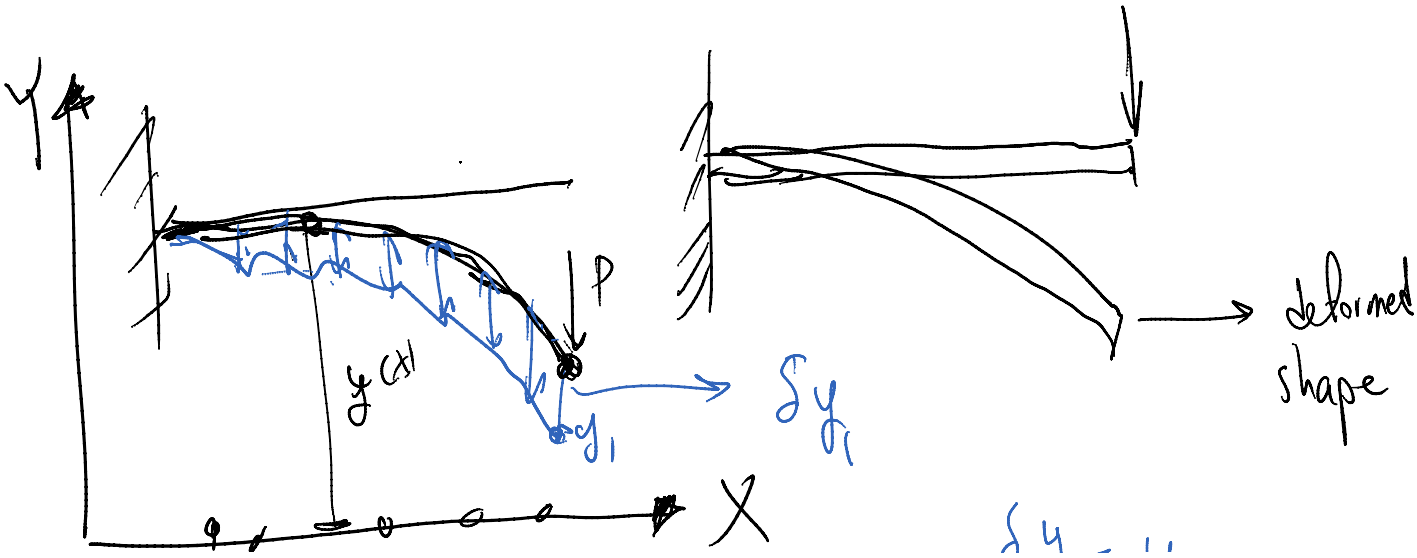
$\Pi = mgy$



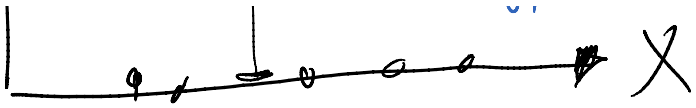
The system is identified by 1 scalar value x_A

Perturbation of that is # " " Δx_A

How about the continuum case?



δy ...



$\delta y = y_1 - y$

y_1: perturbed function

y: solution

In continuum setting the perturbation is a FUNCTION

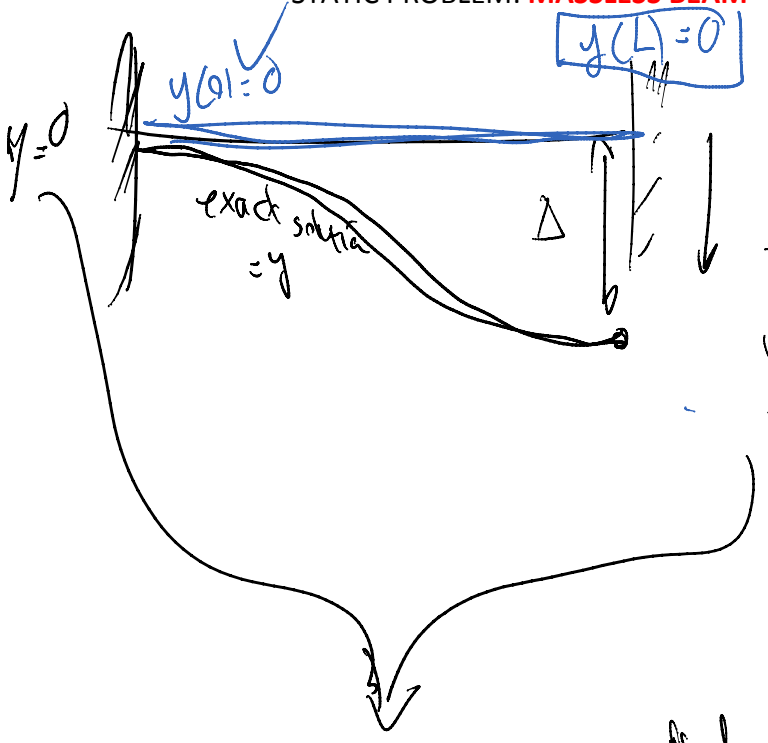
scalar values

$\Pi(y) < \Pi(y_1)$

$\Pi(y)$: solution

$\Pi(y_1)$: a perturbed solution

STATIC PROBLEM: **MASSLESS BEAM**



This is the problem!
Violating **ESSENTIAL BC**

$\Pi(y) > \Pi(y_1)$

What's wrong?

both are essential

Energy Method: Essential Boundaries and Function spaces

As will be shown, the energy for the beam in the figure is,

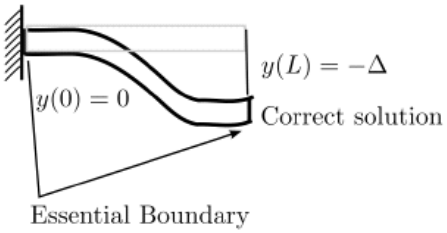
$$\Pi = \int_0^L \frac{1}{2} EI \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \qquad (81)$$

incorrect solutions that do not satisfy essential boundary conditions.

yet their energy is smaller than that of correct solution

The Essential boundary conditions for this problem are:

$$y(0) = 0$$
$$y(L) = -\Delta \qquad (82)$$



The requirements for trial functions are:

- Smoothness: Clearly, the first requirement for the trial functions–for which we evaluate the energy–is that the equation (81) can be evaluated. Since the maximum derivative order is $m = 2$, for now we assume the functions should be in $C^2([0, L])$. We will later discuss how this smoothness requirement can be relaxed.
- Essential boundary conditions: If essential boundary conditions are not enforced for the trial functions, we notice that any of the rigid body configurations (gray rectangles) have zero energy, while the correct solution (shown in black) clearly has a positive energy bases on equation (81). Thus, it is essential to restrict the trial function space to those that satisfy essential boundary conditions.

For energy method satisfaction of essential boundary condition is essential

Remember that for the weighted residual method it was desirable but not mandatory to strongly satisfy essential BC.

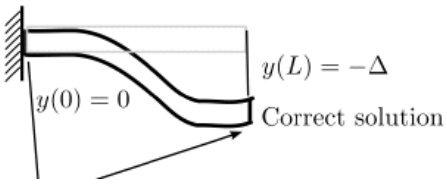
Energy Method: Function spaces

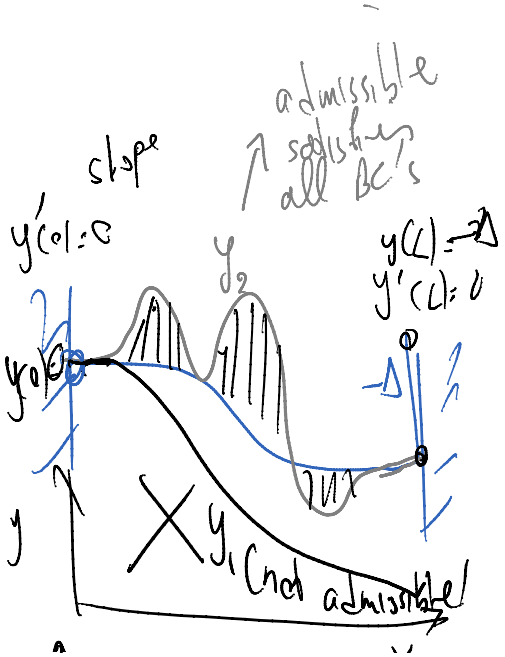
Based on the need to satisfy essential boundary conditions, derivative order in (81), and corresponding essential boundary conditions, the function space for trial functions, $\tilde{y}$ is,

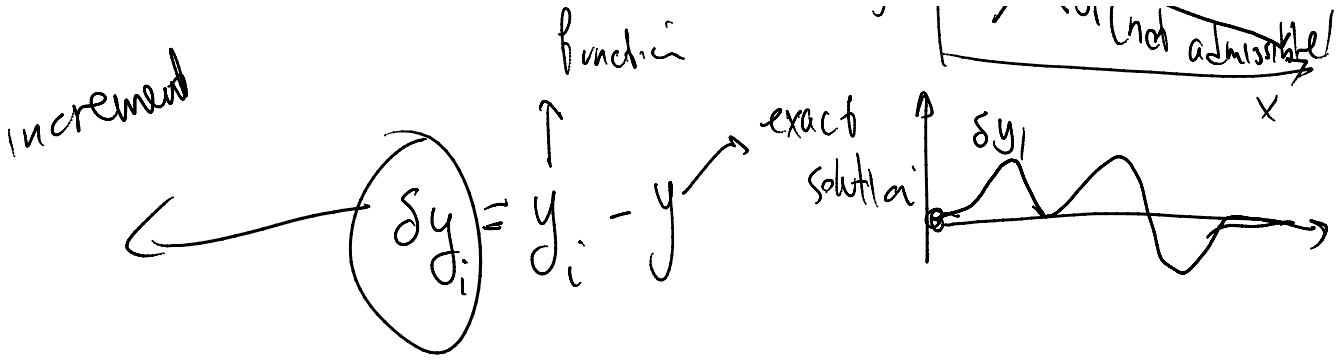
$u'(0) = 0$ $u'(L) = 0$ → slopes

$$\mathcal{V} = \{u \,|u \in C^2([0,\ L]),\ u(0) = 0,\ u(L) = -\Delta\}. \qquad (83)$$

space of solution functions





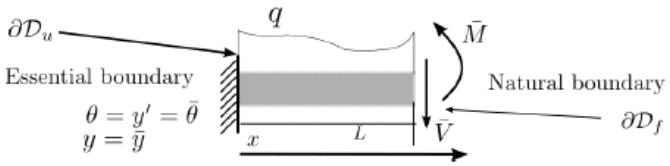


So the increment satisfies the **HOMOGENEOUS** version of ESSENTIAL BOUNDARY CONDITION

$$\mathcal{W} = \{u \mid u \in C^2([0,\ L]),\ u(0) = 0,\ u(L) = 0\}.$$

This is very much like the conditions we had for the solution function and the weight function in the weak statement

Weak Statement (WS)

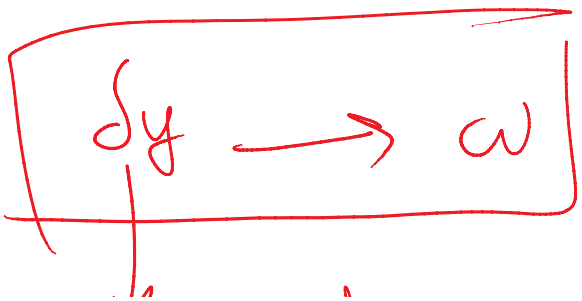


The weak statement for the Euler Bernoulli problem and the BCs in the figure are:

Find $y \in \mathcal{V} = \{u \in C^2(\mathcal{D}) \mid u(0) = \bar{y},\ \dfrac{\mathrm{d}u}{\mathrm{d}x}(0) = \bar{\theta}\}$, such that, (62a)

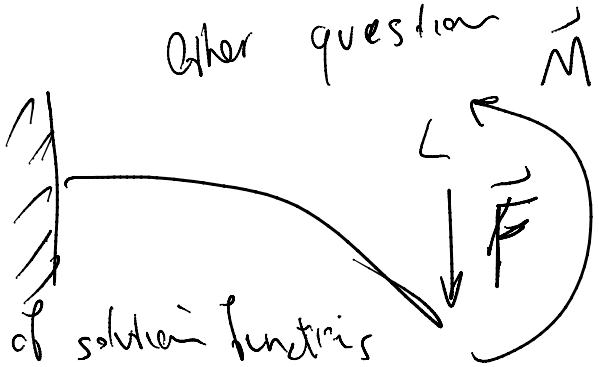
$\forall w \in \mathcal{W} = \{u \in C^2(\mathcal{D}) \mid u(0) = 0,\ \dfrac{\mathrm{d}u}{\mathrm{d}x}(0) = 0\}$ (62b)

$$0 = \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq\right] \mathrm{d}x + \left\{-\frac{\mathrm{d}w}{\mathrm{d}x}\bar{M} + w\bar{V}\right\}_{x=L} \quad (62c)$$



increment becomes our weight function

Other question



space of solution functions

$$V = \{ v \in C^2([0,L]) \mid u(0)=0 \quad u'(0)=0 \}$$

discuss later

$F(u)(L) = \vec{F}$ $\quad M(u)(L) = \vec{M}$

natural

$\frac{d}{dx} EI \frac{d^2u}{dx^2} = \vec{F}$ $\quad EI\, u''(L) = 0$

natural BCs are NOT essential

Total potential energy

$$\Pi = T + V - W = \text{Total energy} \tag{85a}$$

$$T = \int_{\mathcal{D}} \frac{1}{2}\rho \mathbf{v}.\mathbf{v}\, \mathrm{dv} = \text{Kinetic energy} \tag{85b}$$

dm

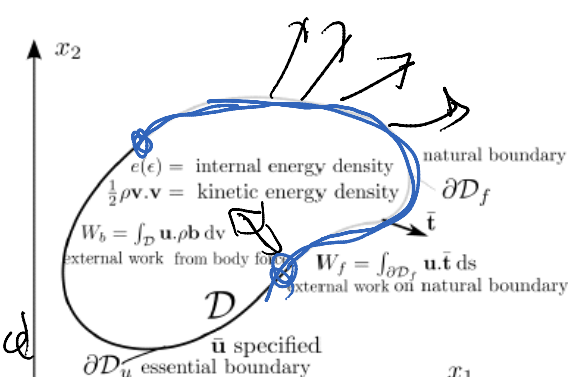
$$V = \int_{\mathcal{D}} e(\epsilon)\, \mathrm{dv} = \text{Internal energy} \tag{85c}$$

$$W = W_b + W_f = \text{External work} \tag{85d}$$

$$W_b = \int_{\mathcal{D}} \mathbf{u}.\rho \mathbf{b}\, \mathrm{dv} \tag{85e}$$

→ external work (body force)

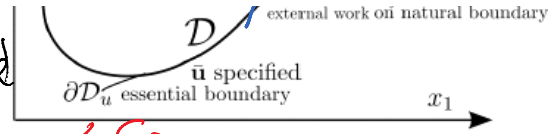
$$W_f = \int_{\partial\mathcal{D}_f} \mathbf{u}.\bar{\mathbf{t}}\, \mathrm{ds} \tag{85f}$$



1D

$$W_b = \int_{\mathcal{D}} \mathbf{u}.\rho \mathbf{b}\, dv \quad (85e)$$

$$W_f = \int_{\partial\mathcal{D}_f} \mathbf{u}.\bar{\mathbf{t}}\, ds \quad (85f)$$



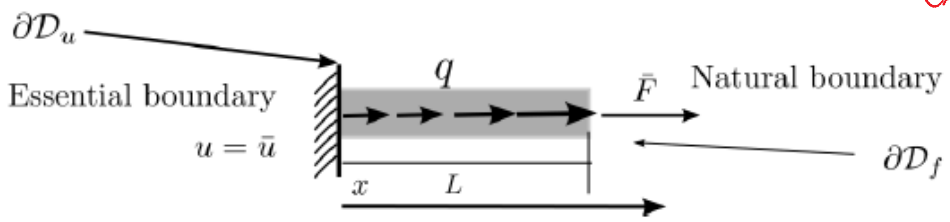
- For static problems $T = 0$.
- Internal energy density, $e(\epsilon) = \frac{1}{2}\epsilon : \sigma(\epsilon) = \frac{1}{2} C_{ijkl}\epsilon_{ij}\epsilon_{kl}$ for linear solid.
- Natural boundary forces are naturally incorporated into the energy (W_f).
- Essential boundary conditions are incorporated into function space:

$$\mathbf{u} \in \mathcal{V} = \{\mathbf{v} \,|\mathbf{v} \in C^1(\mathcal{D}) : \ \forall \mathbf{x} \in \partial\mathcal{D}_u \ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x})\}, \text{ is a solution if}$$
$$\forall \tilde{\mathbf{u}} \in \mathcal{V}, \quad \Pi(\mathbf{u}) \le \Pi(\tilde{\mathbf{u}}). \quad (86)$$

only consider the forces applied on the natural BC's

Let's focus on static problems => V = 0

$$V = \int_{\mathcal{D}} \frac{1}{2} \rho v^2 dV$$



$$\Pi = U + V - W$$

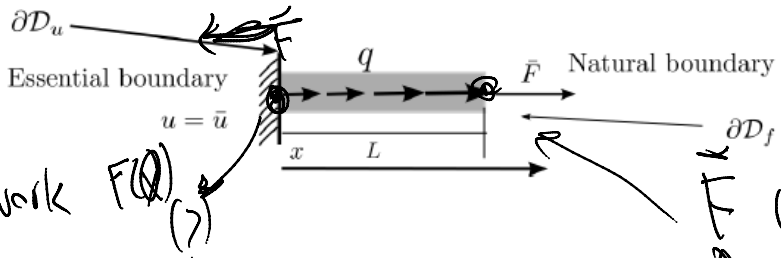
$$U = \int_{\mathcal{D}} e(\varepsilon)\, dV = \int_0^L \int_A e(\varepsilon)\, dA =$$

Constant for the area

$\varepsilon \longmapsto \varepsilon$

$e = \frac{1}{2}\varepsilon\sigma = \frac{1}{2}E\varepsilon^2$ $= \int_0^L A\, e(\varepsilon)\, dx = \int_0^L A E \varepsilon^2 dx \Rightarrow$

$$U = \frac{1}{2}\int_0^L AE\left(\frac{du}{dx}\right)^2 dx$$



external work $F(Q)$ (?)

$\bar{F}\,u(L)$ force

$W = W_b + W_F$

W_b: body force

W_F: external tractions

$$W_b = \int_V u\,\rho b\, dV = \int_0^L u\,\rho A\, q(x)\, dx$$

(ρA)

$$W_F = \bar{F}\,u(L)$$

scalar

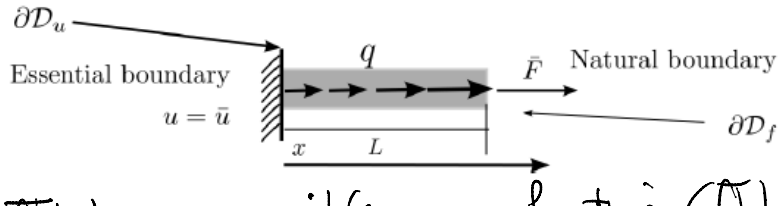
$$\Pi(u) = \int_0^L \frac{1}{2}EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L u(x)q(x)\,dx - u(L)\bar{F} \tag{88}$$

Since $\partial\mathcal{D}_u = \{0\}$ with the essential boundary condition $u(0) = \bar{u}$, the energy statement is,

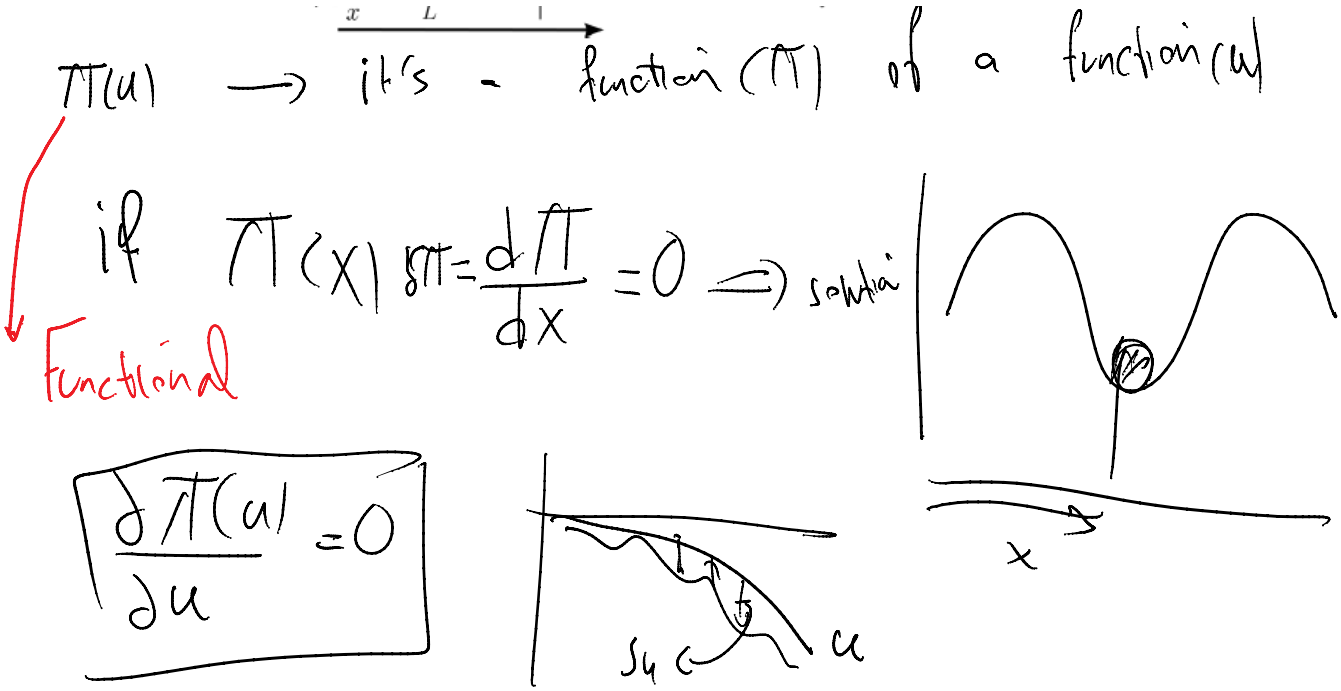
Find $u \in \mathcal{V} = \{v \mid v \in C^1(\mathcal{D}) : v(0) = \bar{u}\}$, such that

$$\forall \tilde{u} \in \mathcal{V}, \quad \Pi(u) \le \Pi(\tilde{u}). \tag{89}$$

75 / 45



Π is called a functional



First variation of a function / extremum condition

Let $f(x)$ be a function from $\mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is the real number set). We are interested in finding the increment to the function value due to change in the function argument x_0:

Change the input

$$x_0 \to x_0 + \Delta x : \; f(x_0) \to ?$$

We adopt the following definitions:

- Total variation: $\Delta f(x_0, \Delta x) = f(x_0 + \Delta x) - f(x_0)$
- First variation: $\delta f(x_0, \Delta x) = \frac{df}{dx}(x_0)\Delta x$

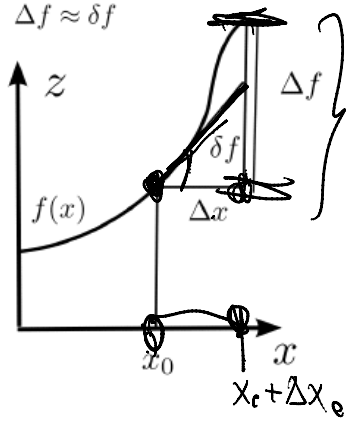
We often drop the arguments x_0 and Δx as shown. For a differentiable function we expect:

$$\Delta f \approx \delta f \text{ for "small" } |\Delta x|$$

Extremum condition (min/max)

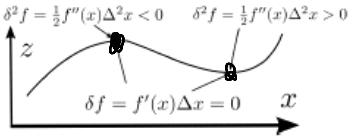
- Necessary condition: $\frac{df}{dx} = 0 \Rightarrow \delta f = 0$ (first variation = 0)
- Whether or not the point is a min/max or otherwise is determined by higher derivatives (variations). Some examples are shown on the right.

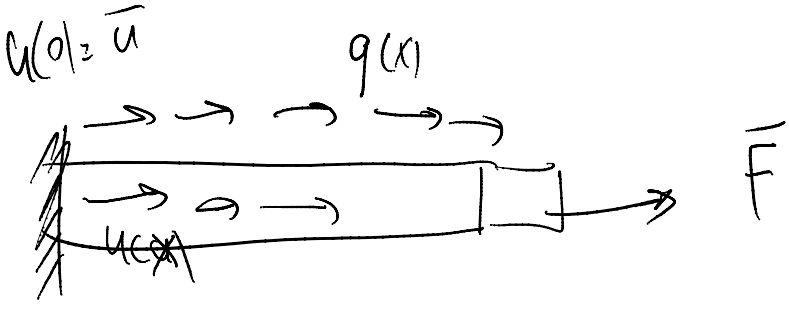
$$\delta f = f'(x_0)\Delta x$$
$$\Delta f = f(x_0 + \Delta x) - f(x_0)$$
$$\Delta f \approx \delta f$$



How much the output changes

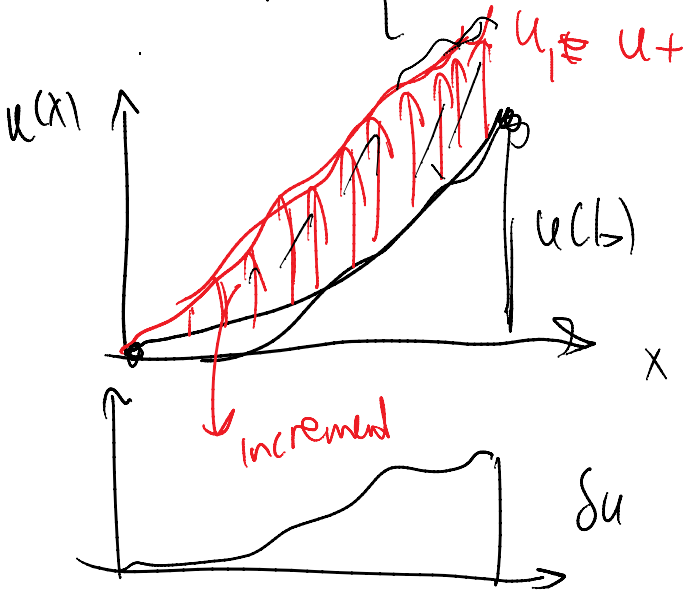
Total variation





1

$$u \in \mathcal{V} = \{ u \in C^1((0,L)) \mid \underbrace{u(0) = \bar{u}}_{\text{essential boundary condition}} \}$$



$$\Pi(u) \leq \Pi(u_1)$$

u: exact solution

$u + \delta u$: another admissible solution

$$\Pi(u_1) = \frac{1}{2}\int_0^L EA\left(\frac{du_1}{dx}\right)^2 - \int_0^L q(x)\,u_1(x)\,dx - u_1(x=L)\,\bar{F}$$

$$u_1 = u + \delta u$$

$$= \frac{1}{2}\int EA\left(\frac{du + d\delta u}{dx}\right)^2 - \int_0^L q(x)\left(u(x) + \delta u(x)\right)dx - \left\{u(x=L) + \delta u(x=L)\right\}\bar{F}$$

$$= \frac{1}{2}\int EA\left\{\left(\frac{du}{dx}\right)^2 + 2\frac{du}{dx}\frac{d\delta u}{dx} + \left(\frac{d\delta u}{dx}\right)^2 - \dots\right.$$

$$\Pi(u+\delta u) = \overbrace{\frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 - \int_0^L q(x)u(x)\,dx - \acute{F}u(L)}^{\Pi(u)}$$

$$- \int_0^L \underbrace{\delta u(x)}_{\delta u}\left\{\underbrace{(EAu')' + q}_{=0}\right\}dx + \underbrace{\left(EAu' - \acute{F}\right)}_{0 =}\delta u(L)$$

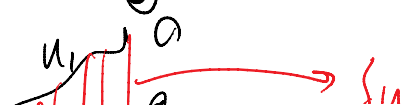
$|x<0$

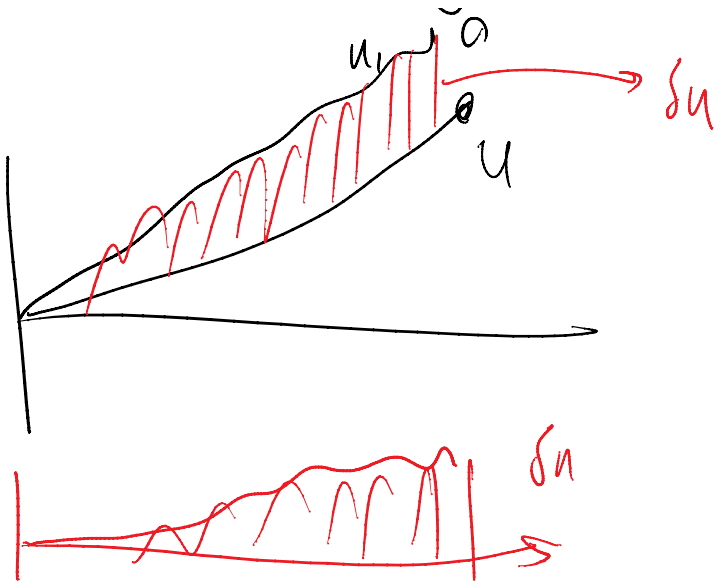
$$+ \int_0^L EA\left(\frac{d\delta u}{dx}\right)^2 dx$$

D.E.
u exact solution
natural BC
u' is exact solution

$$\Pi(u+\delta u) = \Pi(u) + \int_0^L \frac{1}{2} EA\left(\frac{d\delta u}{dx}\right)^2 dx$$

>0





> 0

$$\Delta \Pi(u) = \Pi(u+\delta u)$$

$$\downarrow \text{total variation} \quad - \Pi(u)$$

$$= \int_0^L \frac{1}{2} EA \left(\frac{\partial y}{\partial x}\right)^2 dx > 0$$