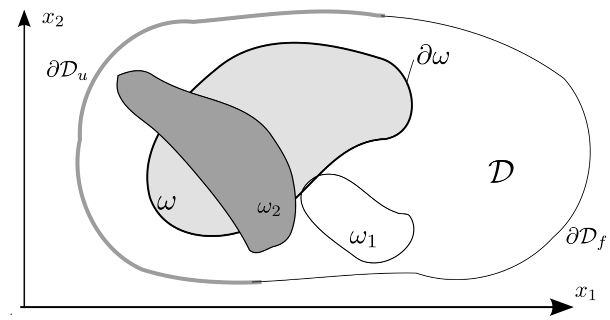
Consider the 2D spatial domain $\mathcal{D}$ with essential (Dirichlet) boundary $\partial\mathcal{D}_u$ and natural (Neumann) boundary $\partial\mathcal{D}_f$ ($\overline{\partial\mathcal{D}_u} \cup \overline{\partial\mathcal{D}_f} = \overline{\partial\mathcal{D}}, \partial\mathcal{D}_u \cap \partial\mathcal{D}_f = \emptyset$) for steady state heat equation. The purpose of this problem is to derive the weak form from two different approaches: balance law approach in problems 1-3 and functional approach in problem 4.



1. **Balance law (30 Points):** The spatial flux $\mathbf{f}$ and source term s for this problem are: *heat flux vector* $\mathbf{f} = \mathbf{q}$ and *heat source (scalar)* $s = Q$. This balance law corresponds to steady state balance of energy when only temperature effects are considered.

   Write the balance law statement for **arbitrary** $\omega \subset \mathcal{D}$.

2. **Strong form (100 Points):** In this step we derive the partial differential equation (PDE) and close the system with constitutive/compatibility equations and boundary conditions.

   (a) PDE: From the balance law derive the PDE. In less than 2-3 lines briefly write the intermediate equations and theorems used to derive the PDE. Why the space of acceptable solutions for the PDE is more limited that than for the balance law.

   (b) BC: Write the expressions for the boundary conditions in terms of primary field temperature T and heat flux vector $\mathbf{q}$ on $\partial\mathcal{D}_u$ and $\partial\mathcal{D}_f$. The applied boundary values are $\bar{T}$ and $\bar{q}$. Note that $\bar{q}$ is scalar and is the outward net flux and pay attention to the fact that $\mathbf{q}$ is a vector when the writing natural BC. Can we solve the system of these boundary conditions and the PDE?

   (c) Constitutive equation: Write the constitutive equation based on Fourier law (relating $\mathbf{q}$ and T). Does Fourier heat flux equation apply to all material types? How do you compare this with a balance law statement?

   (d) Boundary value problem (BVP): Combine all the steps above to obtain the BVP in terms of T.

3. **Weak form (100 Points):** We want to obtain weak statements (integral forms) whose solution is the same as the solution to the BVP above.

   (a) **Weighted Residual (WR) Statement**: As it is the common practice, for the candidate solution T we strongly satisfy the essential boundary condition residual. That is, $\forall \mathbf{x} \in \partial\mathcal{D}_u$: $\mathcal{R}_u(\mathbf{x}) = \bar{T}(\mathbf{x}) - T(\mathbf{x}) = 0$.

      i. Write the interior residual $\mathcal{R}_i$ for $\mathbf{x} \in \mathcal{D}$ and natural BC residual for $\mathbf{x} \in \partial\mathcal{D}_f$.

      ii. Write the Weighted Residual statement.

      iii. What space should the candidate solution function $T(\mathbf{x})$ belong to? Noting this set by $\mathcal{V}_w$, that is $T \in \mathcal{V}_w$, write $\mathcal{V}_w$ in the form $\mathcal{V}_w = \{u \in C^n(\mathcal{D})|$ required boundary conditions (if any)$\}$. That is, specify i) number of derivatives (m) and boundary conditions needed for T.

      iv. What space does the weight function belong to ($w \in \mathcal{W}_w$)? Again write it in the form $\mathcal{W}_w = \{u \in C^n(\mathcal{D})|$ required boundary conditions (if any)$\}$.

(b) **Weak Statement**: Noting that $w\left(\nabla.(\kappa\nabla T)\right) = \nabla.(w\kappa\nabla T) - \nabla w.(\kappa\nabla T)$:

  i. Use the Gauss (divergence) theorem to transform the weighted residual statement to the *weak statement*.
  Hints: 1. Make sure in the WR statement $\mathcal{R}_f$ is added with the right sign so that boundary terms generated by $(w\kappa\nabla T)$ term above cancel some of $\mathcal{R}_f$ terms; 2. After the application of Gauss theorem some boundary terms are generated on $\partial\mathcal{D}_u$. Make judicious choice for the spaces of the functions T or w so that those terms would disappear.

  ii. What space should the candidate solution function $T(\mathbf{x})$ belong to? Noting this set by $\mathcal{V}$ express it in the form $\mathcal{V} = \{u \in C^n(\mathcal{D})|$ required boundary conditions (if any)$\}$. That is, m and BCs should be specified.

  iii. What space does the weight function belong to ($w \in \mathcal{W}$)? Again write it in the form $\mathcal{W} = \{u \in C^n(\mathcal{D})|$ required boundary conditions (if any)$\}$. Note that due to the relation between $\mathcal{W}$ and $\mathcal{V}$, $\mathcal{W}$ is often denoted by $\mathcal{V}_0$.

  iv. Briefly (less than 2-3 lines) discuss how the space of the solution for Weak statement compares to those from BVP and WR as well as that from the balance law (+BCs and constitutive equation).

4. **Functional approach (70 Points)**[1] The solution to the heat equation can alternatively be obtained by minimization of the following functional (no convective BCs):

$$\mathit{\Pi}(T, \nabla T) = \int_{\mathcal{D}} \left\{ \frac{1}{2}\nabla T.(\kappa\nabla T) - TQ \right\} \mathrm{dV} - \int_{\partial\mathcal{D}_f} T(-\bar{q})\,\mathrm{dS} \tag{1}$$

where T strongly satisfies the essential BC:

$$\forall\mathbf{x} \in \partial\mathcal{D}_u : \; T(\mathbf{x}) = \bar{T}(\mathbf{x}) \tag{2}$$

(a) By using the extremum condition $\delta\mathit{\Pi}(T, \nabla T) = 0$ and letting $w := \delta T$ derive the weak statement obtained by balance law approach in 3.b.

(b) Why the weight function strongly satisfy the *homogeneous essential boundary condition*: ($\forall\mathbf{x} \in \partial\mathcal{D}_u : \; w(\mathbf{x}) = 0$)? Hint: $w = \delta T$ and use equation (2).

(c) Briefly less than 2-3 lines compare the functional approach with balance law approach in obtaining the weak statement.

[1] In this case the functional does not correspond to an "energy".