1. **30 + 15 extra credit Points** Sequence of dof numbering has a profound effect on the arrangement of nonzero values for FE models. Finite element stiffness matrices are called band matrix because outside a "band" around the diagonal all the values are zero. The shorter the band the better the matix is both in terms of storage and solution cost. FE codes generally renumber dofs to optimize dof sequencing, *i.e.*, minimizing bandwidth of the matrix. Bandwidth refers to the maximum difference in indices of stiffness matrix for components that are nonzero: bandwidth $= \{\max(|i-j|)|\ k_{ij} \neq 0\}$. To obtain the bandwidth it is assumed all components of element stiffness matrices are nonzero and based on connectivity of elements we find the maximum global dof difference within one element. For example for element A in fig. 1(a) maximum dof difference is 18 - 1 = 17.

   (a) What is the bandwidth for the schemes shown in fig. 1?

   (b) Which scheme is the best?

   (c) **(Extra credit)** Fill out stiffness matrix for scheme C using the matrix provided in fig. **??**. Use 'x' for nonzero value and leave zero entries blank.

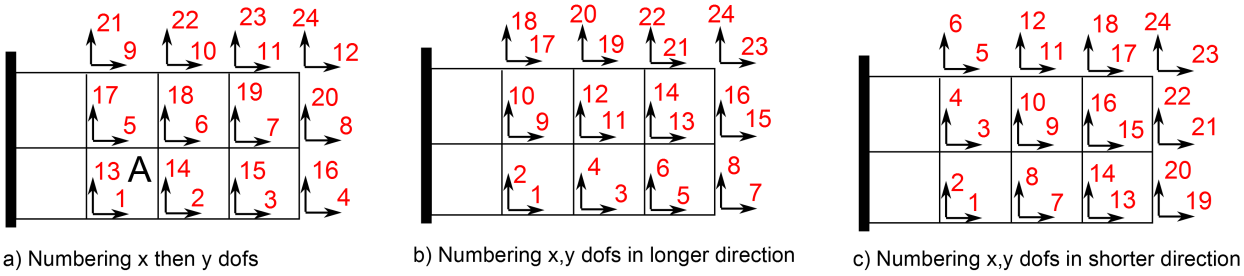
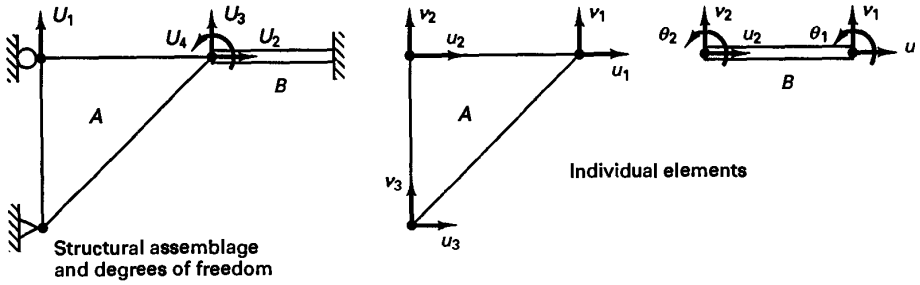


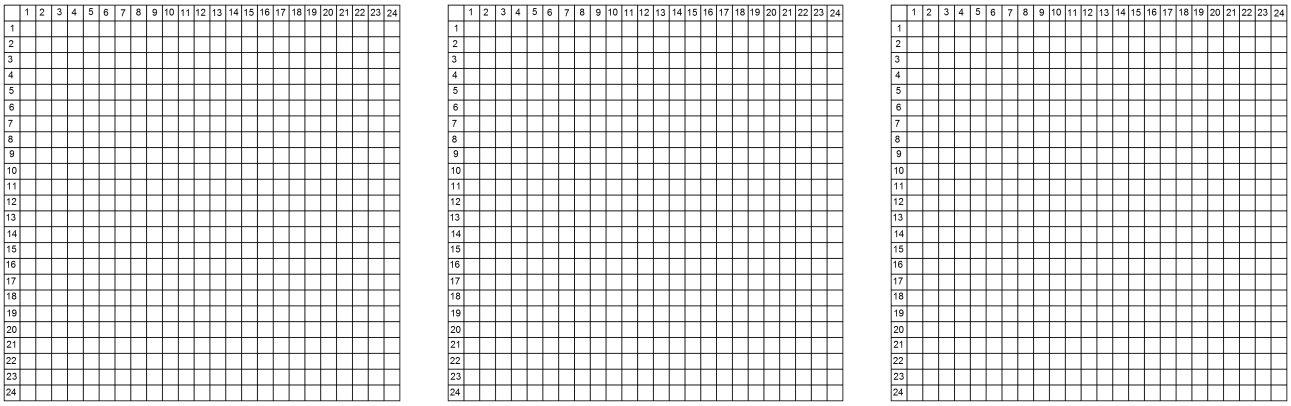
Figure 1: Comparison of the sequence of dof numbering.

2. **20 Points** Assume that the stiffness matrices $\mathbf{K}_A$ and $\mathbf{K}_B$ are already computed. Assemble these element matrices directly into the global stiffness matrix.



$$\mathbf{K}_A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix} \qquad \mathbf{K}_B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{matrix}$$

3. **85 Points** **Static condensation (continuation of HW5)**: In course notes pages 331-334 we discussed how prescribed dofs can be eliminated at the global assembly stage if all dofs (free + prescribed) are assembled into the global system. We compared that approach with common



(a) Sequence A. (b) Sequence B. (c) Sequence C.

Figure 2: Stiffness matrices from fig. 1.

approach of only assembling only the free dofs. The same decomposition of the matrix can be used in "condensing" the internal dofs of higher order elements (those that have the so-called "bubble shape functions") at the element level and only assembling the dofs that are shared between the element level at the global stage. This can substantially reduce the total number of dof and result in much smaller global stiffness matrix to solve. An example can be seen in fig. 3 where the average dof per element (assuming there is one dof per node) reduces by about a factor 2 (below)! The stiffness matrix at the element level can be written as,

$$\mathbf{ka} = \mathbf{f} \quad \text{where } \mathbf{k} = \left[\begin{array}{c|c} \mathbf{k}_{ee} & \mathbf{k}_{ei} \\ \hline \mathbf{k}_{ie} & \mathbf{k}_{ii} \end{array}\right] \quad \mathbf{f} = \left[\begin{array}{c} \mathbf{f}_e \\ \hline \mathbf{f}_i \end{array}\right], \quad \mathbf{a} = \left[\begin{array}{c} \mathbf{a}_e \\ \hline \mathbf{a}_i \end{array}\right] \tag{1}$$

The unknown part that we want to condense and not assemble to the global system is $\mathbf{a}_i$. The equation (1) can be written as

$$\mathbf{f}_e = \mathbf{k}_{ee}\mathbf{a}_e + \mathbf{k}_{ei}\mathbf{a}_i \tag{2a}$$
$$\mathbf{f}_i = \mathbf{k}_{ie}\mathbf{a}_e + \mathbf{k}_{ii}\mathbf{a}_i \tag{2b}$$

By solving for $\mathbf{a}_i$ in (2b) and plugging it into (2a) obtain the "effective" element stiffness matrix $\tilde{\mathbf{k}}$ and force vector $\tilde{\mathbf{f}}$:

$$\tilde{\mathbf{f}} = \tilde{\mathbf{k}}\mathbf{a}_e \tag{3}$$

Note that static condensation does not reduce the order of accuracy of the method; at local element level we solve these small system of equations that can reduce the global number of unknowns. Also $\mathbf{f}$ is the local elemental forces from all sources of forcing such as natural and essential boundary conditions (only external), source term, and nodal concentrated forces (only internal).

- **15 Points** Show the average dofs per element for the 3rd order rectangle in fig. 3 before condensation is 9 and after condensation the average reduced to 5. For this type of analysis we ignore the effect of dofs on the boundaries of the domain (*i.e.,* domain is infinite). This approximation is correct when number of elements in each direction is (very) high.
- **25 Points** Obtain $\tilde{\mathbf{k}}$ and $\tilde{\mathbf{f}}$ in terms of $\mathbf{k}_{ee}, \mathbf{k}_{ei}, \mathbf{k}_{ie}, \mathbf{k}_{ii}$ and $\mathbf{f}_e, \mathbf{f}_i$.

- **45 Points** For second order bar element with constant E, Q and the internal node n_3 being in the center we have (*cf.* fig. 4),

$$\mathbf{k} = \frac{AE}{3L}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \tag{4}$$

using static condensation approach write the expressions for $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{k}}$ when the internal dof u_3 is condensed. Compare $\tilde{\mathbf{k}}$ with the stiffness matrix for a linear bar element and comment on how this effective system is still more accurate given the similarities after condensation to a linear bar element.

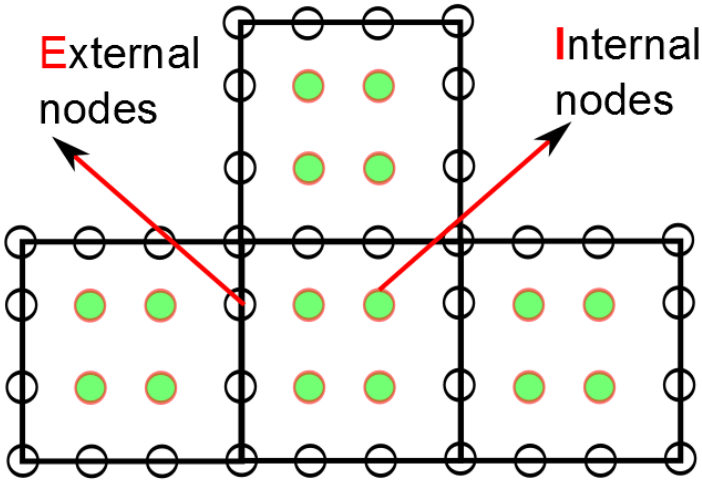


Figure 3: Internal nodes that can be condensed at the element level before global assembly

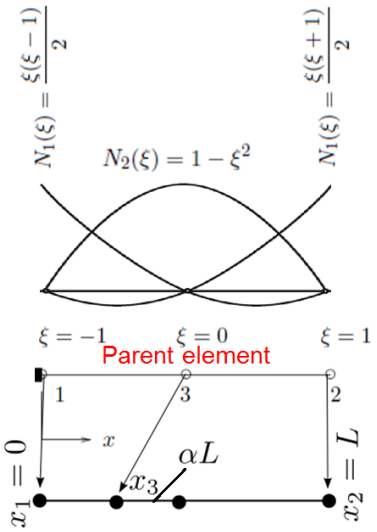


Figure 4: Second order element for 1D problems.

4. **30 Points** In FE method a priori error estimates generally take the form $\mathcal{E} = Ch^{ap+b}$ [1] where

[1] If the solution is not regular enough p must be replaced by $\min(p, s)$ where s is the regularity, *i.e.*, order of continuity

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189

$$\int_a^b f(x)\,dx \simeq C_0 h \sum_{i=1}^{n} W_i f(x_i) + C_1 h^{k+1} f^{(k)}(\xi)$$

Newton-Cotes

n	C_0	W_1	W_2	W_3	W_4	W_5	C_1	k	Name
1	1	1					1/2	1	Rectangle
2	1/2	1	1				–1/12	2	Trapezium
3	1/3	1	4	1			–1/90	4	Simpson
4	3/8	1	3	3	1		–3/80	4	4-point
5	2/45	7	32	12	32	7	–8/945	6	5-point

Figure 5: Gauss and Newton-Cotes quadrature points.

p is the element (polynomial) order C is an unknown constant, $\mathcal{E}$ is an error, and $a > 0$, b, are constants depending on the error type. That is, error convergence is linear with the rate $ap + b$: $(\log\mathcal{E} = \log C + (ap + b)\log h$. The purpose of these estimates are to understand how much the error is expected to decrease when from a current FEM solution we do h-refinement by decreasing element sizes or p-enrichment by increasing element order.[2] Generally, the equation is used for uniform meshes (same element size) and uniform element order. Assume for a simulation a given error $\mathcal{E}$, *e.g.,* numerical energy dissipation, takes the value of $\mathcal{E}_1 = 10^{-4}$ for $h_1 = 0.1$ and $p_1 = 1$ (linear element). Constants a and b are 2 and -1; that is, $\mathcal{E} = Ch^{2p-1}$. We want to estimate how much smaller the element sizes or how larger the element polynomial should be to reduce the error by 4 orders of magnitude. That is to achieve $\mathcal{E}_2 = 10^{-8}$ estimate,

(a) h-refinement: If we keep p fixed ($p_2 = p_1 = 1$) what the new element size h_2 is estimated to achieve the target error of 10^{-8}? How much smaller is the new element size (h_2/h_1)? If the problem is 2D how many more elements is estimated to require for achieve the smaller error ($(n_e)_2/(n_e)_1$)?

(b) p-enrichment: If we keep the element size fixed ($h_2 = h_1 = 0.1$) what is the smallest polynomial that we expect to meet the more stringent error requirement?

(c) Which scheme do you think requires fewer dofs to achieve the raget error, h-refinement or p-refinement? Note that having fewer dofs (with the same error value) for globally coupled static problems generally translates to higher efficiency.

Hint: You can either directly obtain the value C from $\mathcal{E}_1 = Ch^{2p_1-1}$ or divide the equations for 1 and 2 states to eliminate C in calculating h_2 or p_2.

C^s, of exact solution for the given error $\mathcal{E}$. For example if exact solution displacement for a problem is C^4 and we calculate strain energy error $\mathcal{E} = \int_D (\epsilon^h - \epsilon) : \boldsymbol{C} : (\epsilon^h - \epsilon) dV$ since $\epsilon = \frac{1}{2}(\nabla u + \nabla^{\mathrm{T}} u)$, s would be 4 - 1 = 3 (exact ϵ is C^3).

[2] If the elements are too large, these estimates generally do not hold as we are not in asymptotic error convergence range. For example by refining elements the error may reduce much less than estimated.