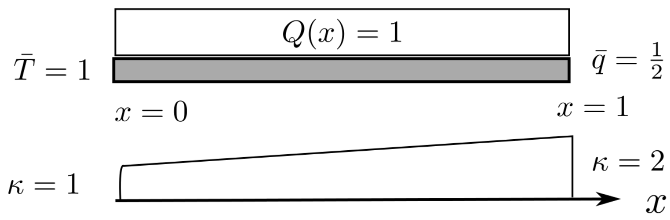

Figure 1: Heat problem

The Weighted Residual (WRS) statement and Weak Statement (WS) for the heat problem in fig.1 are,

$$\text{Find } T \in \mathcal{V} = \{v\,|v \in C^2([0\ 1]),\ v(0) = 1\} \quad \text{such that } \forall w \in \mathcal{W}$$
$$\int_0^1 w\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa\frac{\mathrm{d}T}{\mathrm{d}x}\right) - 1\right)\,\mathrm{d}x + \left[w(x).(\frac{1}{2} + 2\frac{\mathrm{d}T}{\mathrm{d}x})\right]_{x=1} = 0 \tag{1}$$

and

$$\text{Find } T \in \mathcal{V} = \{v\,|v \in H^1([0\ 1]),\ v(0) = 1\} \quad \text{such that}$$
$$\forall w \in \mathcal{V}_0 = \{v\,|v \in H^1([0\ 1]),\ v(0) = 0\}$$
$$\int_0^1 \frac{\mathrm{d}w}{\mathrm{d}x}\kappa\frac{\mathrm{d}T}{\mathrm{d}x}\,\mathrm{d}x = \int_0^1 w(x)\,\mathrm{d}x - \frac{1}{2}w(1) \tag{2}$$

where for weak statement $H^1([0\ 1] = \{u|\int_0^1 u^2(x) + u'^2(x)\mathrm{d}x < \infty\}$ is the correct space that ensures integrals can be evaluated. This space is larger than $C^1([0,1])$ and includes hat functions that are used in first order 1D finite elements.

1. **40 Points** Discrete Weighted Residual Method: In order to solve (1) we are looking for a solution of the form $T^h(x) = \phi_p(x) + a_i\phi_i(x)$ (ϕ_p is a particular solution and ϕ_i are trial functions).

   (a) Describe why $\phi_p(x) = 1$ and $\phi_p(x) = \cos(x)$ are admissible while $\phi_p(x) = \sin(x)$ is not admissible (**5 Points**)?

   (b) Describe why in $T^h(x) = \phi_p(x) + a_i\phi_i(x) = \phi_p + a_0\phi_1 + a_1\phi_1 = \phi_p + a_0.1 + a_1.x$ ($\phi_0 = 1, \phi_1 = x$), $\phi_0 = 1$ is not admissible (**5 Points**)?

   (c) For $T^h(x) = 1 + a_1 x$ ($\phi_p = 1, \phi_1 = x$) find the solution to (1) using **subdomain method**. Make sure to use the right weight function for subdomain method and $n = 1$ (**15 Points**).
   Hint: Instead of forming matrix **K** and **F** and solving for **a**, directly plug $T^h(x) = 1 + a_1 x$ in (1).
   Note that $\kappa(x) = x + 1$ is not constant.

   (d) For the same $T^h(x) = 1 + a_1 x$ solve the problem using **collocation method** with one collocation point at $x = \frac{1}{2}$ (**15 Points**).
   Hint: Again directly plug the solution in (1) and use the right weight function.

2. **30 Points** Discrete Weak Statement:

   - Solve $T^h(x) = 1 + a_1 x$ ($\phi_p = 1, \phi_1 = x$) using **Galerkin** method and the **weak statement** (2) (**15 Points**).
     Note that $\kappa(x) = x + 1$ is not constant.

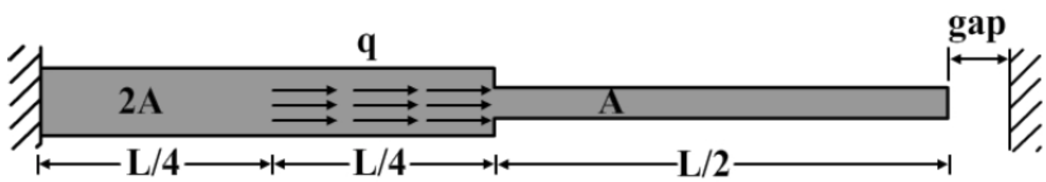

Figure 2: Bar problem

- Why in Galerkin method $\phi_0 = w_0 = \sqrt{x}$ is not an appropriate trial function for the weak statement (2) while $\phi_1 = w_1 = x^{\frac{3}{4}}$ can be used (**15 Points**)?

3. **50 Points** Obtain an approximate solution of the following boundary value problem using spectral Galerkin method (that is trial functions are polynomials over the entire domain):

$$u''(x) + u(x) + x = 0 \tag{3a}$$
$$u(0) = 0 \tag{3b}$$
$$u(1) = 0 \tag{3c}$$

   (a) Derive the weak statement for the problem. (**20 Points**)

   (b) Using **one** appropriate quadratic polynomial trial function, find an approximate solution of the form $u^h(x) = a_0 + a_1 x + a_2 x^2$. (**25 Points**)

   (c) What is the error between approximate solution and the exact solution $u(x) = -x + \frac{\sin(x)}{\sin(1.0)}$ at $x = 0.5$: $e(0.5) = u^h(0.5) - u(0.5)$? (**5 Points**)

4. **80 Points** An axially loaded bar, fig.2, carries a distributed load over one quarter of its length and has a small gap at one end as shown in the following figure. Divide the bar into appropriate number of elements and compute displacements and stresses in the bar. Note that if the load is large enough to close the gap then the gap can be treated as a known displacement boundary condition. Assume $L = 500$ mm, $A = 25$ mm^2, $E = 20,000$ N/mm^2, $q = 400$ N/mm, (a) gap = 1 mm, (b) gap = 20 mm.
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