2016/09/20

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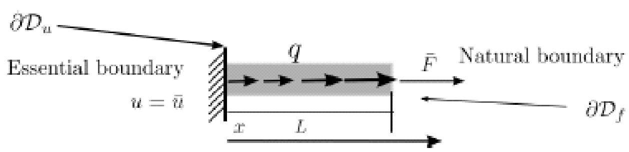
10:06 AM

Energy Method vs. Principle of Virtual Work

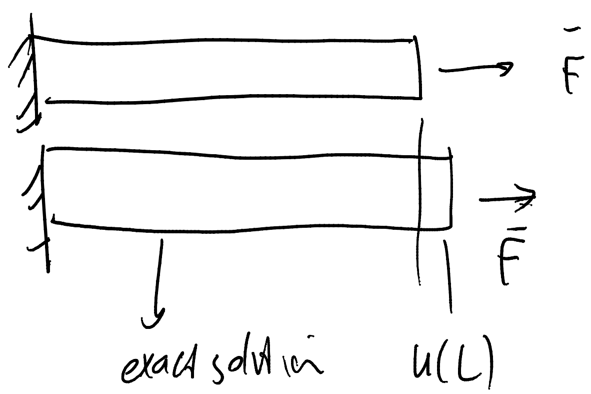
- Principle of virtual work or virtual displacement in solid mechanics states that if $\mathbf{u}$ is the solution to a boundary value problem, the virtual internal and external works produces by admissible virtual displacements $\delta\mathbf{u}$ are equal.
- Virtual displacements $\delta\mathbf{u}$ refer to displacements that are zero at essential boundary values (so that solution displacement plus virtual displacement $\tilde{\mathbf{u}} = \mathbf{u} + \delta\mathbf{u}$ (*cf.* (79)) as another admissible trial function also satisfies essential boundary conditions).
- Virtual Displacement/Virtual work is basically the equation we obtain by minimizing the energy function $\delta\Pi = 0$.
- Similar principles (virtual temperature for heat flow in solids and virtual velocities for fluid flow) are also directly derived from $\delta\Pi = 0$.
- While principle of virtual work can be obtained from $\delta\Pi = 0$, it is often quite easy to directly write and equate internal and external works for a given problem.



Equation (98) can be written as,

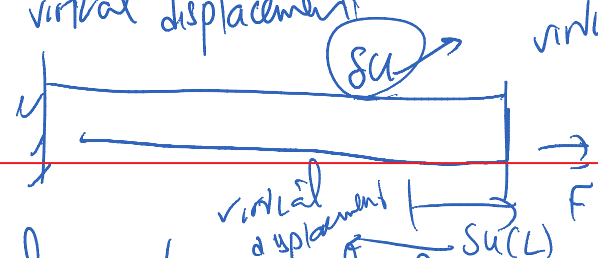
$$\text{Find } u \in \mathcal{V} = \{v \in C^1([0,L]) \mid v(0) = \bar{u}\}, \text{ such that,}$$
$$\forall \delta u \in \mathcal{W} = \{v \in C^1([0,L]) \mid v(0) = 0\}$$

$$\underbrace{\int_0^L \overbrace{\delta u'(x)}^{\frac{\mathrm{d}}{\mathrm{d}x}\delta u}\,\overbrace{EAu'(x)}^{F(u(x))}\,\mathrm{d}x}_{\text{Virtual Internal Work}} = \underbrace{\int_0^L \delta u(x) q(x)\,\mathrm{d}x + \delta u(L)\bar{F}}_{\text{Virtual External Work}} \tag{109}$$



exact solution $u(L)$

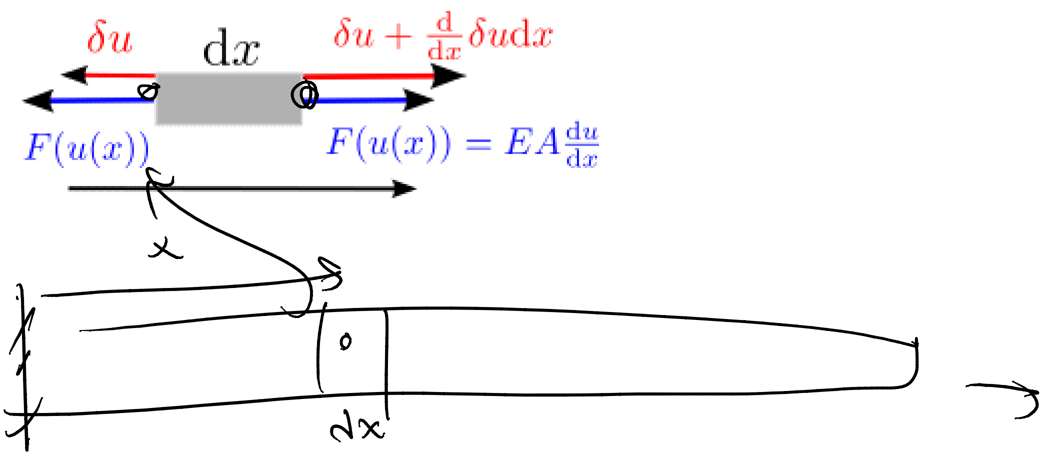
apply virtual displacement δu → virtual displacement



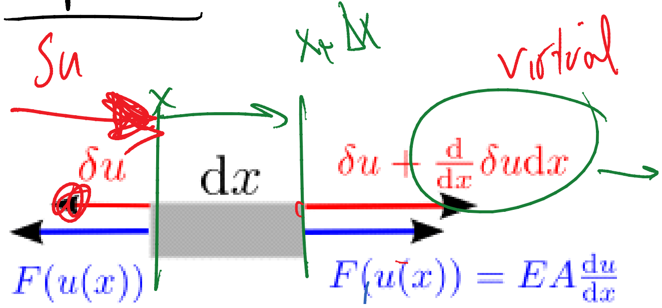
virtual external work = real force × virtual displacement

$$= F \cdot \delta u(L)$$

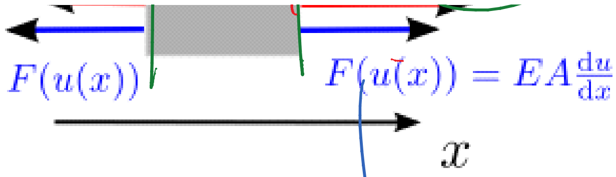
How about virtual internal work?



virtual internal work = work that the virtual displacement δu does on real forces:



virtual displacement increment to virtual displacement due to dx change in position



due to dx change in position

$$dW_{internal} = F(u(x))\left(\delta u + \frac{d\delta u}{dx}dx\right) + \left(-F(u(x))\right)(\delta u)$$

$$= F(u(x))\,\frac{d\delta u}{dx}dx$$

$$= \sigma(x)\underbrace{A}_{\text{area section}}\,\frac{d\delta u}{dx}dx$$

$$= \overbrace{\varepsilon(x)E}^{\sigma(x)}\,A\,\frac{d\delta u}{dx}dx$$

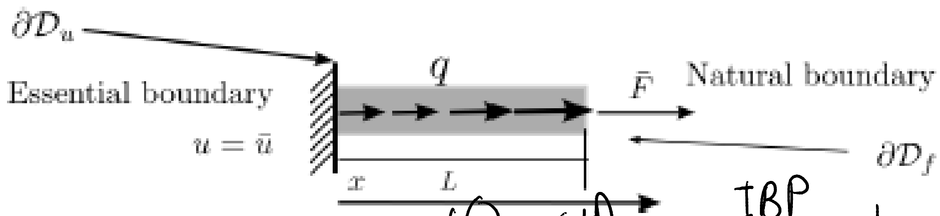
$$= \overbrace{\frac{du(x)}{dx}}^{\varepsilon(x)}\,EA\,\frac{d\delta u}{dx}dx$$

Virtual Internal work =

$$\int dW_{internal} = \int_0^L \delta u'\, EA\, u'(x)\, dx$$

Virtual External work $= \delta u(L)\cdot\bar{F}$

Principal of virtual work states that internal and external virtual works are the same



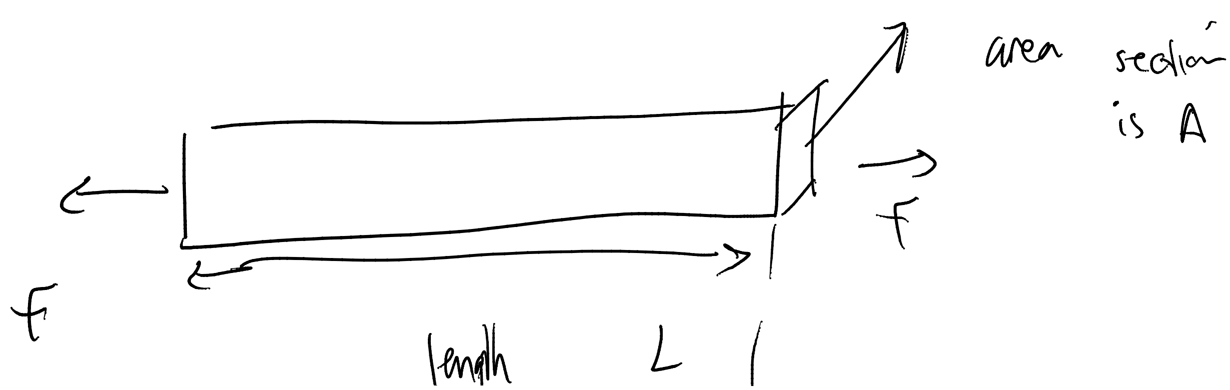
Weak statement ① WR $\xrightarrow{\text{IBP}}$ obtained weak statement
② Energy method δu increment of u $w = \delta u$

$$\boxed{\int_0^L \delta u' \, EA \, u' \, dx = \delta u(L) \cdot \bar{F}}$$

solution

virtual displacement (satisfies the homogeneous version of essential BC)

Physical interpretation of Finite Element Method (FEM)



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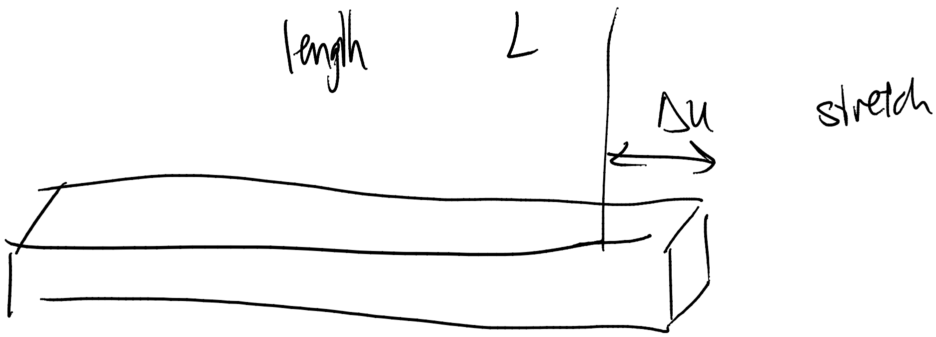


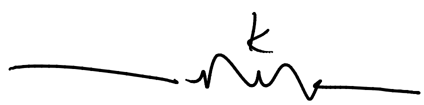
Δu $\quad\quad$ F

$$\sigma = \frac{F}{A}$$

$$\varepsilon = \frac{\sigma}{E} = \frac{F}{AE}$$ constant strain

$$\varepsilon = \frac{\Delta u}{L} \quad \Rightarrow$$

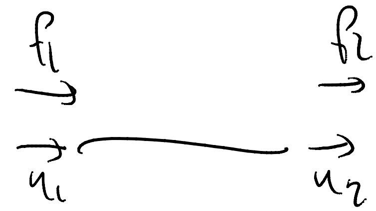
$$\frac{F}{AE} = \frac{\Delta u}{L} \quad \Rightarrow \quad \boxed{F = \overbrace{\underbrace{\frac{AE}{L}}_{\text{stiffness}}}^{k} \Delta u}$$



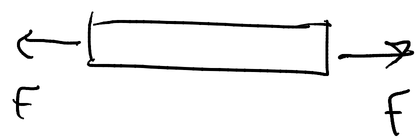
for spring $\quad F = k\Delta u$

$$\boxed{F = \overset{\text{stiffness}}{k}\ \Delta u \qquad k = \underline{AE}}$$ (form I) f_1 $\quad\quad$ f_2

$$\boxed{F = k\,\Delta u \qquad k = \frac{AE}{L}}$$ (form I)



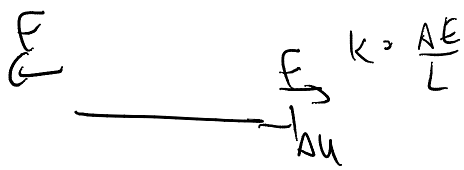
$$\boxed{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}$$ form II

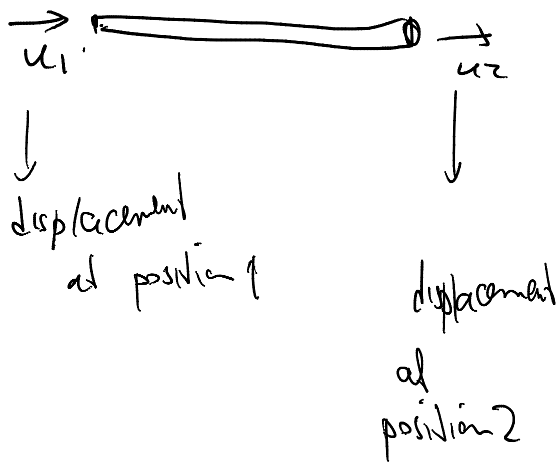


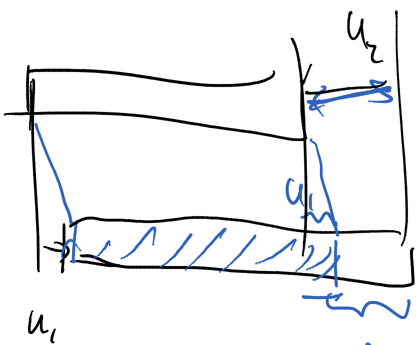
Why should we use form II and what is the advantage of this form?

$$F = k\,\Delta u$$

$$\Delta u = u_2 - u_1$$





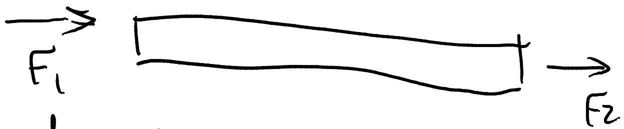


$$\Delta u = u_2 - u_1$$

$$F = k\,(u_2 - u_1) = k \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

F

endpoint forces



F_1 at point 1 $\qquad$ F_2 at point 2

$F_2 = F$

$F_1 = -F$

$F = k\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$

$\Rightarrow$

$F_1 = k\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$

$F_2 = k\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$

$\Rightarrow$

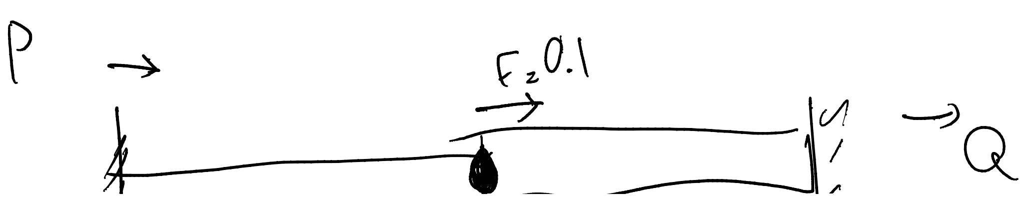
$$\boxed{\begin{bmatrix}F_1\\F_2\end{bmatrix} = k\underbrace{\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}}_{\text{stiffness matrix}}\begin{bmatrix}u_1\\u_2\end{bmatrix}}$$

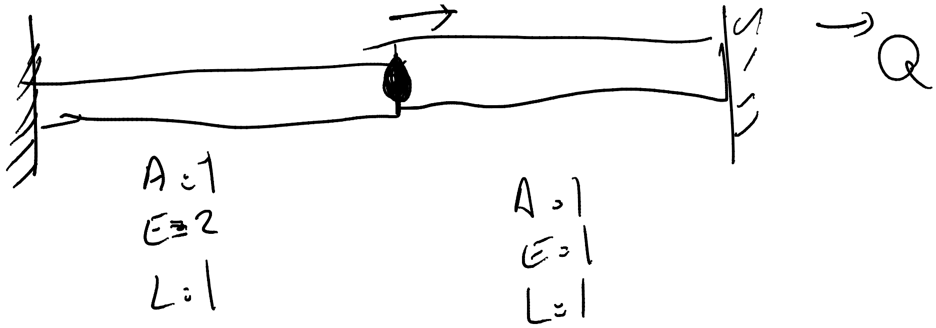
☆

given

what are the end point forces

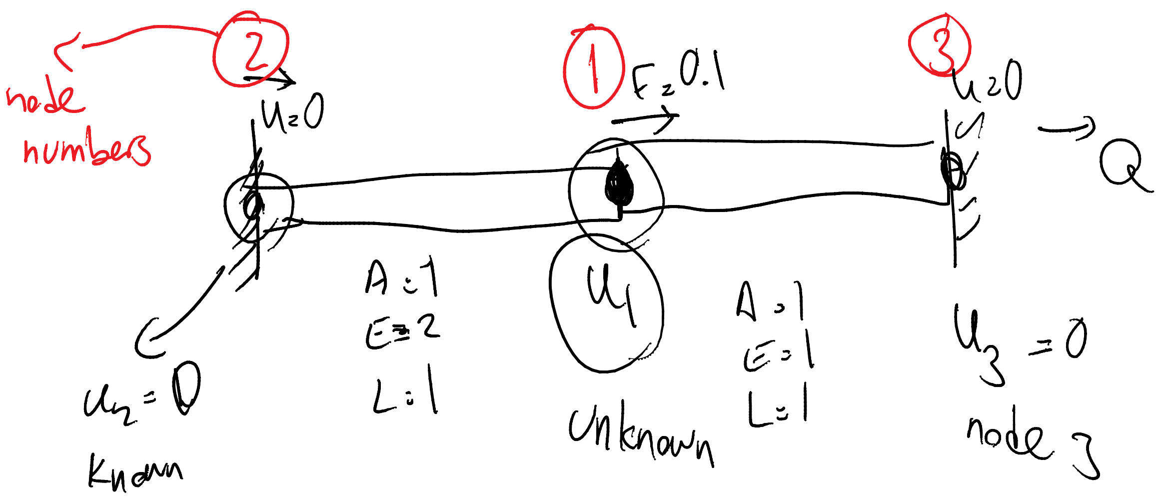
What is the use of this matrix equation?



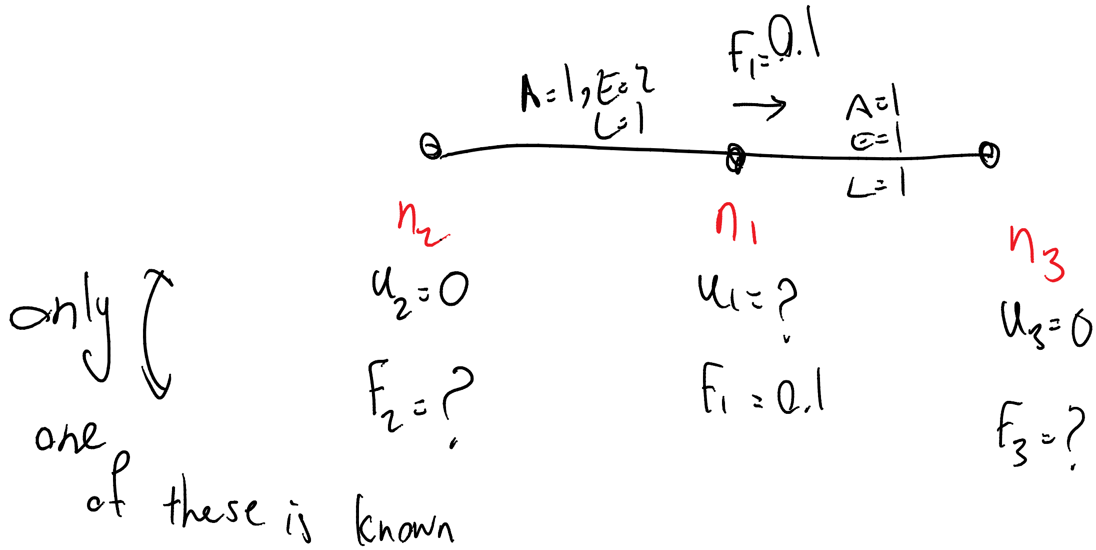


In FE for solid problems displacements are the unknowns of the problem.
After we solve the unknowns we can solve all the forces.

The fewer unknowns displacements, the fewer unknowns of FEM problem and the easier to solve the problem.



1. Number the nodes and elements
   To number nodes, we first number the unknowns then number then those with known displacement



We want to get to this stage:

$$(**)\quad \underbrace{\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}}_{\text{forces}} = \overbrace{\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}}^{\text{stiffness matrix}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\text{displacements}}$$

Forces = Stiffness matrix · displacements

Questions:

1. Can we use equation (*) to obtain equation (**)?

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \overset{\vec{u_1}}{\bullet}\xrightarrow{\hspace{2em}}\overset{\vec{u_2}}{} \quad \vec{F_1} \quad \vec{F_2}$$

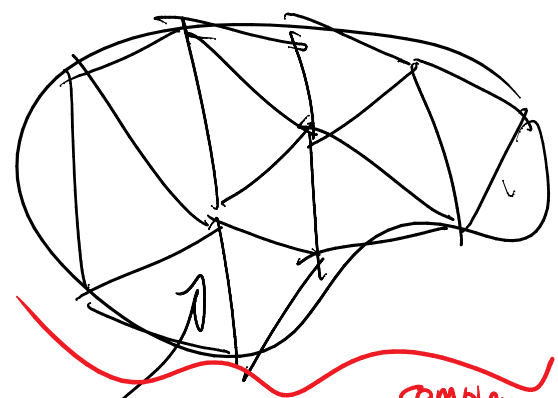
2. If we derive equation (**) how can we solve for the unknowns which are?

U_1, F_2, F_3

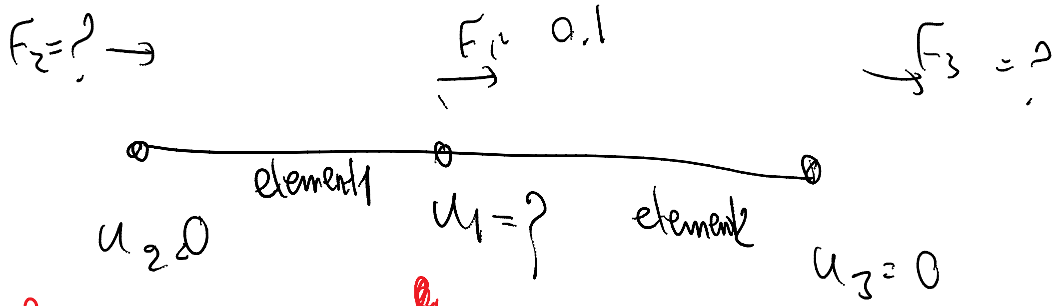
FEM idea

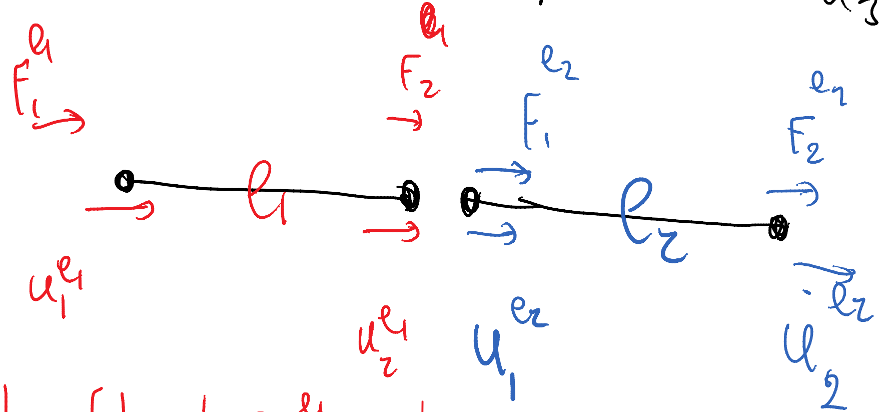
if we know how the force-displacement of a simple system are related

we can obtain force-displacement relation for a complex system



Answering Q1: How do we calculate the 3 x 3 stiffness matrix





$$\begin{bmatrix} F_1^{e_1} \\ F_2^{e_1} \end{bmatrix} = k^{e_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{e_1} \\ u_2^{e_1} \end{bmatrix}$$

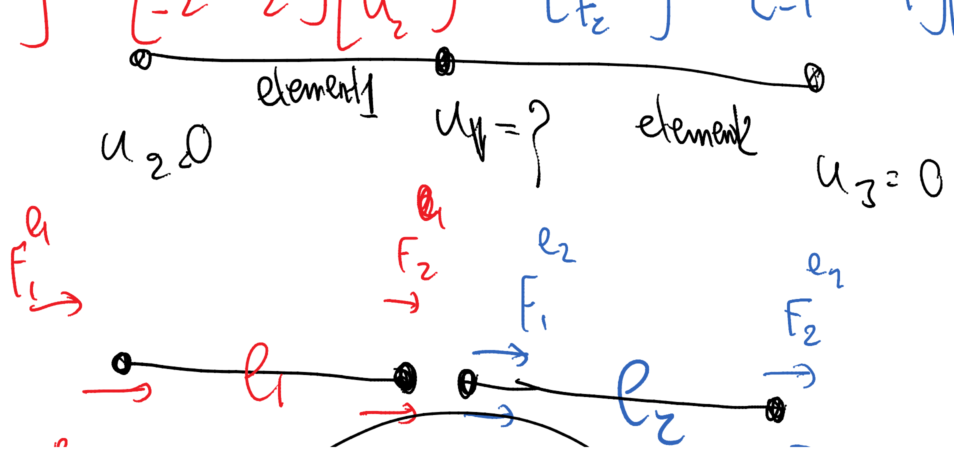
$$k^{e_1} = \frac{A^{e_1} E^{e_1}}{L^{e_1}} = \frac{1 \times 2}{1} = 2$$

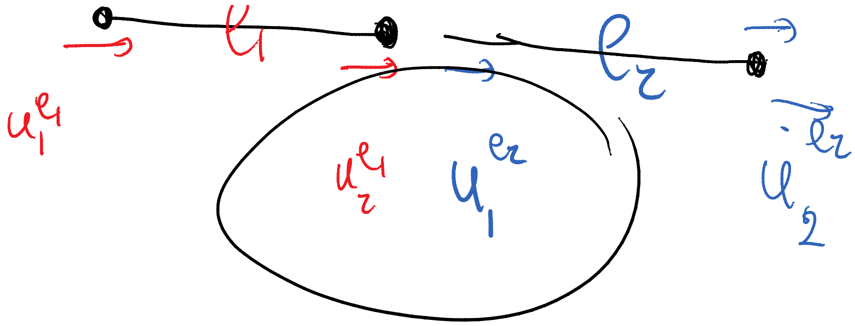
$$\begin{bmatrix} F_1^{e_1} \\ F_2^{e_1} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u_1^{e_1} \\ u_2^{e_1} \end{bmatrix}$$

$$\begin{bmatrix} F_1^{e_2} \\ F_2^{e_2} \end{bmatrix} = k^{e_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{e_2} \\ u_2^{e_2} \end{bmatrix}$$

$$k^{e_2} = \frac{A^{e_2} E^{e_2}}{L^{e_2}} = \frac{1 \times 1}{1} = 1$$

$$\begin{bmatrix} F_1^{e_2} \\ F_2^{e_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{e_2} \\ u_2^{e_2} \end{bmatrix}$$





Relation between $u_2^{e_1}$, $u_1^{e_2}$ & u_1 ?

They are the same

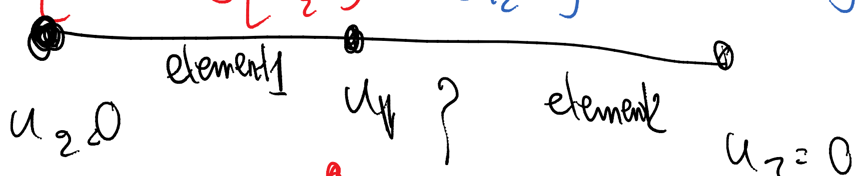
Relation between $F_2^{e_1}$, $F_1^{e_2}$, F_1

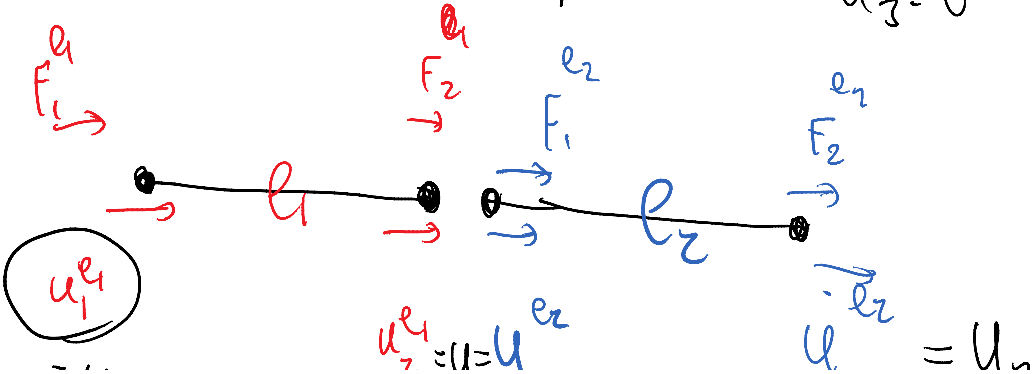
The forces from local elements add up together to give global system force

$$F_2^{e_1} + F_1^{e_2} = F_1$$

Using element matrices instead of F = k Delta u makes the "addition" of forces much easier.

$$\begin{bmatrix} F_1^{e_1} \\ F_2^{e_1} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u_1^{e_1} \\ u_2^{e_1} \end{bmatrix} \qquad \begin{bmatrix} F_1^{e_2} \\ F_2^{e_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{e_2} \\ u_2^{e_2} \end{bmatrix}$$





$$u_1^{e_1} = u_2 \qquad u_2^{e_1} = u = u_1^{e_2} \qquad u_2^{e_2} = u_3$$

$$F_1^{e_1} = F_2 \qquad F_2^{e_1} + F_1^{e_2} = F_1 \qquad F_2^{e_2} = F_3$$

$$F_1 = F_2^{e_1} + F_1^{e_2} = \ ?$$

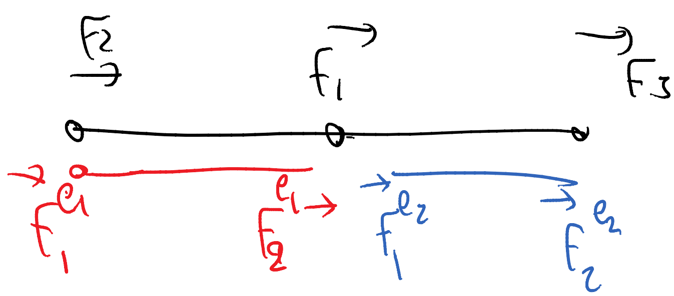
$$\begin{bmatrix} F_1^{e_1} \\ F_2^{e_1} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u_1^{e_1} \\ u_2^{e_1} \end{bmatrix} \qquad \begin{bmatrix} F_1^{e_2} \\ F_2^{e_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{e_2} \\ u_2^{e_2} \end{bmatrix}$$

$$\begin{cases} F_1^{e_1} = 2u_1^{e_1} - 2u_2^{e_1} = 2u_2 - 2u_1 \\ F_2^{e_1} = -2u_1^{e_1} + 2u_2^{e_1} = -2u_2 + 2u_1 \end{cases}$$

element ①

$$\begin{cases} F_1^{e_2} = u_1^{e_2} - u_2^{e_2} = u_1 - u_3 \\ F_2^{e_2} = -u_1^{e_2} + u_2^{e_2} = -u_1 + u_3 \end{cases}$$

Global forces



$$F_1 = F_2^{e_1} + F_1^{e_2} = (-2u_2 + 2u_1) + (u_1 - u_3) = 3u_1 - 2u_2 - u_3$$

$$F_2 = F_1^{e_1} = (2u_2 - 2u_1) = -2u_1 + 2u_2 + 0u_3$$

$$F_2 = F_1^{e_1} = (2u_2 - 2u_1) = -2u_1 + 2u_2 + 0u_3$$

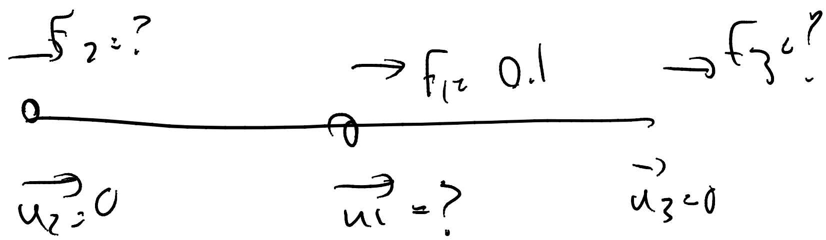
$$F_3 = F_2^{e_2} = (-u_1 + u_3) = -1u_1 + 0u_2 + 1u_3$$

0.1

$$\boxed{\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{matrix} \\ 0 \\ 0 \end{matrix}}$$

Unknown

unknowns



$$F_1 = 3 \cdot u_1 + \begin{bmatrix} -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} u_2 \to 0 \\ u_3 \to 0 \end{bmatrix}$$

$$F_1 = 3u_1 \implies \boxed{u_1 = \frac{0.1}{3} = 0.033}$$

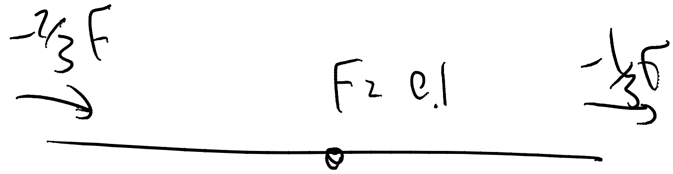
0.1

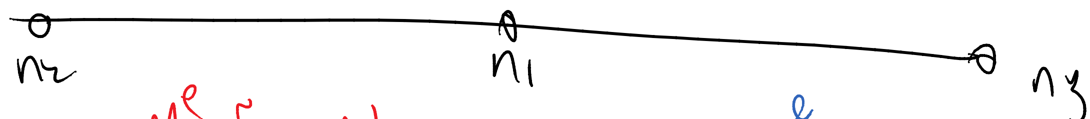
$$\begin{bmatrix} F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} u_1 + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_2 \to 0 \\ u_3 \to 0 \end{bmatrix}$$

$\frac{0.1}{3}$

$\overline{3}$

$$\boxed{\begin{aligned} F_2 &= -\frac{2}{3} \times 0.1 \\ F_3 &= \frac{-1}{3} \times 0.1 \end{aligned}}$$





$M^{e} = [2 \;\; 1]$

$$K^{e1} = \begin{array}{c|cc} & 2 & 1 \\ \hline 2 & 2 & -2 \\ 1 & -2 & 2 \end{array}$$

$M^{e2} = [1 \;\; 3]$

$$K^{e2} = \begin{array}{c|cc} & 1 & 3 \\ \hline 1 & 1 & -1 \\ 3 & -1 & 1 \end{array}$$

$$K = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1+2 & -2 & -1 \\ 2 & -2 & 2 & \\ 3 & -1 & & 1 \end{array}$$

$$K = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 2 & \\ -1 & & 1 \end{bmatrix}$$