

This HW is for your information and is related to balance laws covered in the class. You DO NOT need to return this HW assignment. If returned a maximum of 60 extra credit points will be credited to you (if all solutions are correct). If interested in having the solutions, contact the instructor.

1. Figure 1 shows a possible jump manifold  $\Gamma$  in a domain  $\mathcal{D}$ . The balance law,

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, dS + \int_{\Omega} \mathbf{r} \, dV = \mathbf{0} \quad (1)$$

holds for all  $\Omega \subset \mathcal{D}$ . The flux tensor is denoted by  $\mathbf{F}$  and  $\mathbf{r}$  is the source term. Show that a jump condition across  $\Gamma$  satisfies the condition

$$\mathbf{F}^+ \cdot \mathbf{n}^+ = -\mathbf{F}^- \cdot \mathbf{n}^- \quad \text{which is equivalent to} \quad (2a)$$

$$\llbracket \mathbf{F} \rrbracket|_{\Gamma} \cdot \mathbf{n}^+ = (\mathbf{F}^+ - \mathbf{F}^-) \cdot \mathbf{n}^+ = \mathbf{0} \quad (2b)$$

for all  $\mathbf{x}_0 \in \Gamma$ . We assume that the flux  $\mathbf{F}$  is continuous on both sides of  $\Gamma$ .<sup>1</sup> (20 points)

**Hint:** To prove this equation use the balance law for three domains:  $\Omega^+$ ,  $\Omega^-$ , and  $\Omega = \Omega^+ \cup \Omega^-$ . By subtracting the balance equations on  $\Omega^+$  and  $\Omega^-$  from that on  $\Omega$  obtain an integral equation solely on  $\Gamma \cap \partial\Omega^+$ . Then, “localize” the integral to an arbitrary point  $\mathbf{x}_0$  to obtain the point-wise condition.

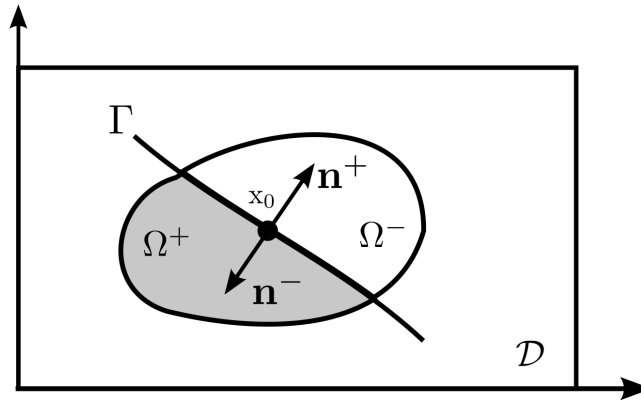


Figure 1: Derivation of the jump conditions from the balance law

2. For elastostatics  $\mathbf{F}$  in equation (1) is  $-\sigma$  and  $\mathbf{r} = \rho\mathbf{b}$ , where  $\sigma$  is the stress tensor and  $\mathbf{b}$  is the body force per unit mass. (20 points)

- (a) Show that the jump condition for elastostatics across a jump manifold (equation (2a)) reduces to

$$\mathbf{t}^+ = -\mathbf{t}^- \quad (\text{Newton's second law for tractions}). \quad (3)$$

**Hint:** Note that traction is given by  $\mathbf{t} = \sigma \cdot \mathbf{n}$  where for traction on each side,  $\sigma$  and  $\mathbf{n}$  are taken from the same side.

<sup>1</sup>While physically jumps can occur in  $\mathbf{F}$ , the strong form of the problem, *i.e.*,  $\nabla \cdot \mathbf{F} - \mathbf{r} = \mathbf{0}$ , may not be valid at the points on the jump manifold as the (partial) derivatives in  $\mathbf{F}$  may not exist.

- (b) Figure 2 shows an example of jump in flux tensor  $-\sigma$  for elastostatics. The top surface is uniformly moved up to generate a uniform strain  $\epsilon_{22} = \bar{\epsilon}$  in both materials. Since they possess different elastic moduli  $E^+$  and  $E^-$ , the stress tensor has a jump across the material interface  $\Gamma$ . Using the solutions provided in the figure, show that in spite of this jump the tractions still satisfy the common action-reaction law at the interface (*cf.* equation (3)).
- (c) For this 2D problem, the strong form is:

$$\begin{aligned}\sigma_{11,1} + \sigma_{12,2} + \rho b_1 &= 0 \\ \sigma_{21,1} + \sigma_{22,2} + \rho b_2 &= 0\end{aligned}$$

Discuss whether any of the partial derivatives in this equation cannot be computed across the jump, given that the stress tensor is discontinuous.

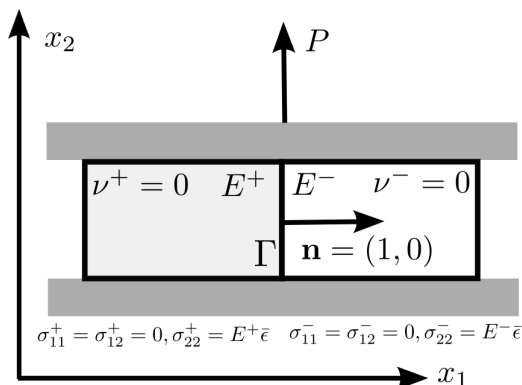


Figure 2: Jump in stress tensor across a material interface.

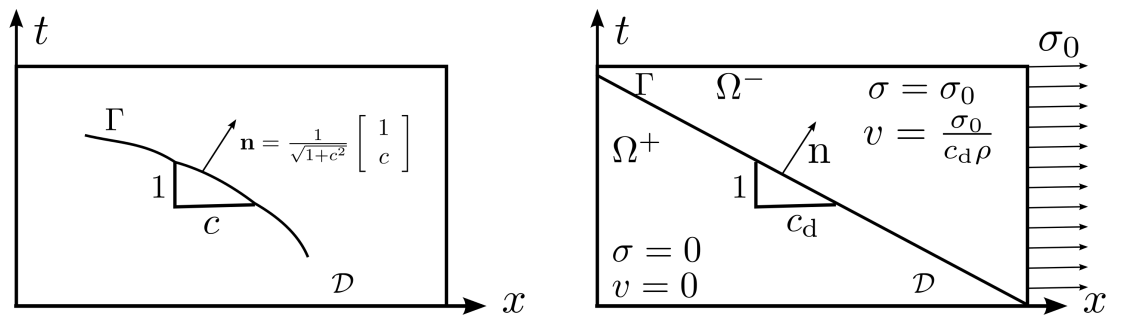
3. Figure 3(a) shows a possible jump manifold in spacetime for 1D setting. In this case the stress matrix and linear momentum  $\mathbf{p} = \rho \mathbf{v}$  reduce to only one component of  $\sigma_{11}$  and  $p_1$ , respectively. For brevity, we drop the these indices. As mentioned, the flux matrix for elastodynamics in spacetime is  $\mathbf{F} = [-\sigma \ p]$ . (20 points)

- (a) Given the equation (2b), write the jump conditions for this problem based on the solutions and spacetime normal vector <sup>2</sup> given in figure 3(a).
- (b) Figure 3(b) shows the exact solution to a 1D wave propagation problem where the right boundary of a semi-infinite bar is loaded by stress  $\sigma_0$ . Here  $c_d = \sqrt{E/\rho}$  is the material dilatational wave speed. Verify that the jump from the two sides + and - satisfy the jump condition obtained in previous step.
- (c) The strong form of this problem is,

$$\sigma_{,x} + \rho b = \dot{p}$$

Given the exact solution in the figure, discuss the validity of the strong form for the points on  $\Gamma$ . Compare this with the elastostatics case in problem 2.c.

<sup>2</sup>As can be seen, in the calculation of the normal vector, we are adding the square of the speed of the manifold  $\Gamma$ ,  $c^2$ , to unity. While the problem can be fixed by using more advanced notations, we do not pursue it further in this course.



(a) Spacetime normal vector  $\mathbf{n}$  for a jump manifold moving with speed  $-c$ .

(b) Exact solution for a problem with applied stress

Figure 3: Jump conditions for elastodynamics