1. Figure 1 shows a possible jump manifold $\mathit{\Gamma}$ in a domain $\mathcal{D}$. The balance law,

$$\int_{\partial\Omega} \mathbf{F}.\mathbf{n}\,\mathrm{dS} + \int_{\Omega} \mathbf{r}\,\mathrm{dV} = \mathbf{0} \tag{1}$$

holds for all $\Omega \subset \mathcal{D}$. The flux tensor is denoted by $\mathbf{F}$ and $\mathbf{r}$ is the source term. Show that a jump condition across Γ satisfies the condition

$$\mathbf{F}^{+}.\mathbf{n}^{+} = -\mathbf{F}^{-}.\mathbf{n}^{-} \quad \text{which is equivalent to} \tag{2a}$$

$$[\![\mathbf{F}]\!]|_{\Gamma}.\mathbf{n}^{+} = (\mathbf{F}^{+} - \mathbf{F}^{-}).\mathbf{n}^{+} = \mathbf{0} \tag{2b}$$

for all $\mathbf{x}_0 \in \Gamma$. We assume that the flux $\mathbf{F}$ is continuus on both sides of Γ. [1] (40 points)

Hint: To prove this equation use the balance law for three domains: Ω^{+}, Ω^{-}, and $\Omega = \Omega^{+} \cup \Omega^{-}$. By subtracting the balance equations on Ω^{+} and Ω^{-} from that on Ω obtain an integral equation solely on $\Gamma \cap \partial\Omega^{+}$. Then, "localize" the integral to an arbitrary point $\mathbf{x}_0$ to obtain the point-wise condition.

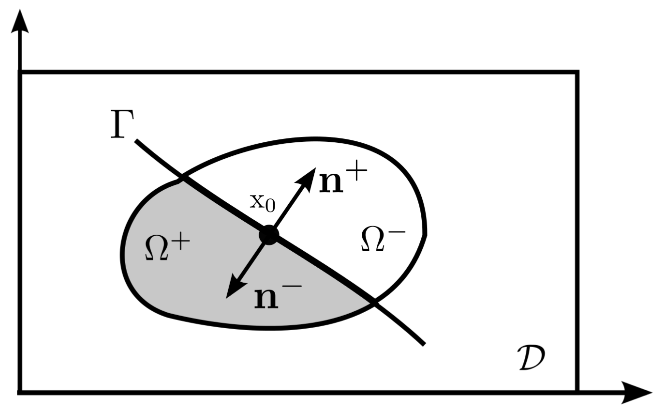


Figure 1: Derivation of the jump conditions from the balance law

2. For elastostatics $\mathbf{F}$ in equation (1) is $-\sigma$ and $\mathbf{r} = \rho\mathbf{b}$, where σ is the stress tensor and $\mathbf{b}$ is the body force per unit mass. (40 points)

   (a) Show that the jump condition for elastostatics across a jump manifold (equation (2a)) reduces to

   $$\mathbf{t}^{+} = -\mathbf{t}^{-} \quad \text{(Newton's second law for tractions).} \tag{3}$$

   Hint: Note that traction is given by $\mathbf{t} = \sigma.\mathbf{n}$ where for traction on each side, σ and $\mathbf{n}$ are taken from the same side.

   (b) Figure 2 shows an example of jump in flux tensor $-\sigma$ for elastostatics. The top surface is uniformly moved up to generate a uniform strain $\epsilon_{22} = \bar{\epsilon}$ in both materials. Since they possess different elastic moduli E^{+} and E^{-}, the stress tensor has a jump across the material interface Γ. Using the solutions provided in the figure, show that in spite of this jump the tractions still satisfy the common action-reaction law at the interface (*cf.* equation (3)).

   (c) For this 2D problem, the strong form is:

   $$\sigma_{11,1} + \sigma_{12,2} + \rho b_1 = 0$$
   $$\sigma_{21,1} + \sigma_{22,2} + \rho b_2 = 0$$

[1] While physically jumps can occur in $\mathbf{F}$, the strong form of the problem, *i.e.*, $\nabla.\mathbf{F} - \mathbf{r} = 0$, may not be valid at the points on the jump manifold as the (partial) derivatives in $\mathbf{F}$ may not exist.

Discuss whether any of the partial derivatives in this equation cannot be computed across the jump, given that the stress tensor is discontinuous.

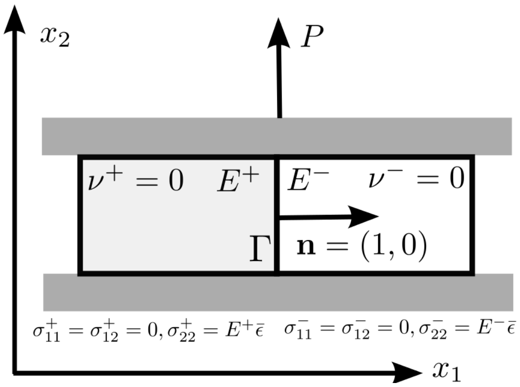


Figure 2: Jump in stress tensor across a material interface.

3. Figure 3(a) shows a possible jump manifold in spacetime for 1D setting. In this case the stress matrix and linear momentum $\mathbf{p} = \rho\mathbf{v}$ reduce to only one component of σ_{11} and p_1, respectively. For brevity, we drop the these indices. As mentioned, the flux matrix for elastodynamics in spacetime is $\mathbf{F} = [-\sigma \ p]$. (40 points)

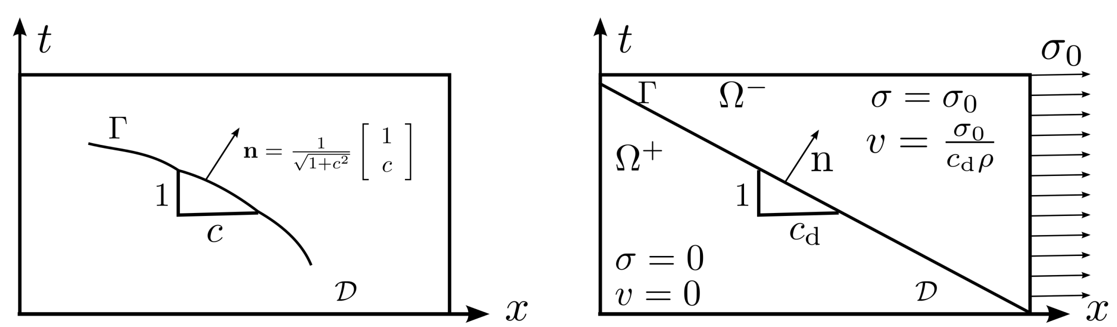
   (a) Given the equation (2b), write the jump conditions for this problem based on the solutions and spacetime normal vector [2] given in figure 3(a).

   (b) Figure 3(b) shows the exact solution to a 1D wave propagation problem where the right boundary of a semi-infinite bar is loaded by stress σ_0. Here $c_{\mathrm{d}} = \sqrt{E/\rho}$ is the material dilatational wave speed. Verity that the jump from the two sides + and − satisfy the jump condition obtained in previous step.

   (c) The strong form of this problem is,

   $$\sigma_{,x} + \rho b = \dot{p}$$

   Given the exact solution in the figure, discuss the validity of the strong form for the points on Γ. Compare this with the elastostatics case in problem 2.c.



(a) Spacetime normal vector $\mathbf{n}$ for a jump manifold moving with speed $-c$.

(b) Exact solution for a problem with applied stress

Figure 3: Jump conditions for elastodynamics

[2] As can be seen, in the calculation of the normal vector, we are adding the square of the speed of the manifold Γ, c^2, to unity. While the problem can be fixed by using more advanced notations, we do not pursue it further in this course.