1. **50 Points** Use a 3 point Gauss and 5 point Newton-Cotes quadrature rule to evaluate the following integral and obtain their respective errors with respect to exact value of the integral $I_e = \tan^{-1}(2) - \tan^{-1}(-1)$. Quadrature points and weights are given in fig. 1.

$$I = \int_{-1}^{2} \frac{\mathrm{d}x}{1+x^2}$$

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189

$$\int_a^b f(x)\,dx \simeq C_0 h \sum_{i=1}^{n} W_i f(x_i) + C_1 h^{k+1} f^{(k)}(\xi)$$

Newton-Cotes

n	C_0	W_1	W_2	W_3	W_4	W_5	C_1	k	Name
1	1	1					1/2	1	Rectangle
2	1/2	1	1				–1/12	2	Trapezium
3	1/3	1	4	1			–1/90	4	Simpson
4	3/8	1	3	3	1		–3/80	4	4-point
5	2/45	7	32	12	32	7	–8/945	6	5-point

Figure 1: Gauss and Newton-Cotes quadrature points.

2. **120 Points** Figure 2 shows a second order element for 1D elastostatic (bar) problem. The node numbering is different from the class as the middle node number is 3 rather than 2 for the purpose of static condensation. The element setup is shown in fig. 2. The Matlab files LoadGPs.m (Gauss point table for number of Gauss points = $1, 2, 3, 4, 5$ and 64) ingegrandSolid1DExample.m (integrand of the stiffness matrix integral (3)) computeK1DSolid.m (Main function for calculating stiffness matrix) are provided. The shape functions that are basically Lagrange functions for the points $\xi = -1, 0, 1$ are,

$$\mathbf{N}^e = \begin{bmatrix} N_1^e(\xi) & N_2^e(\xi) & N_3^e(\xi) \end{bmatrix} = \begin{bmatrix} \frac{\xi(\xi-1)}{2} & \frac{\xi(\xi+1)}{2} & 1-\xi^2 \end{bmatrix} \quad \Rightarrow \tag{1a}$$

$$\mathbf{N}'^e = \frac{\partial \mathbf{N}^e}{\partial \xi} = \begin{bmatrix} \frac{2\xi-1}{2} & \frac{2\xi+1}{2} & -2\xi \end{bmatrix} \tag{1b}$$

For brevity we drop the superscript e. For an "isoparametric" nodal coordinates x_i are mapped to coordinate x the same way that nodal solutions a_i are mapped to element solution u using the shape functions given in (1a). That is,

$$x = \Sigma_{i=1}^{3} x_i N_i(\xi) \quad \Rightarrow \quad J = \frac{\partial x}{\partial \xi} = \frac{L}{2}(4\alpha\xi + 1), \text{ where } \alpha = \frac{x_{\text{ave}} - x_2}{L} \tag{2}$$

The nondimensional value α is a skewness measure of the map between ξ and x and $x_{\text{ave}} = \frac{x_0+x_1}{2}$. Given that $\mathbf{B} = \frac{\partial \mathbf{N}}{\partial x} = \frac{\partial \mathbf{N}}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{J}\frac{\partial \mathbf{N}}{\partial \xi}$, $\mathrm{d}x = J\mathrm{d}x$, $k^e = \int_{x_0}^{x_1} \mathbf{N}^{\mathrm{T}}\mathbf{D}\mathbf{N}\mathrm{d}x$ for constant $D = EA$ we obtain,

$$k^e = AE\int_{-1}^{1} K(\xi)\mathrm{d}\xi, \quad \text{where} \quad K(\xi) = \frac{1}{J(\xi)} \begin{bmatrix} \frac{2\xi-1}{2} \\ \frac{2\xi+1}{2} \\ -2\xi \end{bmatrix} \begin{bmatrix} \frac{2\xi-1}{2} & \frac{2\xi+1}{2} & -2\xi \end{bmatrix} \tag{3}$$

Answer the following questions:

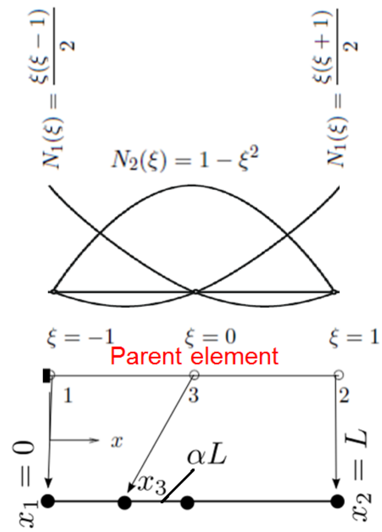

Figure 2: Second order element for 1D problems.

(a) **30 Points** **Full Rank Integration** refers to the (polynomial) order of the integrand if i) geometry is not skewed (*e.g.,* J is constant), ii, iii) material response is linear (as $\sigma = E\epsilon$ for constant E is used here) and homogeneous (*e.g.,* E does not depend on ξ), iv) element has uniform geometry (*e.g.,* constant A here). Basically, it refers to the maximum order of integrand only due to **B** terms.

i. What is full integration order for this second order bar?

ii. How many Gauss points and Newton-Cotes points must be used to exactly evaluate k^e for $\alpha = 0$?

iii. Run the main Matlab function computeK1DSolid.m for non-skewed geometry (Use $\alpha = 0$ as its only input parameter). The function outputs nquad (number of quad points), ke (k^e), eigenVectors, eigenValues, and rankv (rank of k^e) for number of quad points 1 to 5 and 64 respectively with a pause between each set of input waiting for the user to proceed. Comment on the convergence of the matrix as number of quad points increases.

(b) **30 Points** **Rank of the matrix**: The rank of a matrix T, rank(T), is the dimension of its range $R(T) = \{Tx | x \in \mathbb{R}^n\}$ where n the dimension (size) of the matrix. For a full-rank matrix rank(T) $= n$. The kernel or null space of a matrix T is the space of vectors mapped to zero ker(T) $= \{x | Tx = 0\}$. For a square matrix T of dimension n we have dim(ker$T$) + rank(T) $= n$. Clearly, the number of zero eigenvalues is equal to dim(ker$T$) and the corresponding eigenvectors are a basis for the kernel (*i.e.,* their linear combinations makes the space ker(T).

For finite element stiffness matrices $Ka = F$ kernel of the matrix has a physical meaning and denotes a's that are nonzero yet induce zero "loads". In solid mechanics they correspond to rigid body motions where nonzero a's (displacements) result in zero F (nodal forces). A finite element stiffness matrix should satisfy the following two conditions[1]:

- Have all physical null space modes (*e.g.,* zero energy / rigid displacements for solid mechanics). Otherwise, FEM solutions will predict/induce forces when there should not be.

[1] We will discuss what the implications of this may be for large deformation settings in the class.

- No nonphysical rank-deficiency: The stiffness matrix should not have a larger null space than what the physics predicts, otherwise the elements will have nonphysical solutions, *e.g.,* (uncontrolled) rigid displacements under the application of no loads.

Answer the following very briefly:

i. Again by running 2.a.iii observe comment on the null space (dimension - rank) of the stiffness matrices obtained by different integration orders and specify which integration orders do not have the right rank?

ii. For the integration orders that provide the stiffness with the correct rank, how many rigid displacement modes are there and what form of displacement the rigid displacements are? **Hint**: Diagonal values of eigenValues are the eigenvalues and the corresponding columns of eigenVectors are the eigenvectors.

(c) **60 Points** **Skewness**: By increasing $\alpha \to 0.5$ (or decreasing to -0.5) the middle point tends to one of the end points of the bar and the map between ξ and x becomes more nonlinear.

i. Run computeK1DSolid.m for $\alpha = 0.1$ and 0.2, list $k^e(1,1)$ value for all computed number of quadrature points, and comment on the convergence of the values in k^e as the number of quadrature points increase.

ii. What (if any) number of quadrature points can integrate k^e exactly for $\alpha = 0.15$?

iii. If we integrate k^e with full integration order (2.a.i) there will be some errors from quadrature. In general should we increase the number of quadrature points beyond this such that the integral is evaluated very accurately or would a full integration order suffice? What is your justification?

iv. Run computeK1DSolid.m for $\alpha = 0.26$ and $\alpha = 0.32$ and comment on the convergence properties of the matrix as number of quadrature points increases.

v. For a map between parent coordinate ξ and x to be valid $J > 0$ ($\det(J) > 0$ in higher dimensions) at all points (it can be zero at a measure zero set), otherwise the map between ξ and x is not one to one. For example there will be several ξ mapped to one x, material can be mapped inside-out, *etc.*. Find the range of α where this does not happen and can still evaluate stiffness matrix using isoparametric method. Compare this value with $\alpha = 0.26, 0.32$ from previous question and comment on it.

3. **90 Points** In fig. 3 a third order 1D element is shown. Nodal coordinates and parent coordinates are $x_1 = 0$, $x_3 = 2$, $x_4 = 5$, $x_2 = 10$ and $\xi_1 = -1$, $\xi_3 = -\frac{1}{3}$, $\xi_4 = \frac{1}{3}$, $\xi_2 = 1$, respectively. For the following questions only provide function expressions (when needed) and do not simplify them.

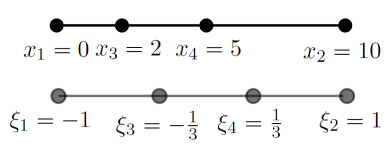

Figure 3: Third order element for 1D problems.

(a) Obtain $N_1(\xi)$ to $N_4(\xi)$ using Lagrange functions.

(b) Write $x = f(\xi)$ using isoparametric formulation (same shape functions map x_1 to x_4 to x (do not simplify).

(c) Write the expression for $J(\xi)$ (do not simplify).

(d) Write the order of integrand, number of Gauss point, and Gauss points for a full integration scheme.

(e) Write the number of Newton-Cotes points, and the points needed for full integration order.

(f) What rank of stiffness matrix should we obtain?
Hint: Consider rigid body motions.

4. **120 Points** **Derivation of Gauss points**: In Bathe's book section 5.5.3 (equations 5.144-5.149) it is shown that Gauss points are the roots of Legendre polynomials (*cf.* fig. 4). This is implied by the following: Equation (5.149) specifies that the (yet unknown) polynomial P of order n is normal to all polynomials of order 0 to $n-1$ for the inner product given by $< f, g >= \int_{-1}^{1} f(\xi).g(\xi)\mathrm{d}\xi$. The discussion beforehand clarifies that the n roots of this polynomial P are in fact the Gauss points of scheme with n points. Given that Legendre polynomials are orthogonal with this inner-product:

$$\int_{-1}^{1} P_m(\xi)P_n(\xi)\,\mathrm{d}\xi = \frac{2}{2n+1}\delta_{mn} \quad \text{no sum on } n \tag{4}$$

equation (5.149) implies that P in that equation is (a constant factor of) P_n. Finally, (5.150) provides an equation to evaluate weights w_j (α_j in text). For more detailed derivation of this relation refer to 01_Gauss_quadrature_derivation_of_points_weights.pdf.

(a) **35 Points** Using P_3 in fig. 4 obtain Gauss points and Gauss weight values for 3 point Gauss rule and compare your results to the values given in fig. 1.

(b) **85 Points** In many instances we are dealing with more general integrals of the form $I = \int_{-\infty}^{\infty} f(\xi)\rho(\xi)\mathrm{d}\xi$ ($\rho(\xi) \geq 0$). For example Gauss integration is a special case where $\rho = \chi_{[-1\ 1]}$. Also in probability theory expected value of a quantity is defined as $\mathbb{E}(f(\xi)) = \int_{-\infty}^{\infty} f(\xi)\rho(\xi)\mathrm{d}\xi$ where $\rho(\xi)$ is the *probability density function* (PDF) of the random variable ξ. In the context of FEM formulation, the integrals of latter form are encountered in the solution of stochastic PDEs. Ideally we want to derive quadrature rules for these more general cases as $I = \int_{-\infty}^{\infty} f(\xi)\rho(\xi)\mathrm{d}\xi \quad \Rightarrow \mathrm{Q}(I) = \Sigma_{i=1}^{n} w_i f(\xi_i)$ where again ξ_i are quadrature points and w_i are quadrature weights. Given that we can define an inner-product of the form $< f, g >_\rho = \int_{-\infty}^{\infty} f(\xi)g(\xi)\rho(\xi)\mathrm{d}\xi$ we can use any orthonormalization scheme such as Gram-Schmidt to form an orthonormal basis of Q_i (Q_i being a polynomial of order i) for polynomial functions. That is,

$$< Q_i, Q_j >_\rho = \delta_{ij} \quad \text{that is} \quad \int_{-\infty}^{\infty} Q_i(\xi)Q_j(\xi)\rho(\xi)\mathrm{d}\xi = \delta_{ij} \tag{5}$$

Following the proof in Bathe section 5.5.3 and the supplementary document above show that for a quadrature scheme of n points,

$$\xi_i \text{ are the roots of polynomials } Q_n \tag{6a}$$

$$w_i = \int_{-\infty}^{\infty} L_i(\xi)\rho(\xi)\mathrm{d}\xi, \quad \text{where } L_i(\xi) = \frac{\prod_{j=0, j\neq i}^{n}(\xi - \xi_j)}{\prod_{j=0, j\neq i}^{n}(\xi_i - \xi_j)} \text{ are Lagrange polynomials} \tag{6b}$$

for $\quad i = 1, \ldots, n$.
Hint: Equation (5) implies $\int_{-\infty}^{\infty} Q_n(\xi)\xi^k\rho(\xi)\mathrm{d}\xi = 0, k = 1, \ldots, n-1$ (Why?) and compare it to Bathe's equation (5.149).

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$\frac{1}{8}(63x^5 - 70x^3 + 15x)$
6	$\frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$
7	$\frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$
8	$\frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$
9	$\frac{1}{128}(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x)$
10	$\frac{1}{256}(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)$

Figure 4: Legendre polynomials (Source: http://en.wikipedia.org/wiki/Legendre_polynomials