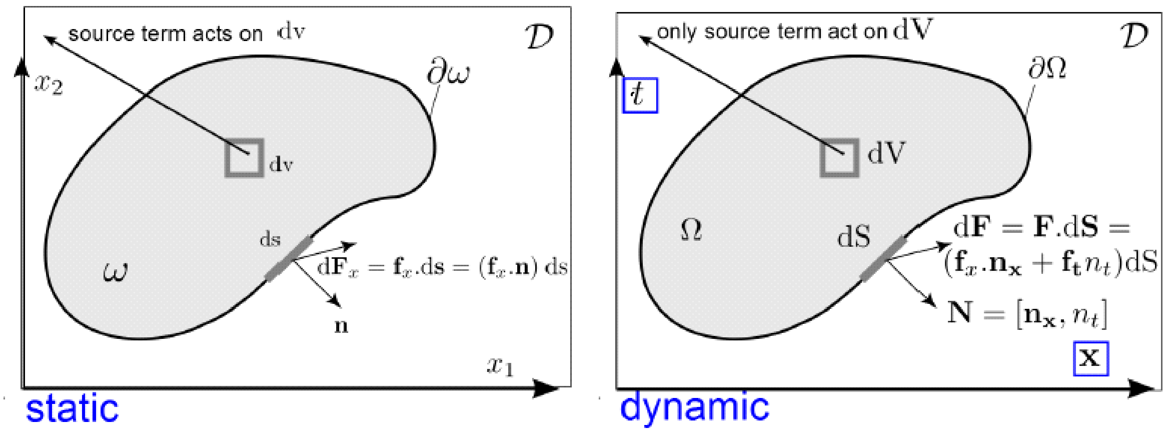
2016/08/30

Tuesday, August 30, 2016
9:58 AM

Any balance law can be written in the following format:



$$\forall\omega \subset \mathcal{D} : \int_{\partial\omega} (\mathrm{f}_x.\mathbf{n})\,\mathrm{ds} - \int_{\omega} \mathbf{r}\,\mathrm{dv} = \mathbf{0} \qquad \forall\Omega \subset \mathcal{D} : \int_{\partial\Omega} (\mathbf{F}.\mathbf{N})\mathrm{dS} - \int_{\Omega} \mathbf{r}\,\mathrm{dV} = \mathbf{0}$$

where $\mathbf{F} = [\mathrm{f}_x|\mathrm{f}_t]$ (*cf.* (15) and (16)). So, in either case a balance law takes the general form of:

$$\boxed{\forall\Omega \subset \mathcal{D} : \int_{\partial\Omega} (\mathrm{f}.\mathrm{n})\mathrm{ds} - \int_{\Omega} \mathrm{r}\,\mathrm{dv} = \mathbf{0}} \qquad (20)$$

21 / 457

spatial flux density

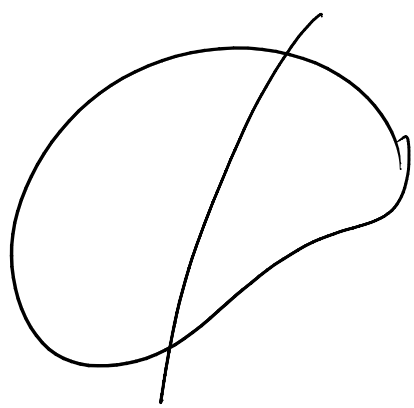
$F = \begin{bmatrix} f_x \\ f_t \end{bmatrix}$

density of preserved quantity

Any balance law can be written in this form:

1. Flux on boundary
2. Source term inside the domain

This session:
Obtain partial differential equations from balance laws



Balance laws

<=>

1. PDE
2. Jump conditions

2. Jump conditions

$\sigma_{22} \neq 0$

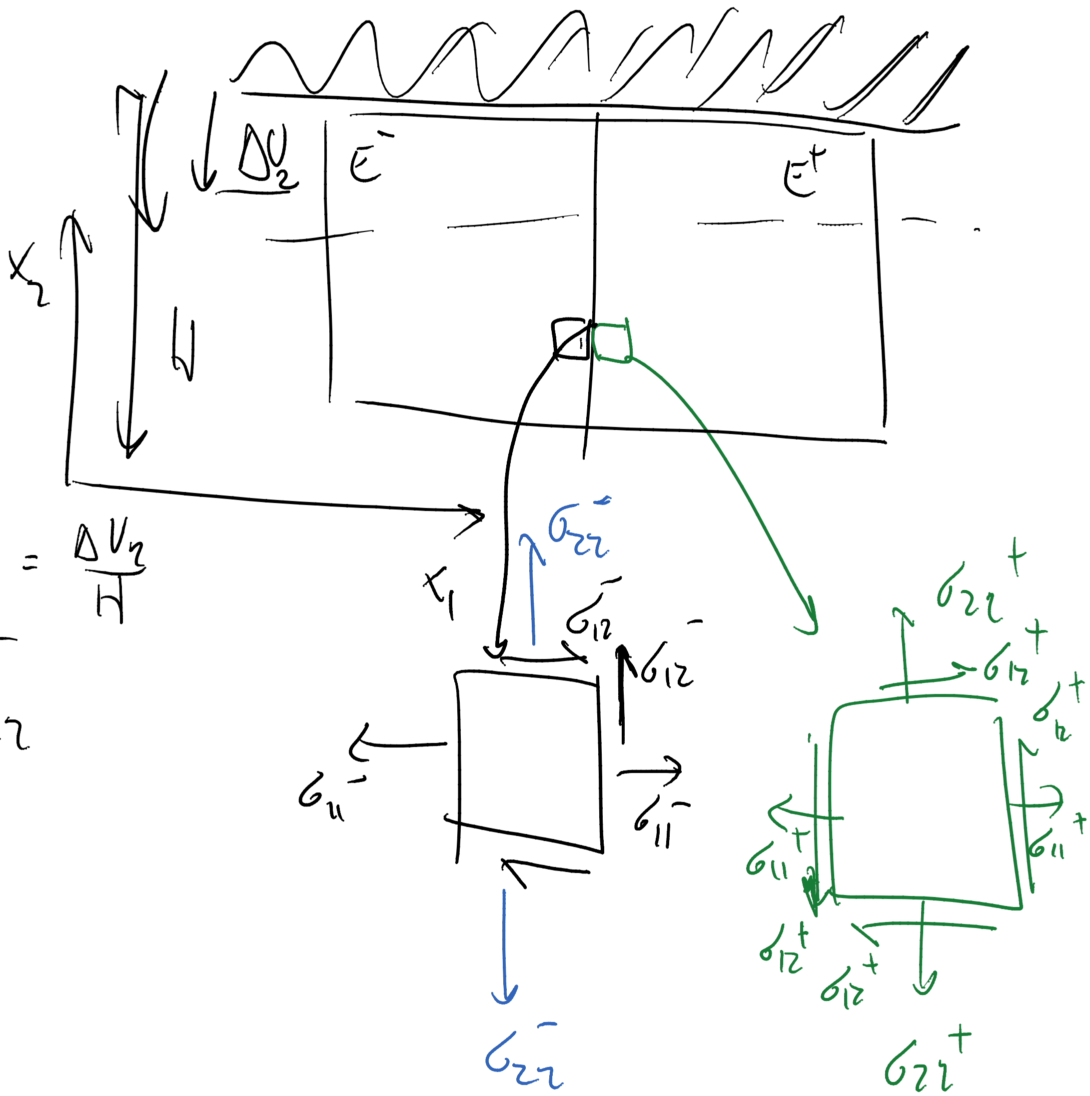
$E^- \neq E^+$

$\epsilon_{22}^- = \epsilon_{22}^+ = \frac{\Delta v_2}{H}$

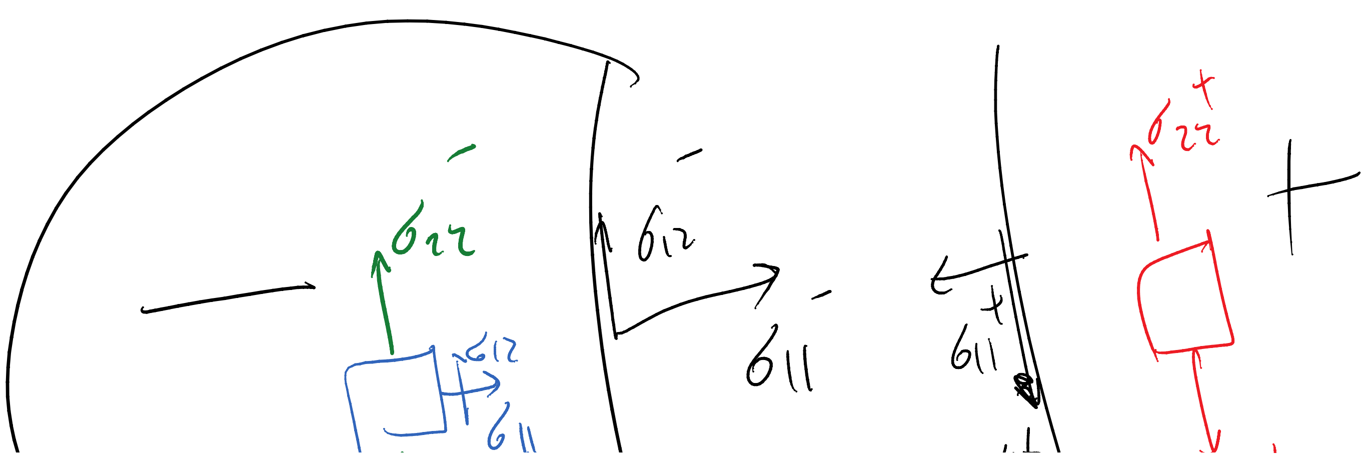
$\sigma_{22}^+ = E^+ (\;) \epsilon_{22}^+$

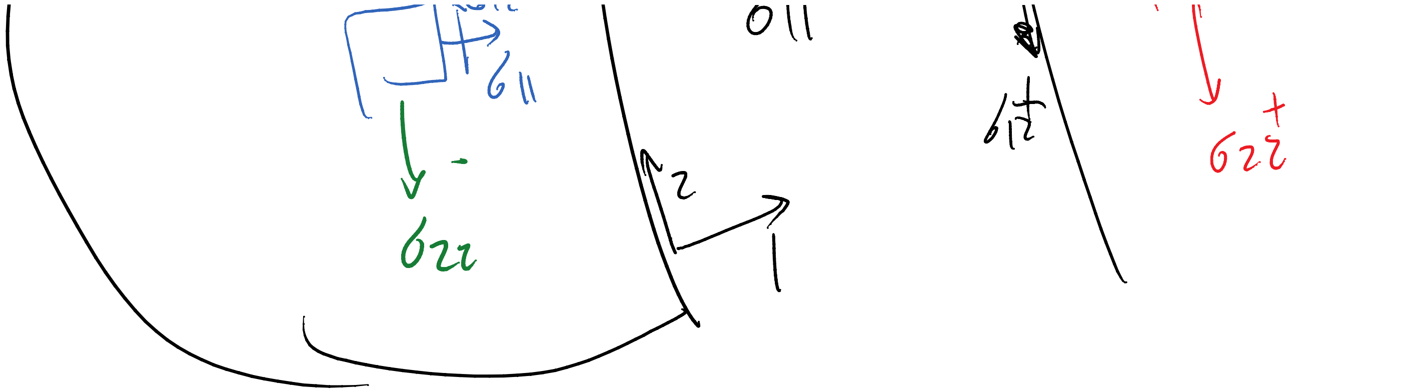
poisson ratio terms

$\sigma_{22}^- = E^- (\;) \epsilon_{22}^-$



Spatial flux (stress for balance of linear momentum in this case) can have a jump across an interface!





Newton's third law $\qquad \sigma_{11}^- = \sigma_{11}^+ \qquad$ normal tractions

action-reaction $\qquad \sigma_{12}^- = \sigma_{12}^+$

Observations:

1. σ_{22} does not appear at the interface through the traction vector
2. σ_{22} can suffer jump at an interface

$$\text{Balace law} \xrightarrow{\text{give}} \begin{cases} 1.\ \text{PDE} \\ 2.\ \text{Jump conditions} \end{cases}$$

Solid Mechanics

Static case $\qquad [\sigma]\cdot n = 0$

$$\sigma^+ \cdot n - \sigma^- \cdot n = 0$$

traction vectors $\longleftarrow$ $\vec{t}^{\,+} - \vec{t}^{\,-} = 0$

In general, especially for dynamic problems these jump conditions are not trivial

Balance laws provide us a systematic way to obtain these jump conditions

- We are not going to talk much more about the jump conditions from this point on. HW0 (no need to return it) has some interesting examples of jump conditions.

In this course (talking about continuous / conventional FEM) we are mostly interested in PDEs rather than jump conditions.

How to we derive PDEs from balance laws

General balance

$$\forall \omega \subseteq D \quad \int_{\partial \omega} f \cdot \vec{n}\, ds - \int_{\omega} r\, dV = 0$$

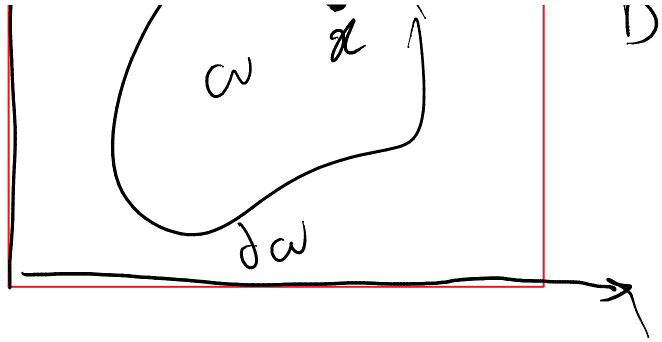
flux density (f)

source term (r)

PDE is $\forall x \in D$

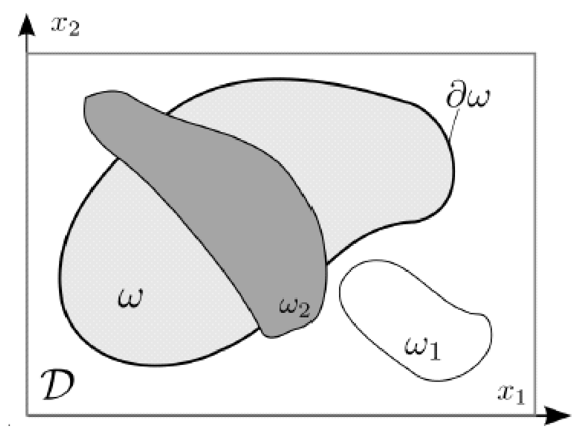
PDE is $\forall x \in D$

$$\nabla \cdot f - r = 0$$



Path to obtain a point-wise equation (strong form)

While balance law hold for arbitrary volumes ω:



$$\forall \omega \subset \mathcal{D} : \int_{\partial\omega} \mathbf{F}.\mathrm{dS} - \int_{\omega} \mathrm{r}\ \mathrm{dV} = 0$$

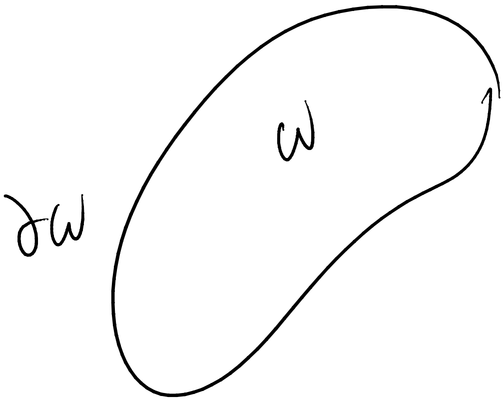
the fact that they are always in integral form makes it practically difficult to obtain the solution to a problem. We systematically derive a point-wise equation (*i.e.*, a differential equation) which has a more limited solution space. We discuss the additional conditions that make the two approaches equivalent.

Divergence theorem:

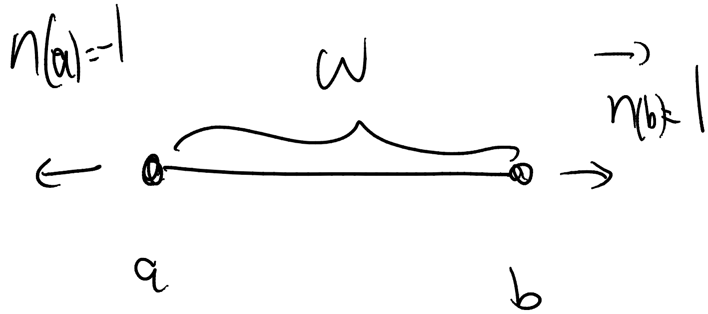
vector

2nd order tensor (matrix)

$$\int_{\partial\omega} f \cdot \vec{n}\, ds = \int_{\omega} \nabla f\, dV$$



1D version



$$\overbrace{\int_{\partial\omega} f n \, ds}^{\text{boundary integral}} = \int_{\omega} \nabla \cdot f \, dV$$

$$\int_{\partial\omega} f n \, ds = f(b)n(b) + f(a)n(a)$$

$$= f(b)(+1) + f(a)(-1)$$

$$= f(b) - f(a)$$

$$\int_{\omega} \nabla \cdot f \, dV = \int_a^b f_{,x} \, dx$$

derivative w.r.t. x

$$\int_{\partial\omega} f n \, ds = \int_{\omega} f \, dv \quad :$$

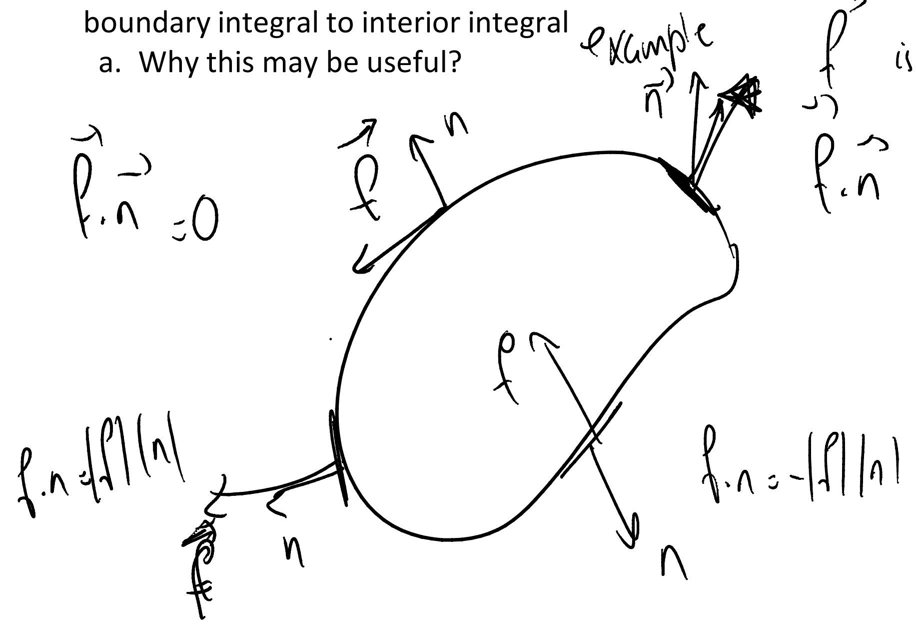
$$\boxed{\text{in 1D} \qquad \underrightarrow{f(b) - f(a)} = \int_a^b f_{,x} \, dx}$$

F = antiderivative of f $\qquad F' = f$

$$F(b) - F(a) = \int_a^b f(x) \, dx$$

1. In 1D we turn interior integral to boundary integral that results in two end point evaluation of the function

2. In 2D/ 3D many times we go through the other direction and turn boundary integral to interior integral
   a. Why this may be useful?

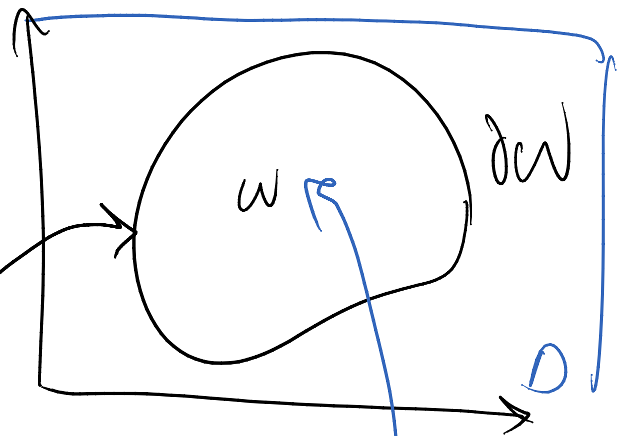


$$\int_{\partial\omega} f \cdot n \, ds$$

i. It may be easier to evaluate the surface integral by turning it to a volume integral (no need to calculate normal vector, …)

ii. Sometime we need to make all integrals surface integrals or volume integrals

Reason ii is why we use divergence theorem for our balance law



$$\forall \omega \subset D \quad \int_{\partial\omega} \underbrace{f \cdot n}_{\text{flux}} \, ds - \int_{\omega} \underbrace{r}_{\text{source term}} \, dV = 0$$

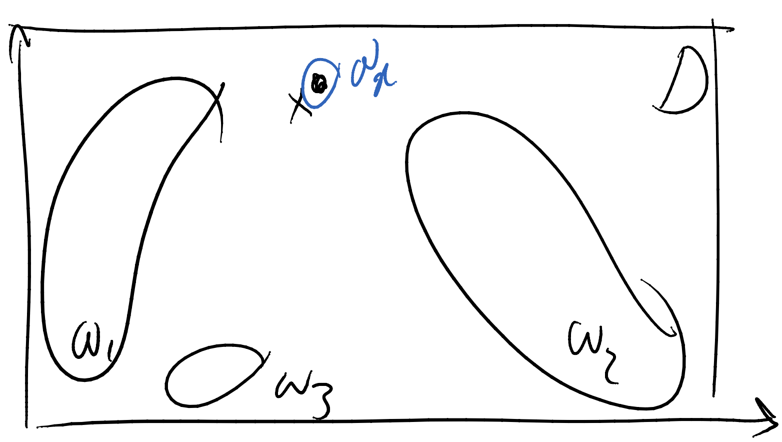
Divergence theorem
$$\int_{\partial\omega} g \cdot n \, ds = \int_{\omega} \nabla \cdot g \, dV$$

Apply divergence theorem to the first term

$$\forall \omega \qquad \int_\omega \nabla \cdot f \, dV - \int_\omega r \, dV = 0$$

$$\Rightarrow$$

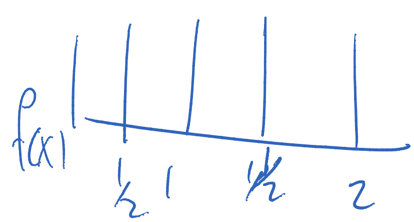
$$\boxed{\forall \omega \qquad \int_\omega \left(\nabla \cdot f - r \right) dV = 0}$$



$$\forall x \qquad \nabla \cdot f - r = 0$$

We use localization theorem to obtain the PDE

- Localization theorem says that if the integral of a continuous function is zero for arbitrary domains inside D then the integrand must be zero.



$$\int f \, dx = 0 \qquad f \neq 0$$

$$\int_{\omega} f \, dx = 0 \qquad f \neq 0$$

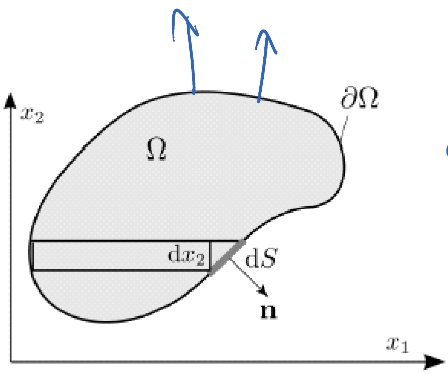
$\omega \subseteq [0,2]$ at $\frac{1}{2}, 1, \frac{3}{2}, 2$

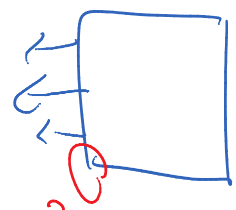
but f is not continuous

Formal overview of the material covered above (divergence theorem and localization theorem)

Transfer of boundary to interior integral higher dimensions

- Ω is compact and closed.
- $\partial\Omega$ is piecewise smooth.
- Normal vector $\mathbf{n}$ is defined almost everywhere (a.e.) and is pointing outward.
- tensor field (scalar, vector, matrix, ...):
  - $\mathbf{F}_{,i} = \partial\mathbf{F}/\partial x_i$ exists everywhere and
  - $\boxed{\text{is continuous.}}$





$$\boxed{\int_{\partial\Omega} \mathbf{F} n_i \, dS = \int_{\Omega} F_{,i} \, dV} \qquad (18)$$

should be continuous

? It's still fine that we cannot define normal vector here

Apply for all i

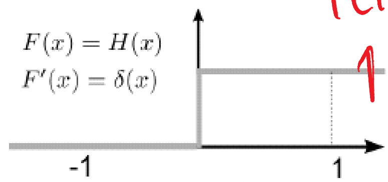
$$\int_{\partial\Omega} F \cdot \vec{n} \, dS = \int_{\Omega} \nabla \cdot F \, dV$$

$$\nabla \cdot F = F_{,1} + F_{,2} + F_{,3}$$

Q1: What if f,i does not exist at all points and is not continuous?

Comment on condition $F'(x) = f(x)$ for all points

- $F(x)$ should be differentiable at all points. Consider the functions $F(x) = H(x)$ and $f(x) = 0$. $F'(x) = f(x)$ everywhere except at 0.



$$F(1) - F(-1) = \int_{-1}^{1} F_{,x}\, dx$$

$$1 - 0$$

$$= 0$$

$f' = 0$ almost every where

$$\left.\begin{array}{ll} \int_{-1}^{1} f(x) = \int_{-1}^{1} 0 & = 0 \\ F(1) - F(-1) & = 1 \end{array}\right\}$$

$$\Rightarrow \int_{-1}^{1} f(x) \neq F(1) - F(0)$$

- The problem stems from the fact that F is not differentiable at zero.
- Loosely speaking $F'(x) = \delta(x)$, the so-called delta Dirac "function". Then,

$$\int_{-1}^{1} F'(x) = \int_{-1}^{1} \delta(x) = 1 = F(1) - F(0)$$

- Delta dirac is in fact not a function. It's a distribution (generalization of a function), which is infinitely differentiable.
- Physically, jump conditions similar to this example correspond to regions where the strong form would not hold.

18 / 457

should exist & should be continuous

Apply divergence theorem to a balance law

$$\boxed{\forall \Omega \subset \mathcal{D} : \int_{\partial\Omega} (\mathbf{f}.\mathbf{n}) \mathrm{ds} - \int_{\Omega} \mathbf{r}\, \mathrm{dv} = \mathbf{0}}$$

$$\int_{\partial\Omega} f.n\, ds = \int_{\Omega} (\nabla . f)\, dv$$

$\forall \Omega$

$$\int_{\Omega} (\nabla . f - r)\, dv = 0$$

assuming r is continuous

$\Rightarrow$ is continuous

So now we have integral of a continuous function that is zero for arbitrary domain Omega.

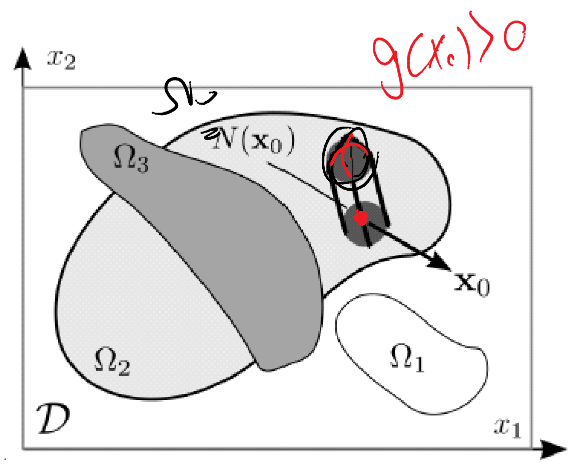
Can we say the function itself (i.e. the integrand) is zero?

Yes, we use the Localization theorem.

Localization theorem

Localization theorem states that if the integral of a continuus function is zero for all subsets of $\mathcal{D}$, then the function is zero:

$$\boxed{\forall \Omega \subset \mathcal{D} : \int_{\Omega} \mathbf{g}(\mathbf{x})\, \mathrm{dv} = \mathbf{0} \quad \Rightarrow \quad \forall x \in \mathcal{D} : \; g(\mathbf{x}) = 0} \tag{21}$$



Let's assume $g(x_0) \neq 0$ (*e.g.*, $g(x_0) > 0$). Since $g(\mathbf{x})$ is continuus, there is a neighborhood of $\mathbf{x}_0$ ($N(\mathbf{x}_0)$) that $g(x) > 0$. We choose an Ω that is only nonzero inside $N(\mathbf{x}_0)$. Then, $\int_{\Omega} g(\mathbf{x})\, \mathrm{d}V > 0$. Thus, $g(\mathbf{x}_0)$ cannot be nonzero and the function g is identically zero.

Balance law $\quad \int_{\partial\Omega} f \cdot n \, ds - \int_{\Omega} r \, dV = 0 \quad \Rightarrow$

$\forall \Omega \subseteq D \quad \int_{\Omega} (\underbrace{\nabla \cdot f}_{\text{Continuous}} - \underbrace{r}_{\text{Continuous}}) \, dV = 0 \quad$ Localization $\Rightarrow$

$$\boxed{\nabla \cdot f - r = 0}$$

PDE

$$\nabla \cdot f - r = 0$$

Optional HW Jump condition $\boxed{[f] \cdot n = 0}$

Next step:

Now that we have the PDE, we want to close the system (have equal number of equations to unknowns) and define boundary conditions

Example:
Elastostatics:

$$\nabla \cdot f - r = 0$$

f: outward spatial flux

$f = -\sigma$ $\quad$ $r = \rho b$

because

tractions add to forces "linear momentum" of a body

$$\boxed{\nabla \cdot \sigma + \rho b = 0}$$

unknown $\quad$ known

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\begin{bmatrix} & & \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$(\nabla \cdot \sigma)_i = \sigma_{ij,j} \Longrightarrow$$

$i = 1,2,3$

repeated

summation convention

$$(\nabla \cdot \sigma)_1 = \sigma_{1\overset{j,j}{1,1}} + \sigma_{12,2} + \sigma_{13,3}$$

2

3

$$(\nabla \cdot \sigma) + \rho b = \begin{bmatrix} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} \\ \sigma_{21,1} \quad \sigma_{22,2} \quad \sigma_{23,3} \\ \sigma_{31,1} \quad \sigma_{32,2} \quad \sigma_{33,3} \end{bmatrix} + \begin{bmatrix} \rho b_1 \\ \rho b_2 \\ \rho b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

\# eqn = 3

\# unknowns = 6 $\quad$ σ is symmetric

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ \text{sym} & & \sigma_{33} \end{bmatrix}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$ strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$ strain tensor

normal strain

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ & \epsilon_{21} & \epsilon_{22} \\ \text{sym} & & \epsilon_{33} \end{bmatrix}$$

$\epsilon_{11} = u_{1,1}$

$\epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1})$

shear strain

6 new unknowns

1D $\quad \sigma = E\,\epsilon$

Elastic modulus

2D & 3D

$$\bar{\bar{\sigma}}_{3\times3} = \mathbb{C}_{3\times3\times3\times3}\ \bar{\bar{\epsilon}}_{3\times3}$$

2nd order tensor

4th order Elasticity tensor

$$C_{ijkl} = \lambda\left(\delta_{ij}\delta_{kl}\right) + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

Lame's parameters

for an isotropic material

C_{ijkl} are known from the type of material

$$\sigma_{3\times 3} = C_{3\times 3\times 3\times 3}\ \epsilon_{3\times 3}$$

Type	Equation	n_e	new unknowns	n_u	$N_e - N_u$
Balance law	$\sigma_{ij,j} + \rho b_i = 0$	3	$\sigma_{ij} = \sigma_{ji}$, $i, j \in \{1, 2, 3\}$	6	3
Constitutive equation	$\sigma_{ij} = C_{ijkl} E_{kl}$	6	$E_{kl} = E_{lk}$	6	3
kinematic compatibility	$E_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$	6	u_k	3	0

Now we can close the system

- Dynamic case

Closing the system of equations (Dynamics)

Strong form (23) of balance of linear momentum for dynamics is:

$$\nabla.[-\sigma|\mathrm{p}] - \rho \mathrm{b} = 0, \quad \Rightarrow \nabla.\sigma - \frac{\partial \mathrm{p}}{\partial t} + \rho \mathrm{b} = 0 \quad \Rightarrow \quad \sigma_{ij,j} + \rho b_i = \dot{p}_i \qquad (25)$$

where $\mathrm{f} = [-\sigma|\mathrm{p}]$, $\mathrm{r} = \rho \mathrm{b}$, and $\nabla(.) = [\nabla_x(.)|\nabla_t(.)] = (\frac{\partial(.)}{\partial x_1} + \frac{\partial(.)}{\partial x_2} + \frac{\partial(.)}{\partial x_3}) + \frac{\partial(.)}{\partial t}$.

Type	Equation	n_e	new unknowns	n_u	$N_e - N_u$
Balance law	$\sigma_{ij,j} + \rho b_i = \dot{\mathrm{p}}$	3	$\sigma_{ij} = \sigma_{ji}$, $i, j \in \{1, 2, 3\}$	6	6
			p_j	3	
Constitutive equation	$\sigma_{ij} = C_{ijkl} E_{kl}$	6	$E_{kl} = E_{lk}$	6	6
	$p_j = \rho v_j$	3	v_j	3	
kinematic compatibility	$E_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$	6	u_k	3	0
	$v_j = \dot{u}_j$	3			

3 unknowns

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Example Heat eqn.: (1eq) $\nabla . q - Q = 0$

heat flux density

heat source

$q = -k \nabla T$

temperature

Conductivity matrix (Scalar)

unknown

$\nabla . (-k \nabla T) - Q = 0$