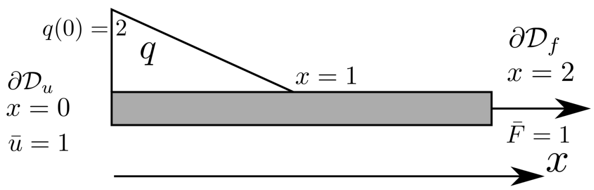
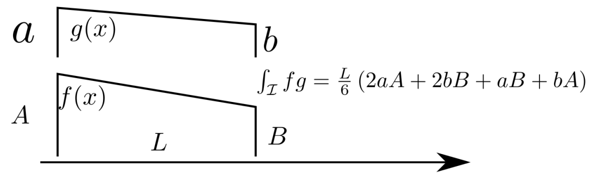


(a) Schematic of the 1D bar example.



(b) Integration of product of linear by linear functions.

Consider the 1D bar example shown with the following boundary value problem:

$$\begin{cases} \frac{\mathrm{d}\sigma(u(x))}{\mathrm{d}x} + q(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) + q(x) = 0 & \text{Strong form} \\ u(0) = \bar{u} & \text{Essential BC on } \partial\mathcal{D}_u = \{0\} \\ F(2) = EA\frac{\mathrm{d}u(x)}{\mathrm{d}x} = \bar{F} & \text{Natural BC on } \partial\mathcal{D}_f = \{2\} \end{cases} \tag{1}$$

Material and load properties are,

$$q(x) = \begin{cases} 2 - 2x & x < 1 \\ 0 & 1 \le x \le 2 \end{cases} \qquad E(x) = 1, \quad A(x) = 1, \quad \bar{u} = 1, \quad \bar{F} = 1$$

More details on this problem can be found in course notes equations (177) to (179).

1. Starting with the interior residual and residual on natural boundary derive weighted residual statement (equation (180) in course notes). (**10 Points**)

$$\text{Find } u \in \mathcal{V} = \{v |\, v \in C^2([0\ 2]), v(0) = \bar{u} = 1\} \ :\forall w \in \mathcal{W} = C^0([0\ 2])$$
$$\int_0^2 w\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + q(x)\right)\,\mathrm{d}x + w^f(1 - \frac{\mathrm{d}u}{\mathrm{d}x}(2)) = 0 \quad (\text{no restrictions for } w \text{ on } \partial\mathcal{D}_u) \tag{2}$$

2. Apply integration by parts to (2) to obtain the weak statement for $w^f = \mathbf{w}$ (equation (187) in course noted). In the process determine how certain boundary terms are eliminated. (**10 Points**)

$$\text{Find } u \in \mathcal{V} = \{v |\, v \in C^1, v(0) = \bar{u} = 1\} \ :\forall w \in \mathcal{V}_0 = \{v |\, v \in C^1, v(0) = \bar{u} = 0\}$$
$$\int_0^2 \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int_0^1 w(x).q(x)\,\mathrm{d}x + w(2) \tag{3}$$

3. For 4 unknowns ($n = 4$):

   (a) Solve the problem for 1) Subdomain method (S); 2) Collocation method (C), 3) Finite Difference (FD), 4) Least Square (L), 5) Galerkin method with $\boldsymbol{\phi} = [x\ x^2\ x^3\ x^4]$ (G), and 6) Galerkin method with Finite Element hat functions (FE) shown in figure 3a . Trial functions for 1, 2 and 4 is also $\boldsymbol{\phi} = [x\ x^2\ x^3\ x^4]$. For your solutions use $\phi_p = 1$. For each case clearly indicate $\mathbf{K}$, $\mathbf{F}$ and the solution $\mathbf{a}$ (for finite difference method the linear matrix equation corresponds to unknown values $u_1^{FD} = u^{FD}(0.5)$, $u_2^{FD} = u^{FD}(1)$, $u_3^{FD} = u^{FD}(1.5)$, and $u_4^{FD} = u^{FD}(2)$). Use weighted residual method (2) for S, C, L and weak statement (3) for G, FE.($6 \times 25 =$ (**150 Points**)).

   (Hint: For FE case there are many integrands that are products of two linear functions. The formula in figure (b) can be used to evaluate such integrals.)

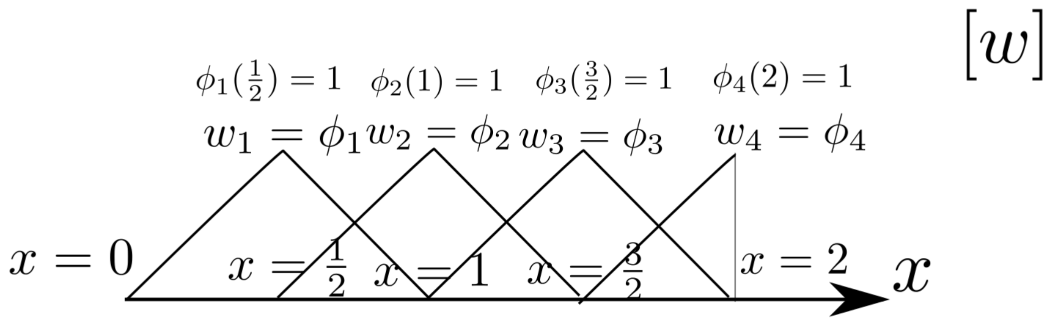


Figure 1: Trial and Weight functions for $n = 1$ 1D Finite Element Method.

Method	$\boldsymbol{\phi}$	**w**	Can WR (2) be used and why?	Can WK (3) be used and why?
S	$[x\ x^2\ x^3\ x^4]$			
C	$[x\ x^2\ x^3\ x^4]$			
L	$[x\ x^2\ x^3\ x^4]$			
G	$[x\ x^2\ x^3\ x^4]$			
FE	figure 3a			

(b) Summarize the unknowns based on the given solution orders above in the matrix U: (**5 Points**)

$$U = \begin{bmatrix} a_1^S & a_2^S & a_3^S & a_4^S \\ a_1^C & a_2^C & a_3^C & a_4^C \\ u_1^{FD} & u_2^{FD} & u_3^{FD} & u_4^{FD} \\ a_1^L & a_2^L & a_3^L & a_4^L \\ a_1^G & a_2^G & a_3^G & a_4^G \\ a_1^{FE} & a_2^{FE} & a_3^{FE} & a_4^{FE} \end{bmatrix} \tag{4}$$

4. Fill out table (4). If extra space needed the three questions about **w** and applicability of WR and WK can be directly addressed in text. For L (least square) also list $\mathbf{w}^f$ (weight functions on $\partial\mathcal{D}_f = \{2\}$. (**50 Points**)

   Special attention should be paid for the applicability of WK (3) to S and C (if not explain) and WR (2) to FE.

5. In Weighted Residual statement (2) it is stated that the weight functions (in $\mathcal{W}$) should be continuus (C^0). Reviewing the weight functions in (4) and your solutions in problem 3. In which approach (S, C, L) the weight functions are not C^0. Is this a problem? (**15 Points**)

6. In Weak statement (3), it is stated that the solution, hence the trial functions ϕ_i, should have continuus derivative every where (C^1). Are the trial functions for Finite Element method (*cf.* figure 3a) C^1? If not, does this cause a problem in evaluating the weak statement? (**15 Points**)

7. Matlab data processing: Run the provided Matlab function WRM1D.m from the course web page [1] with the input argument U in (4) to generate various plots for $n = 1$ to $n = 4$ for all the method discussed above.

   (a) Submit all the plots generated by the function electronically. (**10 Points**)

   (b) File "WRM-error-method6-FE.png" displays the error between Finite Element solution and exact solution ($\Delta u_{FE} = u_{FE} - u$) versus spatial position ($x = 0$ to $x = 2$) for $n = 1$ to

[1] http://rezaabedi.com/wp-content/uploads/Courses/FEM2015-ME517/WRM1D.m

$n = 4$). Discuss the especial property of the error at finite element nodes (*e.g.*, $x = 2$ for $n = 1$, $x = 1, 2$ for $n = 2$, $x = \frac{2}{3}, \frac{4}{3}, 2$ for $n = 3$, and $x = \frac{1}{2}, 1, \frac{3}{2}, 2$ for $n = 4$). (**5 Points**)

(c) In collocation and finite difference methods we enforce the equations at discrete points. While the solutions at these points do not match the exact solution?

Files "WRM-solutions(n)Unknown.png" ($n = 1$ to $n = 4$) display the solutions. (**10 Points**)

(d) In Least square method, we minimize R^2 for the entire admissible function space $\{x^1, \ldots, x^n\}$ ($n = 1$ to $n = 4$). We however observe that other methods (for example Galerkin with the same trial functions (G)) is a more accurate solution. This can be observed in "WRM-solutions(n)Unknown.png" "WRM-errors(n)Unknown.png" file. Explain why this is not a contradiction? (**10 Points**)

(e) For Finite element methods errors between discrete and continuum solutions (here denoted by e) can be expressed as,

$$e = Ch^{\alpha} \quad \Rightarrow \log(e) = \log(C) + \alpha \log(h)$$

That is in a log-log plot, $\log(e)$ should be approximately linear versus $\log(h)$ with the slope of α. Refer to file "WRM-all-errors-log-h.png" and verify if the data set for Finite Element is linear and from the plot provide an approximate value for α. (**10 Points**)

(f) From the same file "WRM-all-errors-log-h.png" discuss which method has the fastest decrease of the error versus h? Does the rate of the error convergence (slope of $\log(e)$ versus $\log(h)$) increases as h decreases for this method? (**5 Points**)

8. **Extra credit problems:** (**65 Total Points**)

In Finite Element method (and many other numerical methods) the solution of algebraic equation $\mathbf{Ka} = \mathbf{F}$ is major source of computational costs. If the matrix $\mathbf{K}$ is diagonal, the solution is trivial ($a_i = \frac{F_i}{K_{ii}}$, no summation on i). In particular, for identity $\mathbf{K}$:

$$\mathbf{K} = \mathbf{1} \ \wedge \ \mathbf{Ka} = \mathbf{F} \quad \Rightarrow \quad \delta_{ij} a_j = F_i \quad \Rightarrow \quad \boxed{\mathbf{a} = \mathbf{F}} \tag{5}$$

Thus, for identity $\mathbf{K}$ we circumvent the solution of a linear system. From (3) we observe that for weak statement,

$$K_{ij} := \mathcal{A}(\phi_i, \phi_j) = \int_0^2 \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \frac{\mathrm{d}\phi_j}{\mathrm{d}x} \, \mathrm{d}x \tag{6}$$

where the bilinear operator $\mathcal{A}(\phi_i, \phi_j)$ has the properties of an inner product and for any given basis for $\{\phi_1, \phi_2, \ldots, \phi_n\}$ we can form an orthonormal basis using Gram Schmidt method. That is, for the new basis system $\mathbf{K}$ is equal to an identity matrix.

There are many orthogonal function basis in mathematics. For example, Legendre polynomials satisfy the following orthogonality condition:

$$\int_{-1}^{1} P_m(\xi) P_n(\xi) \, \mathrm{d}\xi = \frac{2}{2n+1} \delta_{mn} \quad \text{no sum on } n \tag{7}$$

These functions can be obtained from the recursive equation:

$$P_0(\xi) = 1 \tag{8a}$$

$$P_1(\xi) = \xi \tag{8b}$$

$$(n+1) P_{n+1}(\xi) = (2n+1) \xi P_n(\xi) - n P_{n-1}(\xi) \tag{8c}$$

For our purpose, the first four functions are given by,

$$\begin{array}{cc} n & P_n(\xi) \\ 0 & 1 \\ 1 & \xi \\ 2 & \frac{1}{2}(3\xi^2-1) \\ 3 & \frac{1}{2}(5\xi^3-3\xi) \end{array} \tag{9}$$

To use these orthogonal basis for our bar example, we use the coordinate transformation,

$$\xi = x - 1 \quad \text{for } x \in [0\ 2] \tag{10}$$

to map x to $\xi \in [-1\ 1]$ used in definition of Legendre polynomials.

(a) Show that the following trial functions:

$$\boxed{\phi_{i+1}(\xi) = \sqrt{i+\frac{1}{2}}\int_{-1}^{\xi} P_i(\eta)\,\mathrm{d}\eta} \tag{11}$$

- are admissible trial functions for weak statement (3). That is, $\phi_i(x=0) = \phi_i(\xi=-1) = 0$. (**5 Points**)
- are orthonormal with respect to inner product in (6): (**10 Points**)

$$K_{ij} := \mathcal{A}(\phi_i, \phi_j) = \int_0^2 \frac{\mathrm{d}\phi_i}{\mathrm{d}x}(x)\frac{\mathrm{d}\phi_j}{\mathrm{d}x}(x)\,\mathrm{d}x = \int_{-1}^{1} \frac{\mathrm{d}\phi_i}{\mathrm{d}\xi}(\xi)\frac{\mathrm{d}\phi_j}{\mathrm{d}\xi}(\xi)\frac{\mathrm{d}\xi}{\mathrm{d}x}\,\mathrm{d}\xi = \delta_{ij}$$

(Hint: use (7) and (11))

(b) Using (11) and (9) show that, (**10 Points**)

$$\begin{array}{cc} i & \phi_i(\xi) \\ 1 & \sqrt{\frac{1}{2}}(\xi+1) \\ 2 & \sqrt{\frac{3}{8}}(\xi^2-1) \\ 3 & \sqrt{\frac{5}{8}}(\xi^3-\xi) \\ 4 & \frac{1}{8}\sqrt{\frac{7}{2}}(5\xi^4-6\xi^2+1) \end{array} \tag{12}$$

(c) Using (3) we have,

$$F_i = \int_0^1 \phi_i(x).q(x)\,\mathrm{d}x + \phi_i(x=2) = \int_{-1}^{0} \phi_i(\xi).q(\xi)\frac{\mathrm{d}x}{\mathrm{d}\xi}\,\mathrm{d}\xi + \phi_i(\xi=1) \quad \text{for}$$

$$q(\xi) = \begin{cases} -2\xi & \xi < 0 \\ 0 & 0 \le \xi \le 1 \end{cases}$$

Find $\mathbf{F} = \mathbf{a}$ (**20 Points**)

(d) From the values $\mathbf{a}$ above, (12), $u^h = a_i\phi_i + \phi_p$, $\phi_p = 1$, obtain u^h in terms of x and compare it with the Galerkin solution in question 3. (Hint: it is easier to obtain each $a_i\phi_i$ first as a function of ξ then change $\xi \to x$ before plugging into $u^h = a_i\phi_i + \phi_p$.)
(**20 Points**)