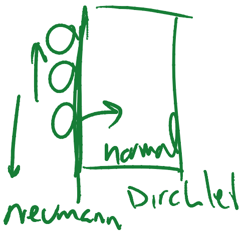
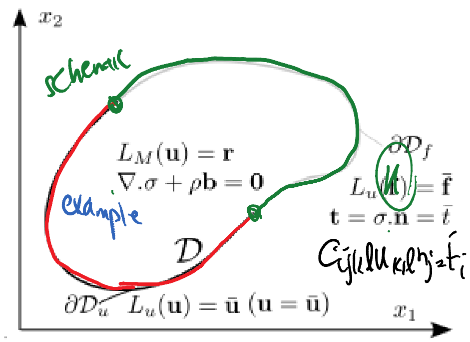
2016/09/08

Thursday, September 08, 2016
10:10 AM

Weighted residual statement:

This is the basis of the weighted residual methods (including FEM)





① Differential equation

$$\underbrace{L_M(u) - r = 0}$$

② Essential (Dirichlet) BC

$$\bar{u} - L_u(u) = 0$$

③ Natural (Neumann) BC

$$\bar{f} - L_u(u) = 0$$

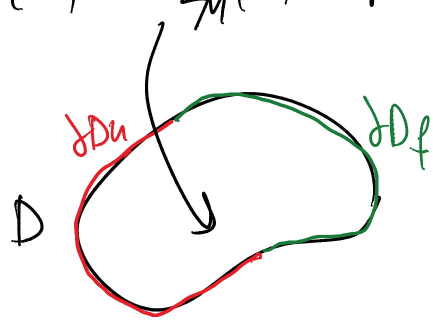
we have the exact solution when all these conditions are satisfied.

Define Residuals

① $\forall x \in D$

$$R_I(u) = L_M(u) - r$$

residual inside the domain



②

$$R_u(u) = \bar{u} - L_u(u)$$

residual on essential boundary

③

$$R_f(u) = \bar{f} - L_f(u)$$

residual on natural boundary

$\boxed{\text{Residual}}$ = error

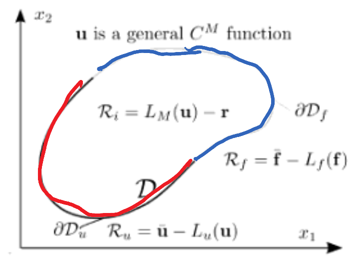
All residuals are zero for the exact solution

weight part: we multiply all residuals by weights & integrate them:

$$wR = \int_V w(\vec{x})\, R_I(u(x))\, dV$$

$$+ \int_{\partial D_u} w(\vec{x})\, R_u(u(x))\, dS$$

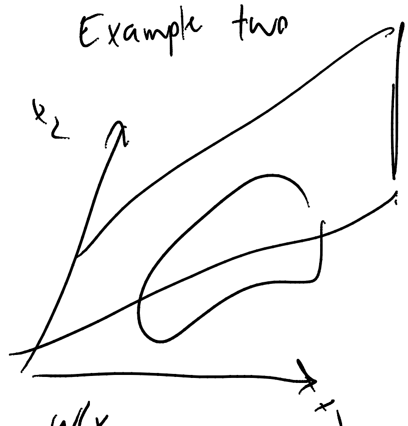
$$+ \int_{\partial D_f} w(x)\, R_f(u(x))\, dS$$



Examples of w's

$w = 1$

Example two



$w(x_1, x_2) = x_1$

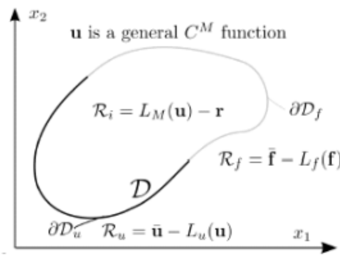
$w(x_1, x_2) = \sin(x_1 + x_2)$

Weighted residual statement

First, we define the following function spaces,

$$\mathcal{W} = C^0(\bar{\mathcal{D}}). \quad (29a)$$
$$\mathcal{V} = C^M(\bar{\mathcal{D}}). \quad (29b)$$

We seek a weak solution, $\mathbf{u} \in \mathcal{V}$, that satisfies,



$$\boxed{\begin{aligned} \forall \mathrm{w} \in \mathcal{W} \ : \int_{\mathcal{D}} \mathrm{w}.\mathcal{R}_i \,\mathrm{dv} + \int_{\partial\mathcal{D}_u} \mathrm{w}.\mathcal{R}_u \,\mathrm{ds} + \int_{\partial\mathcal{D}_f} \mathrm{w}.\mathcal{R}_f \,\mathrm{ds} &= \\ \int_{\mathcal{D}} \mathrm{w}.(L_M(\mathbf{u}) - \mathbf{r}) \,\mathrm{dv} + \int_{\partial\mathcal{D}_u} \mathrm{w}.(\bar{\mathbf{u}} - L_u(\mathbf{u})) \,\mathrm{ds} + \int_{\partial\mathcal{D}_f} \mathrm{w}.(\bar{\mathbf{f}} - L_f(\mathbf{u})) \,\mathrm{ds} &= 0 \end{aligned}}$$

Basically, a weak solution "weakly" satisfies all the PDEs and BCs by requiring the residuals to be zero for arbitrary averaging schemes (through arbitrariness of weight functions w). Next, we want to investigate the relationship between weak and strong solutions.

Equivalence of weak and strong solutions

We do not discuss the existence and uniqueness conditions for either system at the moment. Rather, show that weak and strong solutions are equivalent. That is, a strong solution is a weak solution and wise versa. From previous slides and the definition of a strong solution we have: Function $\mathbf{u} \in \mathcal{V}$ is a strong solution if,

$\forall \mathrm{x} \in \mathcal{D}$:	$\mathcal{R}_i(\mathbf{u}) =$	$L_M(\mathbf{u}) - \mathbf{r} = 0$	PDE	(30a)
$\forall \mathrm{x} \in \partial\mathcal{D}_u$:	$\mathcal{R}_u(\mathbf{u}) =$	$\bar{\mathbf{u}} - L_u(\mathbf{u}) = 0$	Essential BC	(30b)
$\forall \mathrm{x} \in \partial\mathcal{D}_f$:	$\mathcal{R}_f(\mathbf{u}) =$	$\bar{\mathbf{f}} - L_f(\mathbf{f}) = 0$	Natural BC	(30c)

and from previous slide a weak solution $\mathbf{u} \in \mathcal{V}$ satisfies:

$$\boxed{\begin{aligned} \forall \mathrm{w} \in \mathcal{W} \ : \int_{\mathcal{D}} \mathrm{w}.\mathcal{R}_i \,\mathrm{dv} + \int_{\partial\mathcal{D}_u} \mathrm{w}.\mathcal{R}_u \,\mathrm{ds} + \int_{\partial\mathcal{D}_f} \mathrm{w}.\mathcal{R}_f \,\mathrm{ds} &= \\ \int_{\mathcal{D}} \mathrm{w}.(L_M(\mathbf{u}) - \mathbf{r}) \,\mathrm{dv} + \int_{\partial\mathcal{D}_u} \mathrm{w}.(\bar{\mathbf{u}} - L_u(\mathbf{u})) \,\mathrm{ds} + \int_{\partial\mathcal{D}_f} \mathrm{w}.(\bar{\mathbf{f}} - L_f(\mathbf{u})) \,\mathrm{ds} &= 0 \end{aligned}} \quad (31)$$

Reminder:

Strong form: Anything that is "strongly" satisfied at all points (either inside for PDE or on boundary for BCs)

Weak form: Any statement written in integral form

for all weight functions

if we have exact solution

$R_i(u) = L_M(u) - r = 0 \quad \forall x \in D$

$R_u(u) = \bar{u} - L_u(u) = 0 \quad \forall x \in \partial D_u$

$\forall w$

$wR = \int_D w R_i \, dv + \int_{\partial D_u} w R_u \, ds + \int w R_f \, ds$

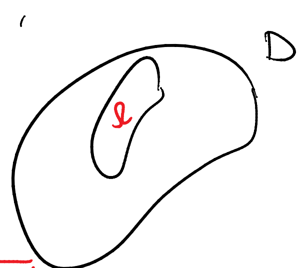
$R_u(u) = \bar{u} - L_u(u) = 0 \quad \forall x \in \partial D_h \quad \Longleftarrow \quad \partial D_h$

$R_f(u) = \bar{f} - L_f(u) = 0 \quad \forall x \in \partial D_f \qquad + \int_{\partial D_f} w R_f \, ds = 0$

The reverse says that if a function satisfies WR for All weight functions then it must be the Exact solution (that is $R_i(u) = 0$, $R_u(u) = 0$, $R_f(u) = 0$ for all points)

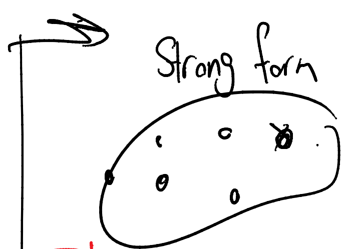
Motivation:

Why WR statement is useful?



$\boxed{\forall \Omega} \quad \int_{\partial\Omega} f \cdot ds - \int_{\Omega} r \, dV = 0$

Weak



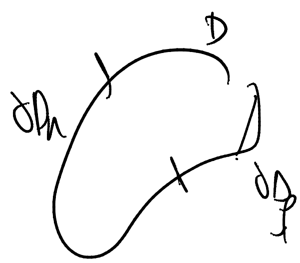
Strong form

$\boxed{\forall x \in D} \; R_i : \nabla \cdot f(u) - r = 0$

$\forall x \in \partial D_h \quad R_u(u) = \bar{u} - L_u(u) = 0$

$\forall x \in \partial D_f \quad R_f(u) = \bar{f} - L_f(u) = 0$

Strong



$\forall \omega$ functions

$\int_D w R_i(u) + \int_{\partial D_h} w R_u(u) + \int_{\partial D_f} w R_f(u) = 0$

ODE

<u>Weak</u>

integration over arbitrary domains is more difficult that integrating over a fixed domain but with different integrands (because of $\forall w$ statement)

2nd point in relation to solving an approximate solution

The idea that if a function satisfies WR for all weight functions must be the solution has profound practical importance:

Eventually in numerical setting we only satisfy WR for a finite number of weight functions (e.g. 1000) because we only have that number of unknowns through the approximate solution we are trying to find.

The hope is that as number of w goes to infinity (e.g. we get close to the actual continuum WR statement) the approximate solution tend to the exact solution. This in fact is the case and a part of the proof has to do with WR statement that stated for infinite weights (all weights) the solution of the WR statement is in fact the exact solution. So as n (number of unknowns/weights) increases in numerical setting we approach continuum WR solution which is nothing but the exact solution to the problem!

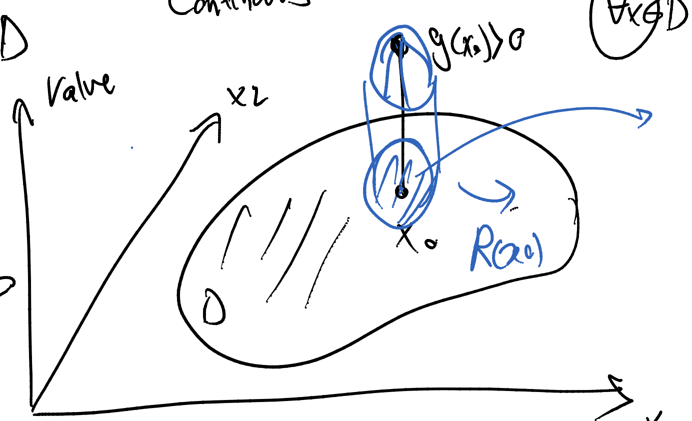
How WR => exact solution proof works

$$\forall w \quad \int_D w(x)\,\underbrace{g(x)}_{\text{continuous}}\,dx = 0 \quad \Rightarrow \quad g(x) = 0 \quad \forall x \in D$$

assume $g(x_0) \neq 0$

$g(x_0) > 0$

try to make a contradiction



$$w = \begin{cases} 1 & x \in R_{G_0} \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_D w(x) g(x) =$$

$$\int 1 . g(x) dv > 0$$

a contradiction

$$\int_D 1 \cdot g(x)\,dV > 0$$

$R(x_0) > 0$

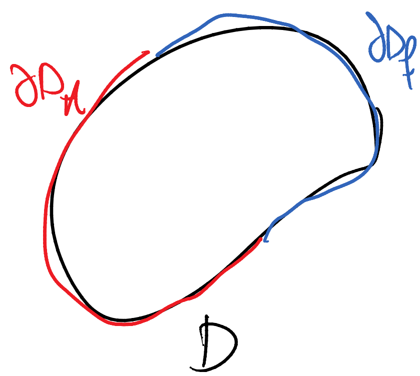
$$\forall w \int_D w(x)\, g(x)\,dx = 0 \implies g(x) = 0$$

continuous

$$\forall w \int_D w(x) \underbrace{R_i(u(x))}_{g(x)}\,dV$$

$$+ \int_{\partial D_u} w(x) R_u(u(x))\,dS$$

$$+ \int_{\partial D_f} w(x) R_f(u(x))\,dS = 0$$



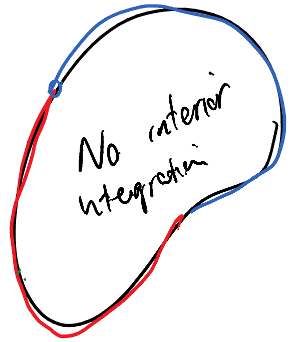
1. choose functions w that are 0 on ∂D_u & ∂D_f to get rid of 2nd & 3rd lines

$w = 0$ on ∂D

$$\forall w \in W_{0} \int_D w \underbrace{R_i(u(x))}_{g(x)}\,dV = 0 \implies \boxed{R_i(u(x)) = 0}$$

$$\implies \forall w \int_{\partial D_u} w\, R_u(u(x))\,dS$$

$$+ \int_{\partial D_u}^{} \ \ \int_{\partial D_f} w \, R_f(u(x))\, ds = 0$$



Now we focus on weights that are zero only on ∂D_f

And show Rf is zero in a similar fashion.

Many different numerical methods are formulated by choosing which errors (residuals) are satisfied strongly and which ones are satisfied weakly.

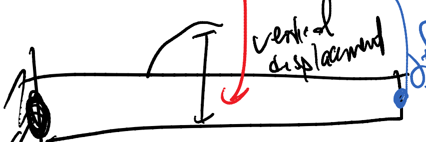
For example in conventional FEMs we
- Strongly satisfy Essential BCs
- Weakly (meaning by WR method) satisfy Natural BC and PDE

Motivation:
Why in conventional FEMs we strongly satisfy essential BC?

Example from beam problem

residual

Term	Domain	Weight function	Weight order	Residual function	Residual order
Interior	$\partial\mathcal{D}$	w energy dual	0	$\mathcal{R}_i = \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - q$	4
Natural Boundary	$\partial\mathcal{D}_f$	$\mathrm{w}_f = \begin{bmatrix} -w' \\ w \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\mathcal{R}_f = \begin{bmatrix} \bar{M} - M(y) \\ \bar{V} - V(y) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
Essential Boundary	$\partial\mathcal{D}_u$	$\mathrm{w}_u = \begin{bmatrix} -M(w) \\ V(w) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	$\mathcal{R}_u = \begin{bmatrix} \bar{y}' - y' \\ \bar{y} - y \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$y(0) = \bar{y}$

$\theta(0) = y'(0) = \bar{\theta}$

∂D_u

∂D_f

$x = L$

for example
for a cantilever beam

$\bar{y} = 0 \quad \bar{\theta} = 0$

$\bar{M} = M = EI\frac{d^2y}{dx^2}$

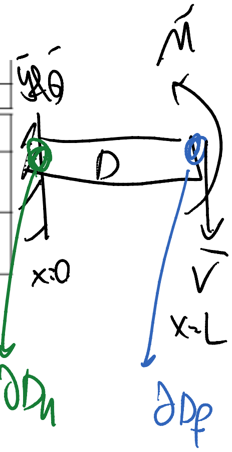
$\bar{V} = V = \frac{d}{dx}M = \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)$

$$V(w) = \frac{d}{dx}M(w) = \frac{d}{dx}EI\frac{d^2}{dx^2}w$$

$$M(w) = EI\frac{d^2 w}{dx^2}$$

Now that we have all the weights and residuals we can write the weighted residual statement

Term	Domain	Weight function	Weight order	Residual function	Residual order
Interior	$\partial\mathcal{D}$	w	0	$\mathcal{R}_i = \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - q$	4
Natural Boundary	$\partial\mathcal{D}_f$	$\mathbf{w}_f = \begin{bmatrix} -w' \\ w \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\mathcal{R}_f = \begin{bmatrix} \bar{M} - M(y) \\ \bar{V} - V(y) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
Essential Boundary	$\partial\mathcal{D}_u$	$\mathbf{w}_u = \begin{bmatrix} -M(w) \\ V(w) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	$\mathcal{R}_u = \begin{bmatrix} \bar{y}' - y' \\ \bar{y} - y \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$$WR = \int_0^L w(x)\left\{\overbrace{\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - q}^{R_i}\right\} dx$$

↳ D

$$+ \int_{\partial D_f} \left\{\begin{bmatrix} -w' \\ w \end{bmatrix} \cdot \begin{bmatrix} \bar M - M(y) \\ \bar V - V(y) \end{bmatrix}\right\}\Bigg|_{x=L}$$

to evaluate @ x = L

$$+ \int_{\partial D_u} \begin{bmatrix} -M(w) \\ V(w) \end{bmatrix} \cdot \begin{bmatrix} \bar y' - y' \\ \bar y - y \end{bmatrix}\Bigg|_{x=0} = 0$$

evaluation @ x = 0

$$\int_0^L \underset{\text{0 derivative}}{w} \left\{ \underbrace{\frac{d}{dx^2}\left(EI\frac{d^2y}{dx^2} + q\right)}_{\text{4th order derivative on } y} \right\} dx$$

$$+ -w'(x=L)\left(\bar M - EI\, y''(x=L)\right)$$

$$+ w(x=L)\left(\bar V - \frac{d}{dx} EI\, y''\right)(x=L)$$

$$+ W(x=L)\left(V - \frac{d}{dx} EI\, y''\right)(x=L)$$

$$-\left(\frac{d}{dx}\left(EI\, w''\right)\right)\Big|_{x=0} \left(\bar{y}' - y'\right)(x=0)$$

$$+ \left(EI\, w''\right)\Big|_{x=0} \left(\bar{y} - y(x=0)\right) = 0$$

orders of derivative

orders of derivatives

Term	Domain	Weight		Residual	
		function	order	function	order
Interior	$\partial\mathcal{D}$	w	0	$\mathcal{R}_i = \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - q$	4
Natural Boundary	$\partial\mathcal{D}_f$	$\mathbf{w}_f = \begin{bmatrix} -w' \\ w \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\mathcal{R}_f = \begin{bmatrix} \bar{M} - M(y) \\ \bar{V} - V(y) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
Essential Boundary	$\partial\mathcal{D}_u$	$\mathbf{w}_u = \begin{bmatrix} -M(w) \\ V(w) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	$\mathcal{R}_u = \begin{bmatrix} \bar{y}' - y' \\ \bar{y} - y \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

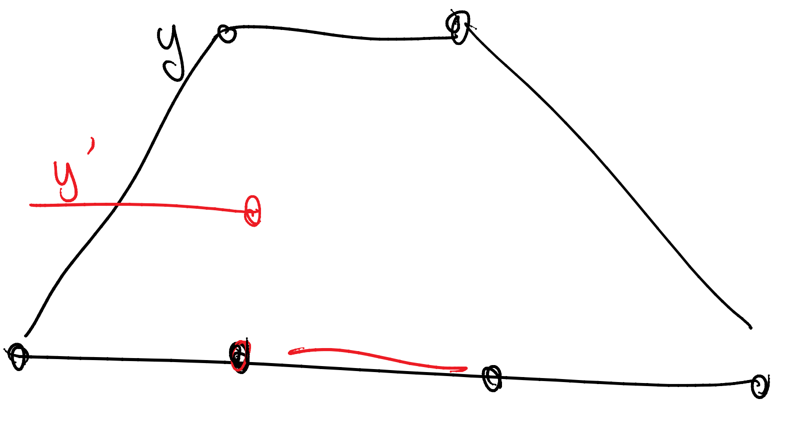
where $M(u) = EI\frac{d^2u}{dx^2}$ and $V(u) = \frac{dEI}{dx}\left(\frac{d^2u}{dx^2}\right)$.

$w, w', w'', w^{(3)}$ on boundary

Solution should have 4 derivative

$y, y', y'', y^{(3)}, y^{(4)}$

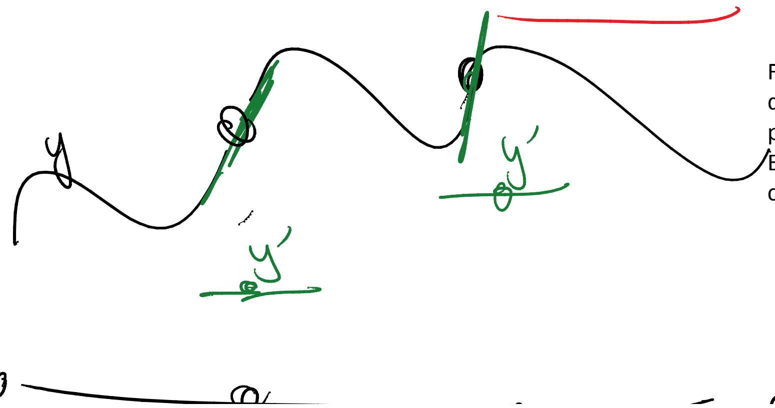
bar problem



The challenge of having high differentiation in WR is that eventually the functions that we choose to approximate the solution should have that many derivatives. However, these functions in FEMs are formed as piece-wise polynomial functions (one polynomial per element) which makes it difficult to accommodate having such high number of differentiation.

For example for a piece-wise linear solution we do not even have first derivative at all points and second derivative does not exist similarly.

beam problem



For the beam problem discussed above we need 4 derivatives on solution y which makes forming these piecewise polynomials even more difficult because at Element boundaries they need higher level of continuity

The challenge of having many derivatives for y and w is that forming functions that have so many derivatives in piecewise polynomial space that is used for FEM is very difficult!

How can we fix this problem?
Step 1. get red of essential BC in WR so that the order of derivatives of w decreases (we need to instead satisfy them strongly)

Term	Domain	Weight		Residual	
		function	order	function	order
Interior	$\partial\mathcal{D}$	w	0	$\mathcal{R}_i = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - q$	4
Natural Boundary	$\partial\mathcal{D}_f$	$\mathrm{w}_f = \begin{bmatrix} -w' \\ w \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\mathcal{R}_f = \begin{bmatrix} M - M(y) \\ \bar{V} - V(y) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
Essential Boundary	$\partial\mathcal{D}_u$	$\mathrm{w}_u = \begin{bmatrix} -M(w) \\ V(w) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	$\mathcal{R}_u = \begin{bmatrix} \bar{y}' - y' \\ \bar{y} - y \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

get rid of this row

Then we get

Dropping the WR term on Essential Boundary $\partial\mathcal{D}_u$.

We noticed that in the original statement $w \in C^3(\mathcal{D})$ because of the weighted residual term on the essential boundary $\partial\mathcal{D}_u$. Suppose we could modify the weighted residual to (dropping the integral on $\partial\mathcal{D}_u$).

$$\forall \mathrm{w} \in \mathcal{W}^{\mathrm{WRS}} : \int_{\mathcal{D}} \mathrm{w}.\mathcal{R}_i \,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathrm{w}.\mathcal{R}_f \,\mathrm{ds} = 0 \tag{42}$$

We will shortly discuss what the space $\mathcal{W}^{\mathrm{WRS}}$ would be. The weight and residual functions are:

Term	Domain	Weight		Residual	
		function	order	function	order
Interior	$\partial\mathcal{D}$	w	0	$\mathcal{R}_i = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - q$	4
Natural Boundary	$\partial\mathcal{D}_f$	$\mathrm{w}_f = \begin{bmatrix} -w' \\ w \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\mathcal{R}_f = \begin{bmatrix} M - M(y) \\ \bar{V} - V(y) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Solution

weight

R_i

$$\int_0^L w \left(\frac{d}{dx^2}\left(EI \frac{d y}{dx^2} + q \right) \right) dx$$

Step 2: Next class
Use integration by parts to transfer derivatives from solution y to weight w

0th order   4th order