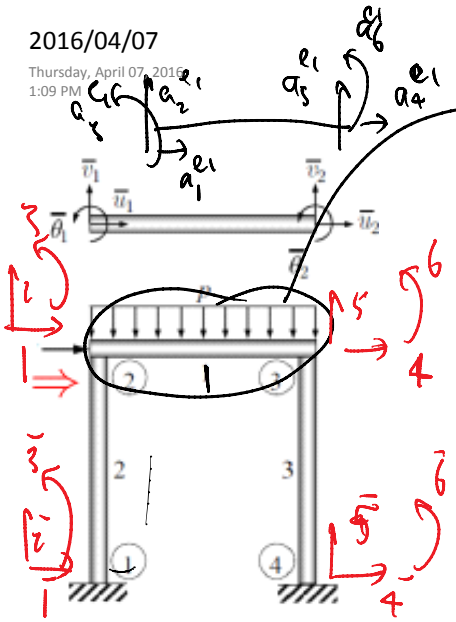
2016/04/07

Thursday, April 07, 2016
1:09 PM



LEM = [2,3]

dofMap = [1 2 3 4 5 6]

loop over nodes

e node 1 → global node 2 (LEM[1]=2)

[1 2 3]

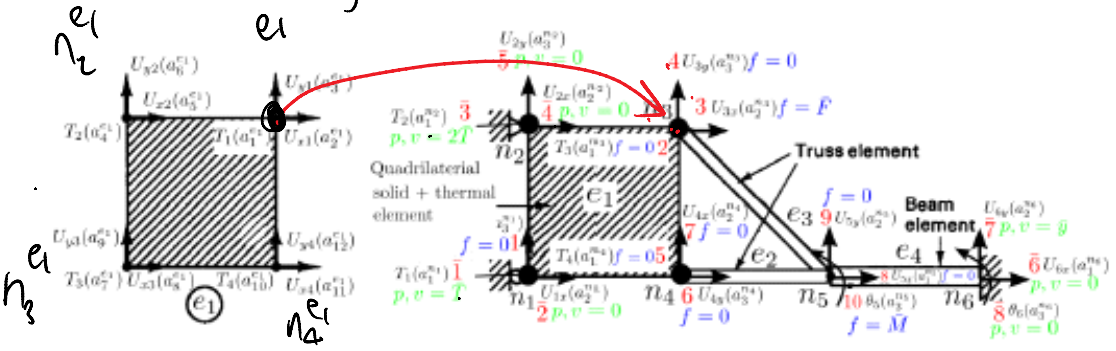
node 2 element → global node 3 (LEM[2]=3)

pos for global node 3

[4 5 6]

[1 2 3 | 4 5 6]

What is missing at element level



dofMap = [2 3 4 3̄ 4̄ 5̄ 1̄ 2̄ 1 5 6 7]

edofs = [$a_1^{e_1}$ - - - $a_{12}^{e_1}$]

a^e = [$T_1^{e_1}$ $U_{x_1}^{e_1}$ $U_{y_1}^{e_1}$ | $T_2^{e_1}$ $U_{x_2}^{e_1}$ $U_{y_2}^{e_1}$ | - - | $T_4^{e_1}$ $U_{x_4}^{e_1}$ $U_{y_4}^{e_1}$]

$$= \left[T_1^{e_1} \; U_{x_1}^{e_1} \; U_{y_1}^{e_1} \,\middle|\, T_2^{e_1} \; U_{x_2}^{e_1} \; U_{y_2}^{e_1} \,\middle|\, \cdots \,\middle|\, T_4^{e_1} \; U_{x_4}^{e_1} \; U_{y_4}^{e_1} \right]$$

why needed

because $f_D = K^e a^e$

$$\left[\underbrace{0 \; 0 \; 0}_{\text{free}} \,\middle|\, 2T \;\; 0 \;\; 0 \,\middle|\, \bar{T} \;\; 0 \;\; 0 \,\middle|\, 0 \; 0 \; 0 \right]$$

still unknown

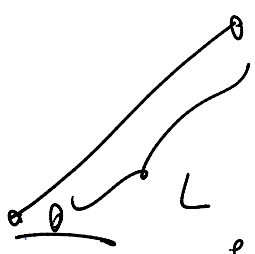
free

Step 10: Compute element stiffness/force

Element has three different specifiers:

1. Element type: *e.g.*, truss or bar element; 1st, 2nd order, *etc.*.
2. Geometry: Geometry of element (*e.g.*, L^e for trusses).
3. Material properties: Properties needed such as A and E for trusses.

stiff for a truss element



$c = \cos\theta$

$s = \sin\theta$

$$k^e = \frac{AE}{L} \left[\begin{array}{c|c} k_b & -k_b \\ \hline -k_b & k_b \end{array} \right]$$

$$k_b = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

Geometry properties of the element

We access the nodal coordinates by LEM map (takes us from local to global node ids)

- In global system we store nodal coordinates

$$\mathrm{LEM}(e_3) = [3\ 5] \Rightarrow \left.\begin{array}{ll} \mathbf{X}_1 & = [x_3\ y_3] \\ \mathbf{X}_2 & = [x_5\ y_5] \end{array}\right\} \Rightarrow \left.\begin{array}{ll} L_x & = x_5 - x_3 \\ L_y & = y_5 - y_3 \end{array}\right\} \Rightarrow$$

$$\boxed{\begin{array}{ll} L^{e_3} & = \sqrt{L_x^2 + L_y^2} \\ c^{e_3} & = \frac{L_x}{L^{e_3}} \\ s^{e_3} & = \frac{L_y}{L^{e_3}} \end{array}}$$

Same concept is used for element material / geometric properties

Step 10: Compute element stiffness/force: Forces

$$f_e = f_r + f_N - f_D$$

f_r → Source term

f_N → natural Neumann boundary

Element specific

Virtual

f_D → Dirichlet Essential BCs

Always the same formula

$f_D = k^e a^e$

$$f_e = f_o - f_D$$

$f_D = k^e a^e$

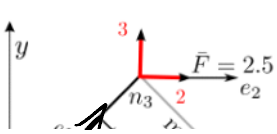
user specific

virtual

$$\mathbf{f}_e^e = -\mathbf{f}_D^e + \mathbf{f}_o^e = \text{Total element force}$$

- The calculation of $\mathbf{f}_o^e$ is specific to a given element (and its physics) and should have its own particular implementation.
- The calculation of $\mathbf{f}_D^e$ and $\mathbf{f}_e^e$ (summation of two forces) is generic on the other hand.
- This is the motivation for the definition of these functions for element class on slide 394:
  1. Calculate $\mathbf{f}_o^e$: sum of all forces, but $\mathbf{f}_D^e$ (virtual)
  2. Calculate $\mathbf{f}_D^e = \mathbf{k}^e \mathbf{a}^e$
  3. Calculate $\mathbf{f}_e^e = -\mathbf{f}_D^e + \mathbf{f}_o^e$

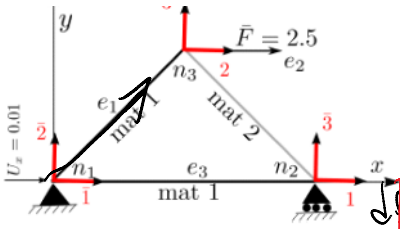
How is the assembly done?



1 2 3

$K = 2$

$f = 2$



$$K = \begin{array}{c} \\ 2 \\ 3 \end{array}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \qquad f = \begin{array}{c} \\ 2 \\ 3 \end{array}\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

free columns get stiffness matrix

e_1

$$k^{e_1} = \frac{AE}{L}\begin{array}{c|cccc} & \bar{1} & \bar{2} & 2 & 3 \\ \hline \bar{1} & c^2 & cs & c^2 & cs \\ \bar{2} & cs & s^2 & cs & s^2 \\ 2 & c^2 & cs & c^2 & cs \\ 3 & cs & s^2 & cs & s^2 \end{array}$$

only use free rows & free columns

$$f_e = \begin{array}{c} \bar{1} \\ \bar{2} \\ 2 \\ 3 \end{array}\begin{bmatrix} f_{e_1} \\ f_{e_2} \\ f_{e_3} \\ f_{e_4} \end{bmatrix} = -\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} \bar{1} & \bar{2} & 2 & 3 \end{array} \\ \begin{bmatrix} c^2 & cs & c^2 & cs \\ cs & s^2 & cs & s^2 \\ c^2 & cs & c^2 & cs \\ cs & s^2 & cs & s^2 \end{bmatrix} \end{array} \begin{bmatrix} a_1^{e_1} \\ a_2^{e_1} \\ a_3^{e_1} \\ a_4^{e_1} \end{bmatrix} + \begin{bmatrix} f_{o_1} \\ f_{o_2} \\ f_{o_3} \\ f_{o_4} \end{bmatrix}$$

prescribed known

unknown

use prescribed columns

$$\underbrace{-k^a e^a}_{= -f_D}$$

not using free column

prescribed columns contribute to f_D

Only consider free rows I of K

$I = \text{dofMap}(i)$
$J = \text{dofMap}(j)$

free columns $\longrightarrow$ $K_{IJ} \mathrel{+}= K_{ij}$ free

prescribed ones $\quad F_I \mathrel{-}= K_{ij} a_j$

prescribed

local dofs

global dofs

```
for e = 1:ne loop over elements
    fee = feo element total force = element all forces except essential force
    for i = 1:nedof loop over rows of ke; nedof = element # dof
        I = dofMap(i) local to global dof map M_i^e
        if (I > 0) I corresponds to a free dof, we skip prescribed dofs
```

continue with free rows

```
fee += feo element total force = element all forces except essential force
for i = 1:nedof loop over rows of ke; nedof = element # dof
    I = dofMap(i) local to global dof map M_i^e
    if (I > 0) I corresponds to a free dof, we skip prescribed dofs
      for j = 1:nedof loop over columns of ke
        J = dofMap(j) global dof corresponding to j
        if (J > 0) now both I and J are free and can add ke(i,j) to global K
          K(I, J) = K(I, J) + ke(i, j)
        else J < 0, prescribed dof j; add contributions of f_D^e = k^e a^e to f_e^e
          fee(i) = fee(i) - ke(i, j) * edofs(j) edofs: element dofs = a^e
        end
      end
      F(I) = F(I) + fee(i)element's total force fee component i'th is computed→added to F(I)
    end
  end
end
```

global dof

continue with free rows

f_e^e k^e a^e

Step 1-11 is assembly

now we have K, F

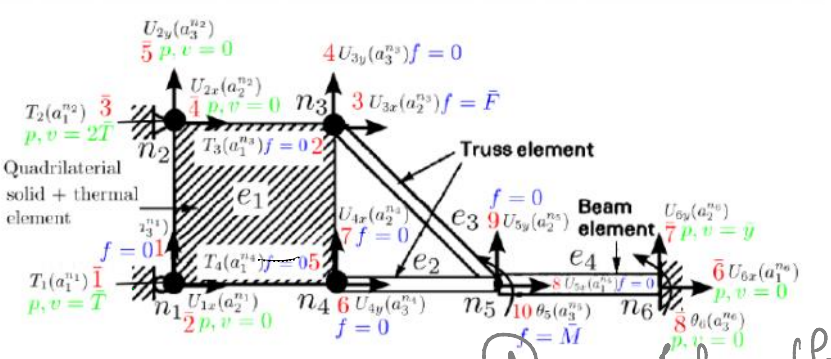
$$Ka = F$$

global free dofs

free dofs $\leftarrow a = K^{-1}F$

we don't invert it

$$K = LU$$



node samples

values of free dofs

forces of prescribed dofs

	value	force	prescribed / free	pos
n_1 dof 1	$\bar{T}$	ρ	true	$\bar{T}$

Nodal dof

				free	
n_1 dof 1	$\bar{T}$	?	true	1
dof 2	0	?	true	2
dof 3	?	0	false	1
n_4 dof 1	?	0	false	5
dof 2	?	0	false	6
dof 3	?	0	false	7

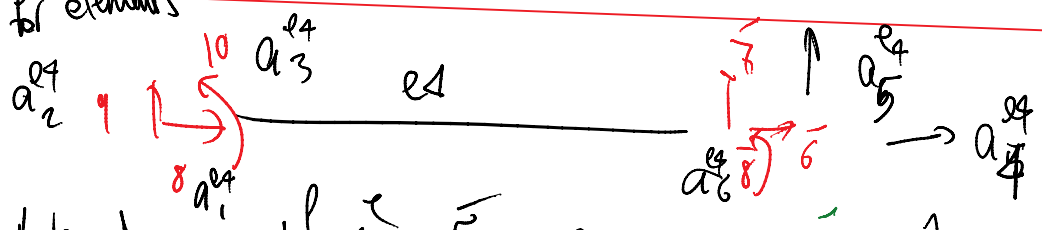
Nodal dof

(n1) unknowns

1. forces for prescribed dofs "Reaction forces"

(n2) 2. Values of free dof

for elements



(e1)

Element level update free dof values

At the element level we need to set free dof

dof $= a^e = [? \; ? \; ? \mid 0 \; \bar{y} \; 0]$

We need element forces e.g. if it's a truss

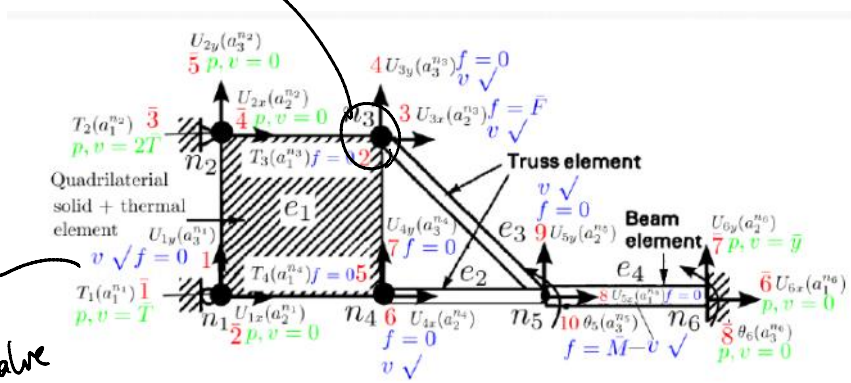
$\bar{T}$

e2 element forces

Step 13: Assign a to nodes and elements: a) nodes

(n2) (e1)

value is n2



value for this is a_2

global free dof

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_8 \end{bmatrix}$$

```
for n = 1:nNodes
  for dofi = 1: node(n).nndof num dof for node (n)
    if node(n).ndof(dofi).p == false free dof
      posn = node(n).ndof(dofi).pos position of dof in global free F
      node(n).ndof(dofi).v = dofs(posn) set free dof val to corresponding val in global dofs (a)
    end
  end
end
```

loop over nodes

loop over dofs of nodes

is the dof is free

get the position

let dof.value = a(pos)

loop over nodes

node1 . . . node2 . .

for node 1

loop over dofs

if free

free

node(1).dof(3).p=false

pos: node(1).dof(3).pos

=1

		value	force	prescribed /free	pos
n_1 dof 1		T	?	skip true	1
dof 2		0	?	skip true	2
dof 3		?	0	false	1
n_4 dof 1		a_5 ?	0	false	5
dof 2		?	0	false	6
dof 3		?	0	false	7

$a = \begin{bmatrix} a_1 \\ \vdots \\ a_5 \\ a_8 \end{bmatrix}$

node(1).dof(3).value = a(pos)

example node 4

dof(1)

node(4).dof(1).p = false

pos = node(4).dof(1).pos =5

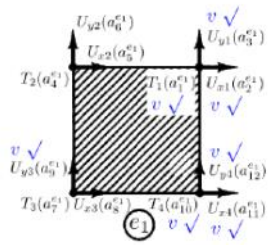
node(4).dof(1).value = a(pos) = a_5

After this step we have All nodes dof's values set (free & prescribed dofs)

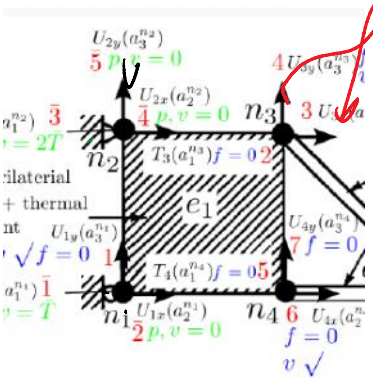
	value	force	prescribe /free	pos
n				

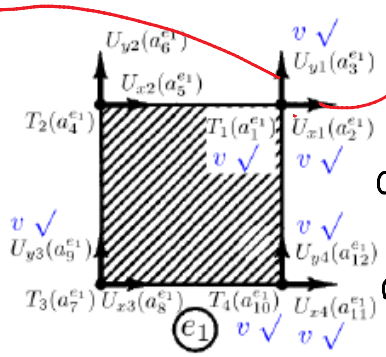
		value	force	prescribed / free	pos
n_1	dof 1	$\bar{T}$	?	true	$\bar{1}$
	dof 2	0	?	true	$\bar{2}$
	dof 3	a_1	0	false	1
n_4	dof 1	a_5	0	false	5
	dof 2	a_6	0	false	6
	dof 3	a_7	0	false	7

Step 13: Assign a to nodes and elements: b) elements




```
for e = 1:ne loop over elements
  for i = element(e).nedof loop over element dofs; nedof = # dof (n^e_dof)
    posn = element(e).dofMap(i) corresponding global position using dofMat (M^e_t)
    if (posn > 0) free dof
      element(e).edofs(i) = dofs(posn)
        set free element dof a^e to corresponding val in global dofs (a)
    end
  end
end
```





want to set those

dofMap = [2 3 4 3 $\bar{4}$ $\bar{5}$ $\bar{1}$ $\bar{2}$ 1 5 6 7]

edofs = [? ? ? | 2 $\bar{1}$ 0 0 $\bar{1}$ 0 ? ? ? ?]

a_2 a_3 a_4 a_1 a_5 a_6 a_7

LEM = [3 2 | 4]

global free dofs

dof values at the element level

$$\text{dof} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

dof's (a) ↓ a have these after the solution | a_3 a_4 a_5 a_6 a_7 a_8 |

$$edofs = [a_2\ a_3\ a_4 \mid 2T\ 0\ 0\ \bar{T}\ 0\ a_1\ a_5\ a_6\ a_7]$$

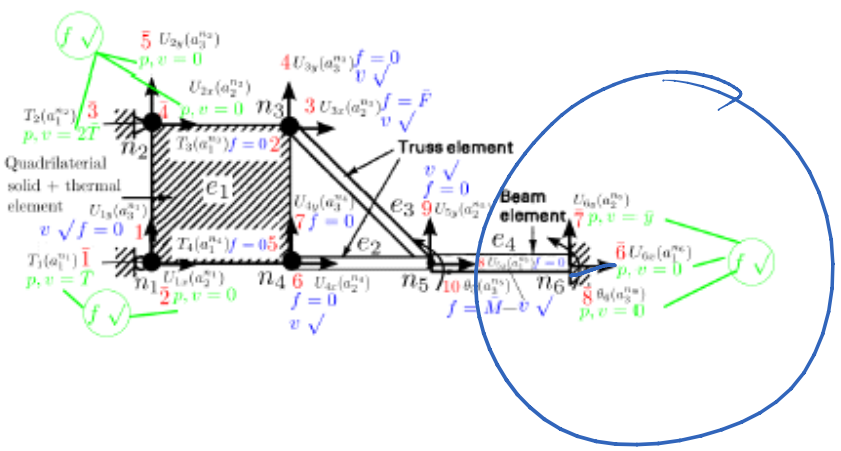
a^e

$$dofMap: [2\ 3\ 4 \mid \bar{3}\ \bar{4}\ \bar{5}\ \bar{1}\ \bar{2}\ 1\ 5\ 6\ 7]$$

At this stage we have all nodes dof values set ✓

All element dof values set ✓

Step 14: Compute prescribed dof forces



nodal unknowns.

group 2

what are the reaction forces

$f_{\bar{6}} = ?$ $f_{\bar{7}} = ?$ $f_{\bar{8}} = ?$

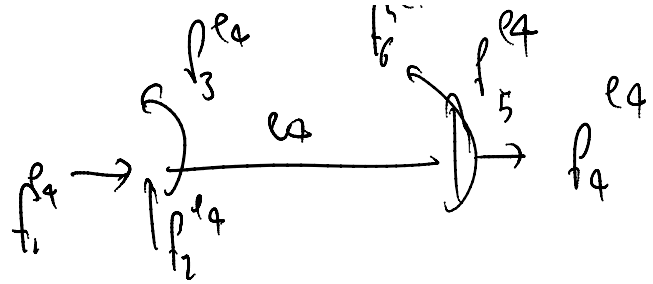
Another thing that we want to obtain

what forces are applied at element nodes

$f_3^{e_4}$ $f_6^{e_4}$ $f_7^{e_4}$ e_4

Let's do this

do this first



The element nodal forces are

$$-f_e^e = -\left(f_0^e - f_D^e\right)$$

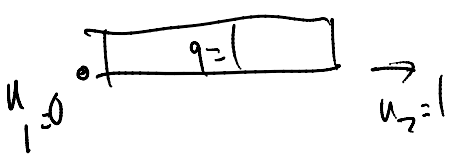
$$= -f_0^e + f_D^e$$

$$f_D^e = k^e a^e$$

other forces

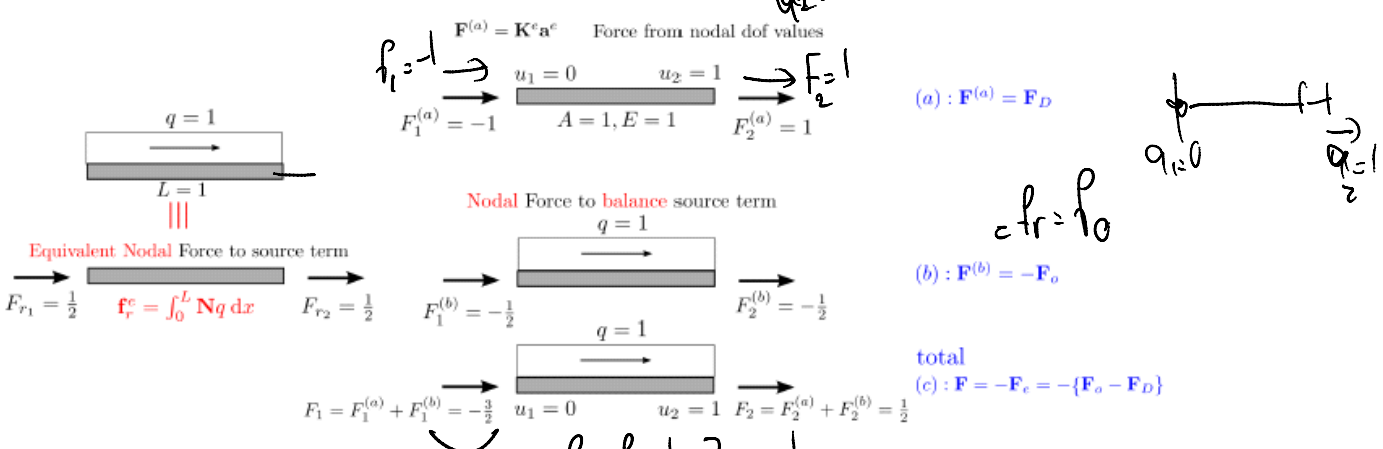
$$f_0^e = f_r^e + f_n^e \quad (\neq 0)$$

to demonstrate this consider the case that $f_0^e = f_r^e$ we only have surce term



$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \underbrace{\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{k^e}\underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{a^e} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$q_1=0$ $q_2=1$

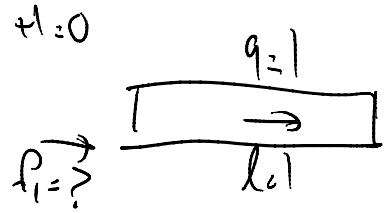


$$f_1 + f_2 + 1 = -\frac{3}{2} + \frac{1}{2} + 1 = 0$$

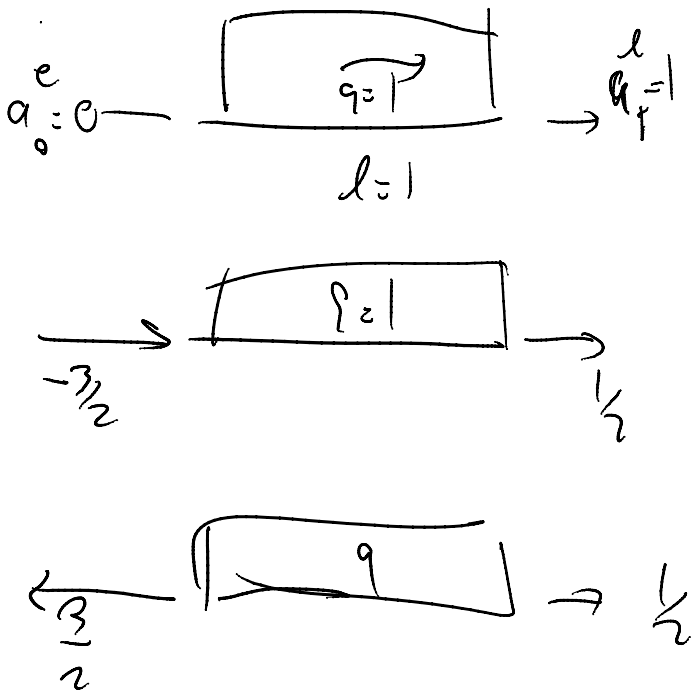
$q_1=0$ $q_2=1$

$= f_r = f_0$

what are element nodal forces



$$f_1 + f_2 + q \cdot l = 0$$



$$f_1 + f_2 + \underbrace{q \cdot l}_{1 \times 1} = 0$$

$$f_1 + f_2 + 1 = 0$$

This demonstrates that element nodal forces are

nodal forces $\quad f^e = k^e a^e - f_0^e = f_D^e - f_0^e = -\underbrace{(f_0^e - f_D^e)}_{f_e^e}$

$$f_e^e = \underbrace{f_r^e + f_N^e}_{f_0^e} - f_D^e$$

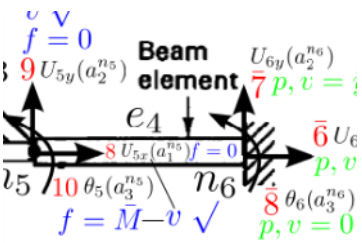
is what is assembled to global for a vector

$$-f_e^e = -f_0^e + \overset{k^e a^e}{f_D^e}$$

are the forces applied on element dofs

```
for e = 1:ne loop over elements
   fee = feo element total force = element all forces except essential force
   for i = 1:nedof loop over rows of ke; nedof = element # dof
      I = dofMap(i) local to global dof map M_t^e
      if (I < 0) I corresponds to a prescribed dof, we skip free dofs
         for j = 1:nedof loop over columns of ke. ALL columns (dofs) of p and f used
            fee(i) = fee(i) - ke(i, j) * edofs(j) edofs: element dofs = a^e
         end
         Fp(-I) = Fp(-I) - fee(i)
            1. element's total force fee component i'th is computed→added to Fp(-I)
            2. -I used because I < 0: prescribed dof
            3. fee is subtracted
      end
   end
end
```

$$f_e^e = f_o^e - f_D^e$$
$$= f_o^e - k^{ee} a^e$$
$$f_e^e = f_o^e - k^{ee} a^e$$



$i = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

dofMap = [8 9 10 $\bar{6}$ $\bar{7}$ $\bar{8}$]

free free free — skip; prescribed

$$f_e^e = f_o^e - k^{ee} a^e$$

i | I | 8 9 10 $\bar{6}$ $\bar{7}$ $\bar{8}$

$$k^e = \begin{array}{c|c} i & I \\ 1 & 8 \\ 2 & 9 \\ 3 & 10 \\ 4 & \bar{6} \\ 5 & \bar{7} \\ 6 & \bar{8} \end{array} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{bmatrix} a_8 \\ a_9 \\ a_{10} \\ 0 \\ \bar{y} \\ 0 \end{bmatrix}$$

skip

$$a^e = \begin{bmatrix} a_8 \\ a_9 \\ a_{10} \\ 0 \\ \bar{y} \\ 0 \end{bmatrix} ✓$$

$i = 4$ $\quad f_{ee}(4) = [k_{41} \; k_{42} \; k_{43} \; k_{44} \; k_{45} \; k_{46}] \cdot [a_8 \; a_9 \; a_{10} \; 0 \; \bar{y} \; 0]$

$i=4$ $\quad f_{ee}^{e}(4) = [k_{41}\ k_{42}\ k_{43}\ k_{44}\ k_{45}\ k_{46}] \cdot [a_8\ a_9\ a_{10}\ 0\ y\ 0]$

$I = 6$ $\quad f_e = f_e^{e} - K^{e} a$

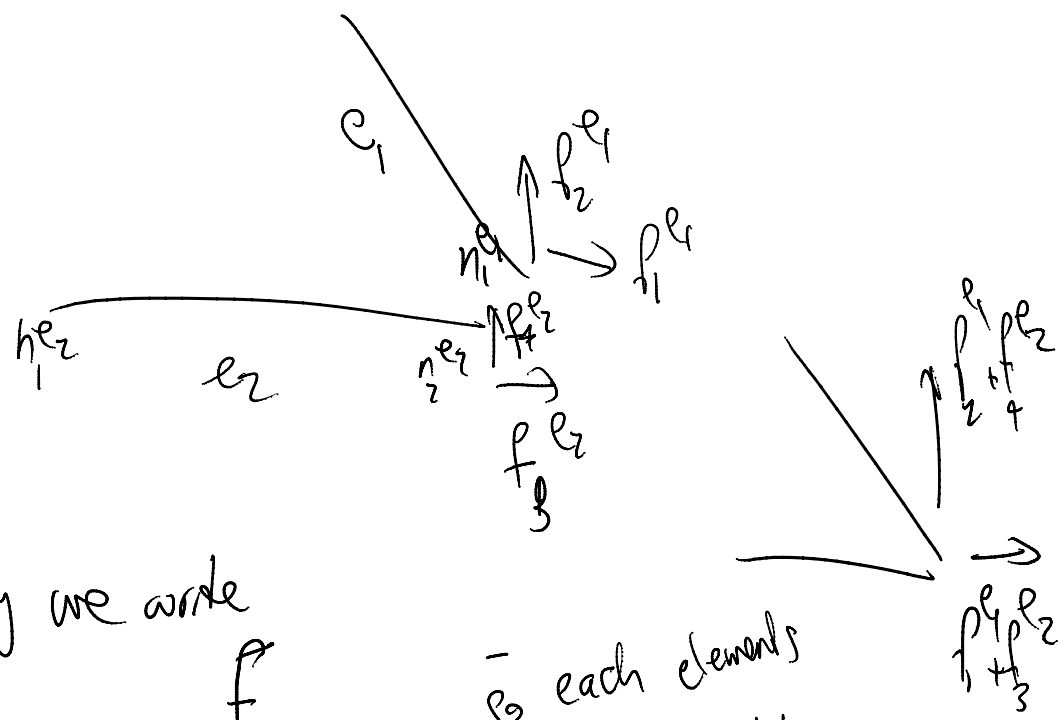
$x = -y \qquad x = x - y$

shorthand

size npdof = 10 vector of reaction forces

F_p

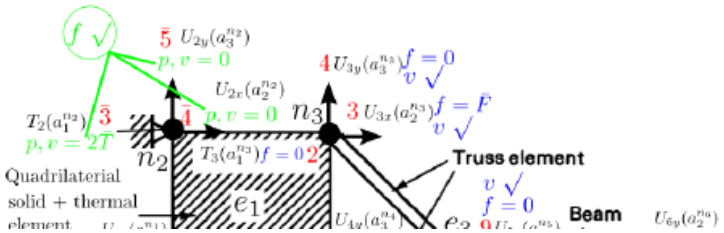
$$F_p(-\bar{6}) = F_p(6) + = -f_e^{e}$$



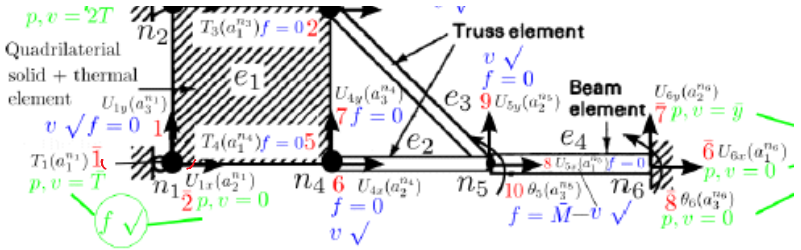
that's why we write f

So each elements adds its contribution to the reaction force

$$F_p(-I) = F_p(-I) - fee(i)$$



After this stage



After this stage we will have

$$F_p = \begin{bmatrix} f_{\bar{1}} \\ f_{\bar{2}} \\ \vdots \\ f_{\bar{8}} \end{bmatrix} \rightarrow \text{known at this moment}$$

The only remaining step is setting nodal prescribed dof forces which are the reaction forces we just computed

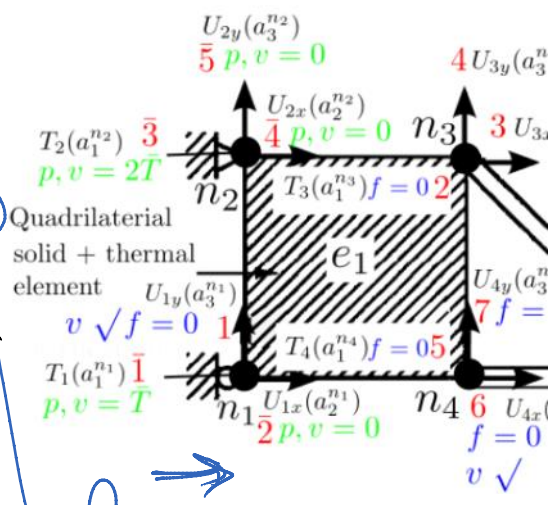
only remaining things to set

		value	force	prescribe /free	pos
n_1 dof 1		$\bar{T}$	f	true	$\bar{1}$
dof 2		0	f	true	$\bar{2}$
dof 3		a_4	0	false	1
n_4 dof 1		a_5	0	false	5
dof 2		a_6	0	false	6
dof 3		a_7	0	false	7

$$F_p = \begin{bmatrix} f_{\bar{1}} \\ f_{\bar{2}} \end{bmatrix}$$

$f_{\bar{2}}$

pos = $\bar{2}$ = 2



$$F_p = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad pos = 2 = -2$$

		value	force	prescribe /free	pos
n_1	dof 1	$\bar{T}$	f_1	true	1
	dof 2	0	f_2	true	2
	dof 3	a_4	0	false	1
n_4	dof 1	a_5	0	false	5
	dof 2	a_6	0	false	6
	dof 3	a_7	0	false	7

Nodal dof table is comple

How to set up nodal prescribed dof forces, *i.e.* reaction forces

```
for n = 1:nNodes
  for dofi = 1: node(n).nndof num dof for node (n)
    if node(n).ndof(dofi).p == true prescribed dof
      posn = node(n).ndof(dofi).pos position of dof in global prescribed force F_p
      node(n).ndof(dofi).f = Fp(-posn)
        1. set prescribed dof force to corresponding force in global Fp (F_p)
        2. posn < 0; prescribed dof
    end
  end
end
```