2016/02/18

Thursday, February 18, 2016
8:14 AM

Last time we derived the weighted residual statement for the 1D example:

- According to derivative orders the weighted residual statement with function spaces is,

  Find $u \in \mathcal{V} = \{v | \, v \in C^2([0\ 2]), v(0) = \bar{u} = 1\}$:$\forall w \in \mathcal{W} = C^0([0\ 2])$

$$\int_0^2 w \left(\frac{d^2u}{dx^2} + q(x) \right) dx + w^f (1 - \frac{du}{dx}(2)) = 0 \quad \text{(no need for } w(\partial\mathcal{D}_f = \{0\}) = 0) \tag{180}$$

Now we want to derive the weak form from that

$$\int_0^2 w u'' = fg\Big|_0^2 - \int_0^2 f' g \, dx$$

$f \quad g'$

$g = u'$

$$= w u'\Big|_0^2 - \int_0^2 w' u' \, dx$$

$$\boxed{\int_0^2 w u'' dx = w(2) u'(2) - w(0) u'(0) - \int_0^2 w' u' dx}$$

$\bar{F} - AE u'$ $\quad R_f = \bar{F} - F$ or $R_f \; F = \bar{F}$

$$\int_0^2 w \left(\frac{d^2u}{dx^2} + q(x) \right) dx + w^f (1 - \frac{du}{dx}(2)) = 0 \quad \text{(no need for } w(\partial\mathcal{D}_f = \{0\}) = 0) \tag{180}$$

this one is chosen to get cancellation in ☆

$$w(2) u'(2) - w(0) u'(0) - \int_0^2 w' u' dx + \int_0^2 w q \, dx +$$

$$w(2) 1 - w(2) u'(2) = 0$$

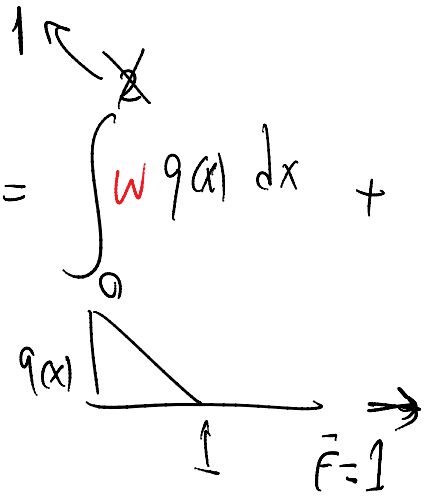
☆

if w satisfies the HOMOGENEOUS ESSENTIAL BC this term is also eliminated

WEAK form w satisfies

This is not need for WRS

$$\int_0^2 w' u' \, dx = \int_0^1 w\, q(x)\, dx + \underbrace{1}_{\tilde{F}}\, w(2)$$



Internal virtual work

$u(x) = \phi_p(x) + \sum a_j \phi_j(x)$

$w = w_i \Longrightarrow$ i^{th} equation

$$\int_0^2 w_i' \left(\phi_p + \sum_{j=1}^{n} a_j \phi_j(x) \right) dx = \int_0^1 w_i\, q(x)\, dx + w_i(2)\, 1$$

stiffness

rest of the terms contribute to the RHS (F)

$$\underbrace{\left[\int_0^2 w_i' \phi_j'(x)\, dx \right]}_{K_{ij}} a_j = \underbrace{\int_0^2 w_i\, q(x)\, dx + w_i(2) - \int_0^2 w_i' \phi_p'\, dx}_{F_i}$$

in $Ka = F$

$K_{ij}\, a_j = F_i$

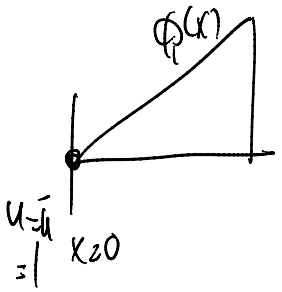
$$K_{ij}\, q_j = F_i$$

$$K_{ij} = \int_0^2 w_i' \phi_j' \, dx \qquad f_i = \int_0^1 w_i\, q(x)\, dx + w_i(1) - \int_0^2 w_i'(x)\phi_p'(x)dx$$

K & F for weak statement

1. Subdomain method

$n = 1$ ↑ unknown $\quad \phi_1(x) = x$



$$\phi(x) = \phi_p(x) + a_1 \phi_1(x)$$

satisfies essential BC

$\phi_1(0) = 0$

ϕ's satisfy homogeneous essential BC

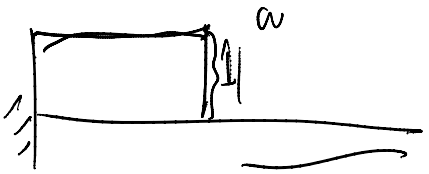
$1 - x$ ✓

$1 - x^2$ ✓

1 use this

$\phi_p(0) = 1$

weight function



we choose $w_1(x) = 1$

$w_1(1) = 1 \neq 0$

for essential BC

So it does not satisfy the homogeneous essential BC

is this a problem? It's fine because we are using NRS not the weak statement

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$\phi_p = 1 \qquad \phi_1(x) = x \qquad u = \phi_p(x) + a_1 \phi_1(x) = 1 + a_1 x$

$w_1 = 1$ for $0 \le x \le 2$

Last time we derived equations for Kij and Fi

$$\sum_{j=1}^{n} \left[\int_0^2 w_i \phi_j'' \, dx - w_i(2)\phi_j'(2) \right] a_j = \int_0^2 w_i \left(-\phi_p''(x) - q(x) \right) dx + w_i(2)\left(-1 + \phi_p'(2) \right)$$

K_{ij} on the left; F_i on the right

Plug in the values of $\phi_1 = x$, $\phi_p = 1$ & $w_1 = 1$ in this expression

$$K = \int_0^2 [\chi_{[0\ 2]}].\{\frac{d^2}{dx^2}[x]\}^T dx - \{[\chi_{[0\ 2]}].\left(\frac{d[x]}{dx}\right)^T\}|_{x=2}$$
$$= \int_0^2 [1].[0]^T dx - \{[1].[1]^T\}|_{x=2} = 0 - 1 = -1$$
$$F = -\int_0^1 [\chi_{[0\ 2]}].(2-2x)\, dx - [\chi_{[0\ 2]}]_{x=2}$$
$$= -\int_0^1 [1].(2-2x)\, dx - [\chi_1]_{x=2} = -2 + 1 - 1 = -2$$

$K a_1 = F \qquad (-1) a_1 = (-2) \Rightarrow a_1 = 2$

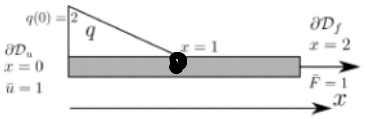
$u^h(x) = \phi_p(x) + a_1 \phi_1(x) = 1 + 2x$

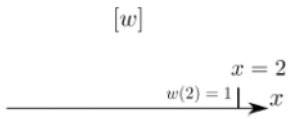
$u^{h'}(x) = 2 \Rightarrow \vec{F}(2) = EA\, u'(2) = 2 \neq 1$

$u^h(x) = 2 \Rightarrow \vec{F}(2) = EA\, u'(2) = 2 \neq 1$

force vector

Bar example, $n = 1$, Collocation method





We need to choose 1 equation which is the PDE inside a bar at one point OR the natural BC at the end of the bar (x = 2)

$u^h = \phi_0(x) + a_1\phi_1(x) = 1 + a_1 x$

($\phi_0(x) = 1$, $\phi_1(x) = x$, a_1 unknown)

1 equation How about satisfying the PDE at $x = 1$ (middle of the bar)

PDE: $(AE u^{h\prime})' + q(x) = 0$ ($AE = 1$)

$u^{h\prime\prime} + q(x) = 0$ @ $x = 1$

$u^h = 1 + a_1 x \Rightarrow u^{h\prime\prime}(1) = 0$

$q(1) > 0$

$u^{h\prime\prime}(1) + q(1) =$

$0 + 0 = 0$

Not a good choice

How about if we satisfy the natural BC

$R_1 = F - F_1 = (1 - AE\, du)_1 - 1 - u'(2)$

$$R_f = \bar{F} - F\big|_{x=2} = \left(1 - \underbrace{AE}_{1}\frac{du}{dx}\right)\Big|_{x=2} = 1 - u'(2)$$

$$u^h = 1 + a_1 x \Rightarrow u' = a_1 \Rightarrow$$

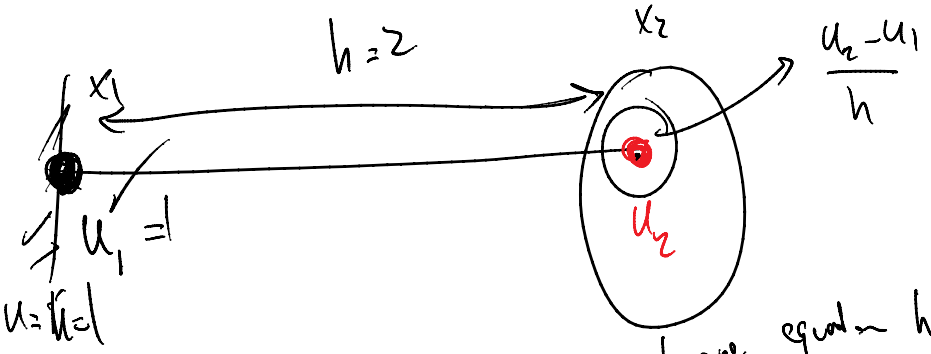
$$R_f(x=2) = 1 - a_1 = 0 \Rightarrow a_1 = 1$$

Collocation Method choosing $x=2$ & Natural BC

$$u_c^h(x) = 1 + x$$

FD similar to Collocation we satisfy equations

@ n points



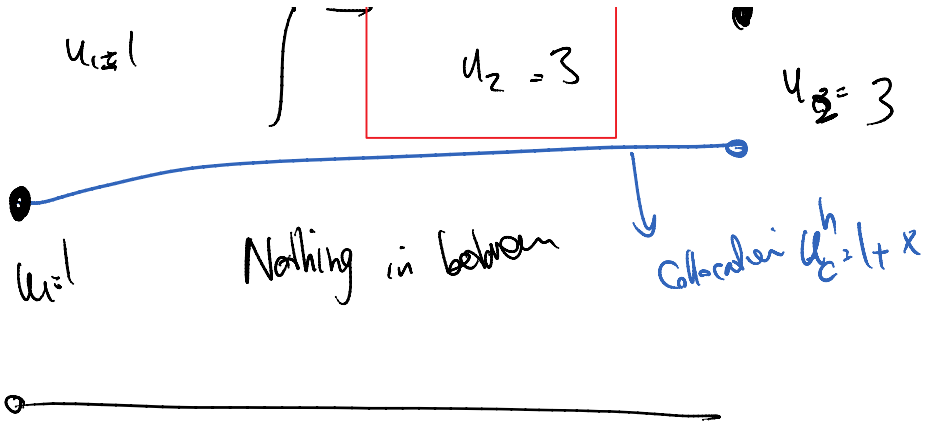
need one equation here

not a good option ← ✗ PDE $u'' + q = 0$ @ x_2 ⇒ ✗ $\frac{u_3 - 2u_2 + u_1}{h^2} + q(2) = 0$

or Natural BC $\bar{F} - F$: $1 - u' = 0$ $1 - \frac{u_2 - u_1}{2} = 0$

$$1 - \frac{u_2 - u_1}{2} = 0 \quad u_1 = 1 \Rightarrow \boxed{u_1 = 1 \quad u_2 = 3}$$

$u_2 = 2$



they both satisfied the natural BC
but FD has point values & derivs are evaluated approximately

Collocation $\underbrace{u = 1 + x}_{\text{a function rather than point value}} \Rightarrow F = 1 = \bar{F}$ satisfied exactly

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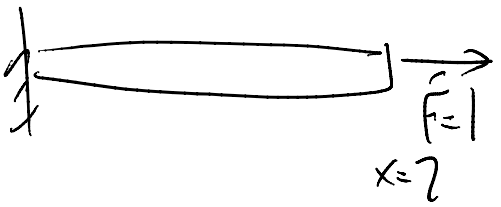
Least square

$$u^h = \underbrace{\phi_p(x)}_{1} + a_1 \underbrace{\phi_1(x)}_{x} = 1 + a_1 x \Rightarrow \begin{cases} u^{h'} = a_1 \\ u^{h''} = 0 \end{cases}$$

$$R_I = (\underbrace{EA}_{1} u^{h'})' + q(x) = u^{h''} + q(x) = q(x)$$

$$R_f = (\bar{F} - F)\Big|_{x=2} =$$

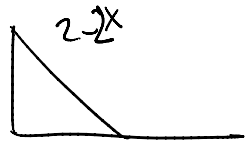


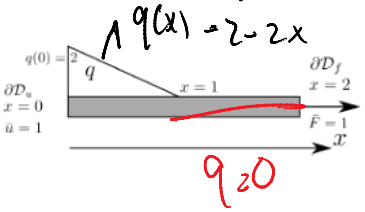
$$1 - \underbrace{AE}_{1} u^{h'}(2) = 1 - a_1$$

$$R^2 = \int^2 R_I^2 \, dx + R_f^2 =$$

$$R^2 = \int_0^{\cdot} R_I^2 \, dx + R_f^2 =$$

$$\int_0^{1} q^2(x)\, dx + (1-a_1)^2 =$$





$$R^2 = \int_0^1 (2-2x)^2 \, dx + (1-a_1)^2$$

$$\boxed{R^2 = \frac{7}{3} - 2a_1 + a_1^2}$$

To have the least $\underbrace{\text{square}}_{R^2}$

$$\frac{\partial R^2}{\partial a_1} = 0 \Rightarrow 2a_1 - 2 = 0 \rightarrow a_1 = 1$$

$$\boxed{u_L^h(x) = 1 + x}$$

Remember the least method can be expressed as a weighted residual method with the following weight function choices

Least Square Method

- Equations (156) and (157) yield,

$$\forall \phi_i \in \mathcal{V}^h : \int_{\mathcal{D}} L_M(\phi_i) \underbrace{(L_M(\mathbf{u}^h) - \mathbf{r})}_{\mathcal{R}_i} \, \mathrm{dv} + \int_{\partial \mathcal{D}_f} (-L_f(\phi_i)) \underbrace{(\bar{\mathbf{f}} - L_f(\tilde{\mathbf{u}}^h))}_{\mathcal{R}_f} \, \mathrm{ds} = 0 \tag{158}$$

- In comparison to (126) for the general statement of weighted residual methods we observe,

$$\forall \mathbf{w} \in \mathcal{W}^h : \int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i(\mathbf{u}^h) \, \mathrm{dv} + \int_{\partial \mathcal{D}_f} \mathbf{w}^f.\mathcal{R}_f(\mathbf{u}^h) \, \mathrm{ds} = 0$$

- Discrete Least Square problem for linear differential operators L_M and L_f is equivalent to a discrete weighted residual statement with the weight functions:

Weight functions corresponding to Least Square Method

$$\mathbf{w} = L_M(\phi) \tag{159a}$$
$$\mathbf{w}^f = (-L_f(\phi)) \tag{159b}$$

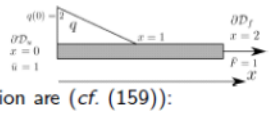
$w_1 = \frac{d^2}{dx^2}\phi_1$

$-AE\frac{d}{dx} \Rightarrow -\frac{d}{dx}$ $\quad w_1^f = -\frac{d\phi_1}{dx}$

on the natural Boundary

Bar example, $n = 1$, Least Square method

- For $n = 1$ our trial function is $[\phi] = \{x\}$ (*cf.* (196a)).



  - For least square method, the corresponding weight function are (*cf.* (159)):

$$\mathbf{w} = L_M(\phi) \quad \Rightarrow \quad \mathbf{w}_1 = \frac{\mathrm{d}^2\phi_1}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 x}{\mathrm{d}x^2} = 0$$
$$\mathbf{w}^f = (-L_f(\phi)) \quad \Rightarrow \quad \mathbf{w}_1^f = -\frac{\mathrm{d}\phi_1}{\mathrm{d}x} = -\frac{\mathrm{d}x}{\mathrm{d}x} = -1$$

$\phi_1(x) = x$

So for LS method, $n = 1$ weight functions are $[w] = 0, [w^f] = -1$.

- We plug $[\phi] = \{x\}$ into (185) to obtain:

$$\mathbf{K} = \int_0^2 \overbrace{[\,0\,]}^{w_1} \{\frac{\mathrm{d}^2}{\mathrm{d}x^2}[x]\}^{\mathrm{T}} \, \mathrm{d}x - \{\overbrace{[-1]}^{w_1^f}.\left[\frac{\mathrm{d}[x]}{\mathrm{d}x}\right]^{\mathrm{T}}\}|_{x=2} = -\{[-1].[1]^{\mathrm{T}}\}|_{x=2} = -(-1) = 1$$

$\mathbf{K}_E$ $\quad \mathbf{R}_\beta$

$$\mathbf{F} = -\int_0^1 \overbrace{[\,0\,]}^{w_1}.(2 - 2x) \, \mathrm{d}x - \overbrace{[-1]}^{w_1^f}_{x=2} = 0 - [-1]_{x=2} = -(-1) = 1 \tag{203}$$

- From $\mathbf{K}\mathbf{a} = \mathbf{F}$ (125) and $u^h = a_j\phi_j + \phi_p$ (117a), and $\phi_p = 1$ (182):

$$\boxed{u_{L1}^h = 1 + x} \tag{204}$$

Galerkin:

$w_i = \phi_i$

$u^h = \phi_p + a_1\phi_1(x)$

$= 1 + a_1 x$

$w_i = \phi_i$ → ϕ_i's by construction satisfy homogeneous essential BC

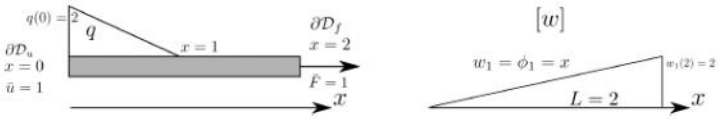
⟹ they can be used as weight functions in weak statement

In Galerkin method where $w_i = \phi_i$.

we can choose both

& weak statement → easier we
weighted residual

Bar example, $n = 1$, Galerkin method



- For $n = 1$ our trial function is $[\phi] = \{x\}$ (*cf.* (196a)).
- For Galerkin method $n = 1$ weight function is $[w] = [w^f] = [\phi]$ (shown in figure).
- We plug $[\phi] = \{x\}$ and $[w] = \{x\}$ into (185) to obtain:

$$\mathbf{K} = \int_0^2 [x].\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}[x]\}^{\mathrm{T}}\,\mathrm{d}x - \{[x].\left(\frac{\mathrm{d}[x]}{\mathrm{d}x}\right)^{\mathrm{T}}\}|_{x=2}$$

$$= \int_0^2 [x].\,[0]^{\mathrm{T}}\,\mathrm{d}x - \{[x].\,[1]^{\mathrm{T}}\}|_{x=2} = 0 - 2 = -2 \qquad (205a)$$

$$\mathbf{F} = -\int_0^1 [x].(2-2x)\,\mathrm{d}x - [x]_{x=2} = -1 + \frac{2}{3} - 2 = -\frac{7}{3} \qquad (205b)$$

- From $\mathbf{K}\mathbf{a} = \mathbf{F}$ (125) and $u^h = a_j\phi_j + \phi_p$ (117a), and $\phi_p = 1$ (182):

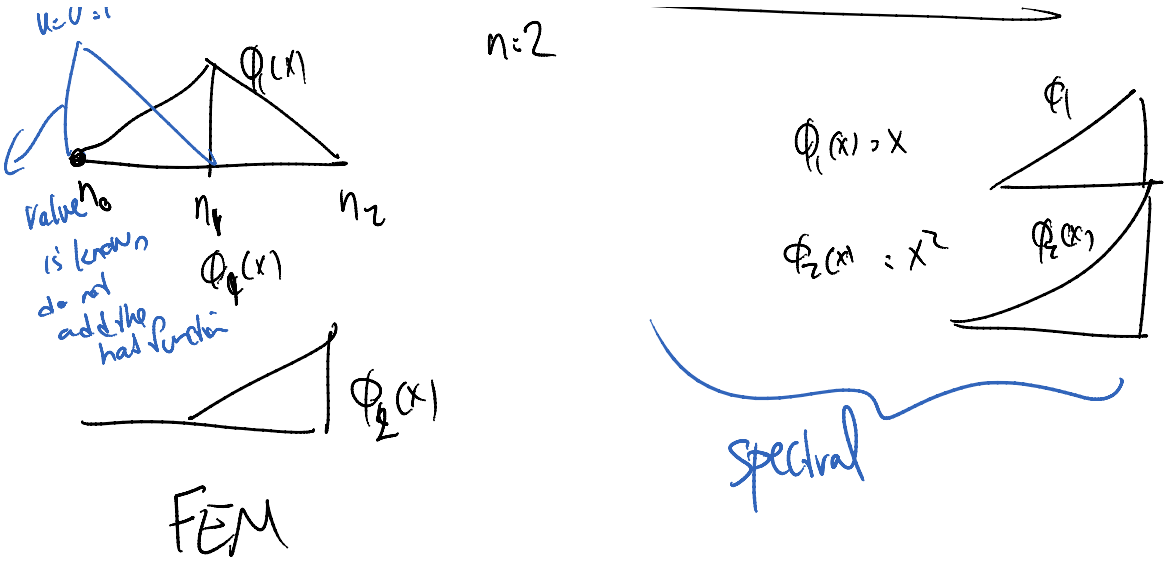
$$\boxed{u^h_{G1} = 1 + \frac{7}{6}x} \qquad (206)$$

$n=1$

$\phi_1(x) = x$

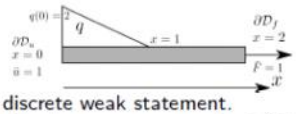
$n=2$

ϕ_1



Bar example, $n = 1$, Ritz (energy) Method

- As discussed before, the application of stationary condition on energy functional at continuum level directly yields the weak statement.
  - Subsequent discretization on this weak statement is a discrete weak statement.
  - The Galerkin solution of this weak system (*e.g.*, shown for $\phi_1 = w_1 = x$ previously) is equivalent to Ritz method:
  - In Ritz method, we first discretize then apply the stationary condition.
  - For $u^h = a_1\phi_1 + \phi_p = a_1 x + 1$ ($\phi_1 = x$) from (194) we get,



$$\begin{aligned}\Pi(a_1) &= \int_0^2 \frac{1}{2}\left(\frac{\mathrm{d}\{a_1x+1\}}{\mathrm{d}x}\right)^2 \mathrm{d}x - \int_0^1 \{a_1x+1\}(2-2x)\,\mathrm{d}x - \{a_1x+1\}(2) \\ &= \int_0^2 \frac{1}{2}{a_1}^2\,\mathrm{d}x - a_1\int_0^1 (2x-2x^2)\,\mathrm{d}x - \int_0^1 (2-2x)\,\mathrm{d}x - (2a_1+1) \\ &= {a_1}^2 - a_1\left(1-\frac{2}{3}\right) - (2-1) - (2a_1+1) = {a_1}^2 - \frac{7}{3}a_1 - 2\end{aligned}$$

$u_h = 1 + a_1 x$

$\Pi(a_1) = \int_0^2 \frac{1}{2} EA\,(u^h)^2\,dx - \left(\int u^h(x) q(x)\,dx + u^h(2)\bar{F} \right)$

U

W

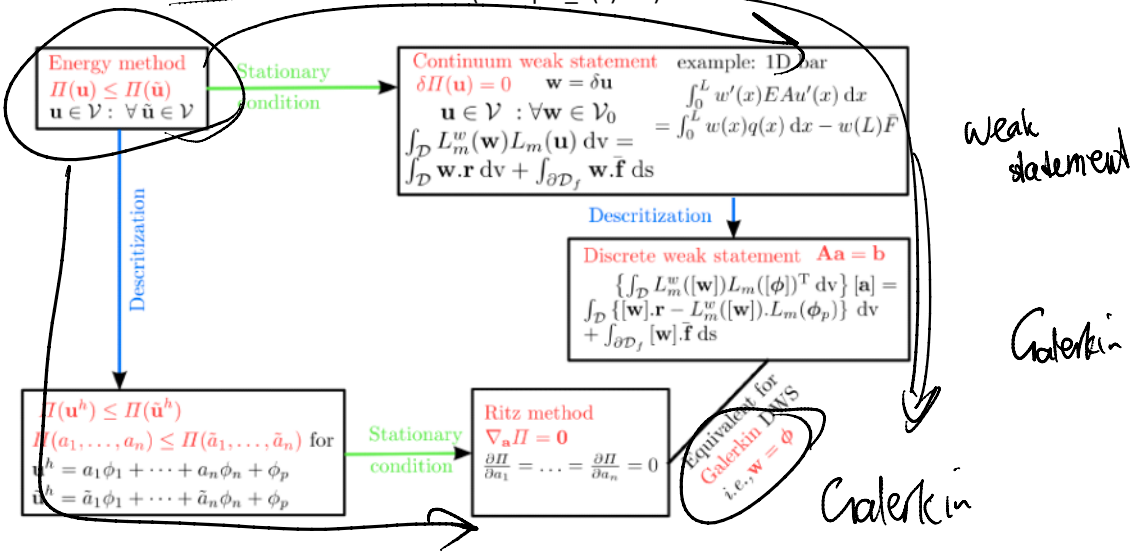
$\Pi(a_1) = a_1^2 - \frac{7}{3}a_1 - 2$

Minimize Π to get the solution

$$\frac{\partial \Pi}{\partial a_1} = 2a_1 - \frac{7}{3} = 0 \quad \Rightarrow \quad a_1 = \frac{7}{6}$$

$$u^h_{Ritz}(x) = 1 + \frac{7}{6}x$$

Gives the same solution as Galerkin method (with phi_1(x) = x)



weak statement

Galerkin

Galerkin

Why?

Ritz