2016/09/29

Thursday, September 29, 2016
10:10 AM

In all the following methods we want to obtain the stiffness K (n x n) and force vector (n x 1) and solve for the unknowns a (n x 1) from the system below

$$\boxed{\mathbf{Ka} = \mathbf{F}}$$

These methods provider different equations for Kij and Fi

For example on slides 104- 107 we obtain K and Fr for WR method

$$\mathbf{K} = \int_{\mathcal{D}} [\mathbf{w}].[L_M(\phi)]^{\mathrm{T}}\,\mathrm{dv} - \int_{\partial\mathcal{D}_f} [\mathbf{w}^f].[L_f(\phi)]^{\mathrm{T}}\,\mathrm{ds} \qquad (132a)$$

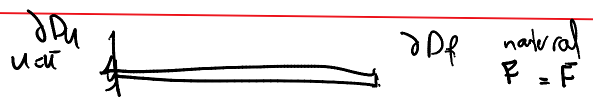
$$\mathbf{F} = \int_{\mathcal{D}} [\mathbf{w}].(\mathrm{r} - L_M(\phi_p))\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} [\mathbf{w}^f].(L_f(\phi_p) - \bar{\mathrm{f}})\,\mathrm{ds} \qquad (132b)$$

or alternatively the individual components are given,

$$K_{ij} = \int_{\mathcal{D}} \mathrm{w}_i L_M(\phi_j)\,\mathrm{dv} - \int_{\partial\mathcal{D}_f} \mathrm{w}_i^f L_f(\phi_j)\,\mathrm{ds} \qquad (133a)$$

$$F_i = \int_{\mathcal{D}} \mathrm{w}_i(\mathrm{r} - L_M(\phi_p))\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathrm{w}_i^f(L_f(\phi_p) - \bar{\mathrm{f}})\,\mathrm{ds} \qquad (133b)$$

The derivation of K and F for WR method:



PDE $R_i = L_M(u) - r = \underbrace{\frac{d}{dx}\left(EA\frac{d}{dx}\right)}_{L_M} u + q$

R_i: inside residual

$R_f = \bar{F} - F = \bar{F} - \underbrace{\left(AE\frac{d}{dx}\right)}_{L_f} u$

Now we do the derivation for general LM and Lf

$u = \underbrace{a_j \phi_j}_{\text{contribution from each } j} + \phi_p$

contribution from each j goes to COLUMN j of K

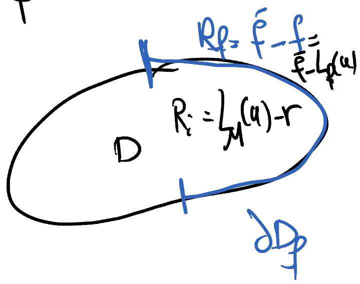
Contribution for ω_i (i'th weight function) goes to ROW i of K

i's weighted residual equation is obtained by replacing ω by ω_i

Find $\{a_1, \ldots, a_n\}$ such that

for $\omega \in \{w_1, \ldots, w_n\}$

$$\int_D \omega \, R_I(u) \, dV + \int_{\partial D_f} w_f \, R_f(u) \, ds = 0$$



equation # i

$$(*) \int_D w_i \underbrace{\left(L_M(u) - r\right)}_{R_I} + \int_{\partial D_f} w_i^f \underbrace{\left(\bar{f} - L_f(u)\right)}_{R_f} ds = 0$$

$$u = \phi_p + \phi_j a_j \quad \text{unknown}$$

(1) $R_I = L_M(u) - r = L_M(\phi_p + \phi_j a_j) - r$

$$= L(\phi_j) a_j + \{L(\phi_p) - r\}$$

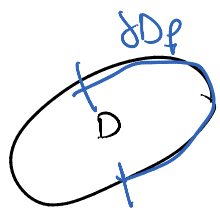
Assumes $L_M(af + bg) = a L_M(f) + b L_M(g)$

Linear Differential operator

(2) $R_f = \bar{f} - L_f(u) = \{\bar{f} - L_f(\phi_p)\} - a_j L_f(\phi_j)$

plug (1) & (2) in eqn (*) to get

$$\int_D W_i \underbrace{\left(\overbrace{L_M(\phi_j) a_j}^{\text{go to stiffness}} + L_M(\phi_p) - r\right)}_{R_I} +$$



$$+ \int_{\partial D_f} W_i^f \left(\overbrace{-L_f(\phi_j) a_j}^{\text{go to stiffness}} + \bar{f} - L_f(\phi_p)\right) ds = 0$$

K_{ij}

∂D_f

$$\boxed{\left\{\overbrace{\int_D w_i\, L_M(\phi_j)\,dV - \int_{\partial D_f} w_i^f\, L_f(\phi_j)\,ds}^{K_{ij}}\right\} a_j = \underbrace{\int_D w_i\,(r - L_M(\phi_p))\,dV - \int_{\partial D_f} w_i\,\bar{f}\,ds}_{F_i}}$$

$$K a = F$$

for $\quad K_{ij} = \int_D w_i\, L_M(\phi_j)\,dV - \int_{\partial D_f} w_i^f\, L_f(\phi_j)\,ds$

$$F_i = \int_D w_i\,(r - L_M(\phi_p))\,dV - \int_{\partial D_f} w_i\,\bar{f}\,ds$$

How to do this in a more concise manner?

$$\omega = \begin{bmatrix} \omega_1 \\ | \\ | \\ \omega_n \end{bmatrix}_{n\times 1}$$

$$L_M(u) = L_M(a_j\phi_j + \phi_p)$$

$$= L_M(\phi_j)\,a_j + L_M(\phi_p)$$

$$= [L_M(\phi_1) \cdots L_M(\phi_n)]\begin{bmatrix} a_1 \\ | \\ | \\ a_n \end{bmatrix} + L_M(\phi_p)$$

$$= L_M(\phi)^T a$$

$$L_M(\phi) = \begin{bmatrix} L_M(\phi_1) \\ | \\ | \\ L_M(\phi_n) \end{bmatrix}_{n\times 1}$$

$$a = \begin{bmatrix} a_1 \\ | \end{bmatrix}$$

$$\ldots_{n\times 1}$$

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

For example: R_I part

$$\int_D \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \left(L_M(u) - r \right) =$$

$$= \int_D w_{n\times 1} \left(\underbrace{\left\{ L_M^T(\phi) \right\}_{1\times n} a_{n\times 1}}_{[L_M(\phi_1) \cdots L_M(\phi_n)]} - r \right) dV$$

$$= \underbrace{\left(\int \underbrace{w_{n\times 1}}_{\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}} \left(L_M^T(\phi) \right)_{1\times n} dV \right)}_{n\times n \text{ matrix}} a_{n\times 1} - \underbrace{\left(\int w_{n\times 1}\, r \, dV \right)}_{n\times 1 \checkmark}$$

This is just a more concise way of finding the final form of the equations:

$$\begin{aligned} \mathcal{R}_i(\mathbf{u}^h) &= a_j L_M(\phi_j) + (L_M(\phi_p) - \mathbf{r}) = [L_M(\phi)]^{\mathrm{T}}[a] + (L_M(\phi_p) - \mathbf{r}) \\ \mathcal{R}_f(\mathbf{u}^h) &= -a_j L_f(\phi_j) + (\bar{\mathbf{f}} - L_f(\phi_p)) = -[L_f(\phi)]^{\mathrm{T}}[a] + (\bar{\mathbf{f}} - L_f(\phi_p)) \end{aligned} \tag{128}$$

- In (127) the row and column matrix product is:

$$[L_M(\phi)]^{\mathrm{T}}[a] = \left[L_M(\phi_1)\ L_M(\phi_2)\ \cdots\ L_M(\phi_n) \right] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_j L_M(\phi_j) \tag{129}$$

- We plug (127) into (126) to obtain,

$$\begin{aligned} &\text{Find } [a] = \left[a_1\ a_2\ \cdots\ a_n \right]^{\mathrm{T}} \text{ such that} \\ &\int_{\mathcal{D}} [\mathrm{w}].\left\{ [L_M(\phi)]^{\mathrm{T}}[a] + (L_M(\phi_p) - \mathbf{r}) \right\} \mathrm{dv} + \\ &\int_{\partial\mathcal{D}_f} [\mathrm{w}^f].\left\{ -[L_f(\phi)]^{\mathrm{T}}[a] + (\bar{\mathbf{f}} - L_f(\phi_p)) \right\} \mathrm{ds} = 0 \quad \Rightarrow \\ &\left\{ \int_{\mathcal{D}} [\mathrm{w}].[L_M(\phi)]^{\mathrm{T}} \mathrm{dv} - \int_{\partial\mathcal{D}_f} [\mathrm{w}^f].[L_f(\phi)]^{\mathrm{T}} \mathrm{ds} \right\} [a] = \\ &\left\{ \int_{\mathcal{D}} [\mathrm{w}].(\mathbf{r} - L_M(\phi_p))\, \mathrm{dv} + \int_{\partial\mathcal{D}_f} [\mathrm{w}^f].(L_f(\phi_p) - \bar{\mathbf{f}})\, \mathrm{ds} \right\} \end{aligned} \tag{130}$$

$[\mathrm{w}]$ and $[\mathrm{w}^f]$ correspond to vectors of weight functions on $\mathcal{D}$ and $\partial\mathcal{D}_f$:

$$[\mathrm{w}] = \begin{bmatrix} \mathrm{w}_1 \\ \vdots \\ \mathrm{w}_n \end{bmatrix} \quad \text{and} \quad [\mathrm{w}^f] = \begin{bmatrix} w_1^f \\ \vdots \\ w_n^f \end{bmatrix} \tag{131}$$

- According to equations (125), and (130), for a given ϕ_p the solution to the discrete weighted residual statement (126) is obtained from $\mathbf{Ka} = \mathbf{F}$:

$$\mathbf{K} = \int [\mathrm{w}].[L_M(\phi)]^{\mathrm{T}} \mathrm{dv} - \int [\mathrm{w}^f].[L_f(\phi)]^{\mathrm{T}} \mathrm{ds} \tag{132a}$$

$$\left(\int \begin{bmatrix} w_1 \\ \vdots \end{bmatrix} [L_M(\phi_1) \cdots L_M(\phi_n)] \right.$$ column 1

- According to equations (125), and (130), for a given ϕ_p the solution to the discrete weighted residual statement (126) is obtained from $\mathrm{K}a = \mathrm{F}$:

$$\mathrm{K} = \int_{\mathcal{D}} [\mathrm{w}].[L_M(\phi)]^{\mathrm{T}}\,\mathrm{dv} - \int_{\partial\mathcal{D}_f} [\mathrm{w}^f].[L_f(\phi)]^{\mathrm{T}}\,\mathrm{ds} \tag{132a}$$

$$\mathrm{F} = \int_{\mathcal{D}} [\mathrm{w}].(\mathrm{r} - L_M(\phi_p))\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} [\mathrm{w}^f].(L_f(\phi_p) - \bar{\mathrm{f}})\,\mathrm{ds} \tag{132b}$$

or alternatively the individual components are given,

$$K_{ij} = \int_{\mathcal{D}} \mathrm{w}_i L_M(\phi_j)\,\mathrm{dv} - \int_{\partial\mathcal{D}_f} \mathrm{w}_i^f L_f(\phi_j)\,\mathrm{ds} \tag{133a}$$

$$F_i = \int_{\mathcal{D}} \mathrm{w}_i(\mathrm{r} - L_M(\phi_p))\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathrm{w}_i^f (L_f(\phi_p) - \bar{\mathrm{f}})\,\mathrm{ds} \tag{133b}$$

$$\int \begin{bmatrix} \\ \\ w_n \end{bmatrix} [L_M(\phi_1) \cdots L_M(\phi_n)]$$ column

$$: \int \begin{bmatrix} w_1 L_M(\phi_1) & w_1 L_M(\phi_2) & \cdots & w_1 L_M(\phi_n) \\ w_2 L_M(\phi_1) & & & \\ & & w_i L_M(\phi_j) & \\ & & & w_n L_M(\phi_n) \end{bmatrix}$$

row i

$$\mathrm{F} = \int_{\mathcal{D}} [\mathrm{w}].(\mathrm{r} - L_M(\phi_p))\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} [\mathrm{w}^f].(L_f(\phi_p) - \bar{\mathrm{f}})\,\mathrm{ds}$$

1 → How essential BC on ∂D_u causes force

2 → Natural BC contribution ∂D_f

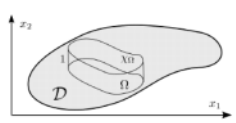
3 → Contribution from source term

Different choices of weight function in WR method creates many different numerical methods!

Overview of the choices:

1. Subdomain method

Weighted Residual Method: Subdomain Method



- We define the characteristic function for the set $\Omega \subset \mathcal{D}$

$$\chi_\Omega(\mathrm{x}) = \begin{cases} 1 & \mathrm{x} \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{134}$$

- In subdomain method, the weight functions are n characteristic functions for sets $\Omega_1, \ldots, \Omega_n$:

$$[\mathrm{w}] = \begin{bmatrix} \chi_{\Omega_1} \\ \chi_{\Omega_2} \\ \vdots\, \chi_{\Omega_n} \end{bmatrix} \tag{135}$$

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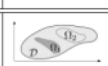
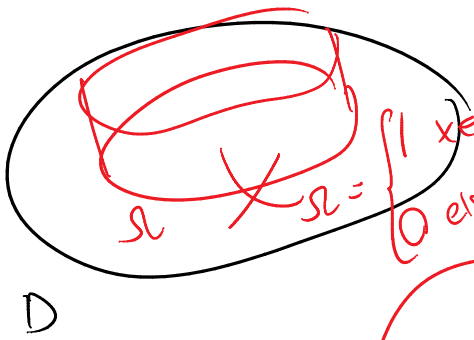
It basically satisfies the balance law for n distinct Ω_1 to Ω_n

Approach	Equation	Figure	Discretization	Discretization method
Balance Law (20)	$\forall \Omega \subset \mathcal{D}: \int_{\partial\Omega} (\mathrm{f.n})\mathrm{ds} - \int_\Omega \mathrm{r}\,\mathrm{dv} = 0$		Change $\forall \Omega$ to $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$	Similar to subdomain method in WRM



$\chi_\Omega = \begin{cases} 1 & x \in \Omega \\ 0 & \text{elsewhere} \end{cases}$

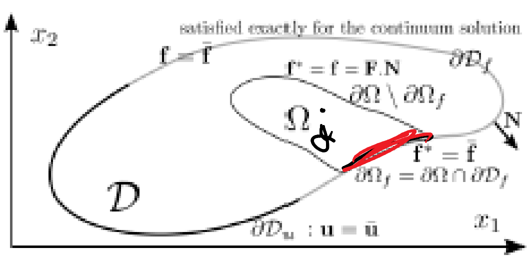
$$\int_D \chi_\Omega R_i \, dv = \int_\Omega R_j \, dv$$

This method only changes the domains of integration from D(∂Df) to

- Based on (134) and (133) the component expressions are:

$$K_{ij} = \int_{\Omega_i} L_M(\phi_j)\,\mathrm{dv} - \int_{\partial(\Omega_i)_f} L_f(\phi_j)\,\mathrm{ds} \tag{137a}$$

$$F_i = \int_{\Omega_i} \mathbf{w}_i(\mathbf{r} - L_M(\phi_\mathbf{p}))\,\mathrm{dv} + \int_{\partial(\Omega_i)_f} (L_f(\phi_\mathbf{p}) - \bar{\mathbf{f}})\,\mathrm{ds} \tag{137b}$$

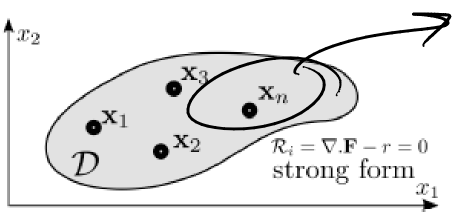


$\partial\Omega_{f_j} = \partial\Omega_j \cap \partial D_f$

2nd choice
Collocation method

We satisfy the PDE & natural BC at n points (n was the number of unknowns)

Weighted Residual Method: Collocation Method



n # to satisfy
the PDE
&/or
natural BC

Eqn # i for $x_i \in D$

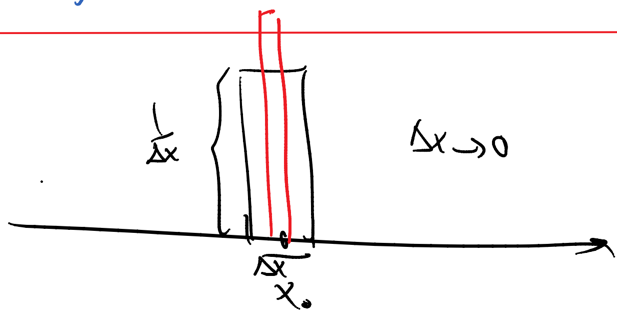
$$R_I(u, x_i) = 0$$

$$\left(L_M(\underbrace{\phi_j a_j + \phi_p}_{u}) - r\right)(x_i) = 0$$

$$L_M(\phi_j(x_i))\, a_j + L_M(\phi_p)(x_i) - r(x_i) = 0$$

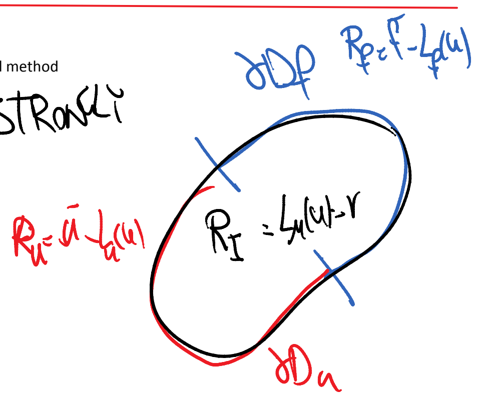
$$K_{ij}\, a_j = F_i \qquad F_i = r(x_i) - L_M(\phi_p)(x_i)$$

Delta function



How to show that a least square method is a weighted residual method

Let's satisfy Essential BC STRONGLY



$$R^2 = \int_D R_I^2(x)\, dV + \int_{\partial D_f} R_f^2(x)\, ds$$

$u =$ exact solution

$$R^2(u) = 0$$

should satisfy essential BC

we have $\underset{\text{exact}}{0 = R^2(u)} \leq R^2(\tilde{u})$

Discrete setting

$$\underset{\text{exact}}{\tilde{u}} = \phi_p + \phi_j \tilde{a}_j$$

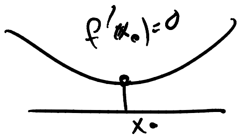
$$U = \phi_p + \phi_j a_j$$

$$R^2(\tilde{u}) = R^2(\tilde{a}_1, \dots, \tilde{a}_n)$$

discrete solution $\quad R^2(u) = R^2(a_1, \dots, a_n)$

Best solution (discrete solution) is when $R^2(a_1 \dots a_n)$ is minimum

$$\underbrace{R^2(a_1, \dots, a_n)}_{\text{Minimum value}} \leq R^2(\tilde{a}_1, \dots, \tilde{a}_n)$$



n variables

$$\nabla R^2 = \begin{bmatrix} \frac{\partial R^2}{\partial a_1} \\ \vdots \\ \frac{\partial R^2}{\partial a_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R^2(a_1, \dots, a_n) = \int_D R_I^2(a_1, \dots, a_n)\, dV + \int_{\partial D_f} R_f^2\, ds$$

to get row i

$$\frac{\partial R^2}{\partial a_i} = \int_D \frac{\partial R_I^2}{\partial a_i}\, dV + \int_{\partial D_f} \frac{\partial R_f^2}{\partial a_i}\, ds$$

$$= \int \left(2 R_I \frac{\partial R_I}{\partial a_i} \right) dV + \int_{\partial D_f} 2 R_f \frac{\partial R_f}{\partial a_i}\, ds = 0$$

factor out the 2's

$$\int_D \frac{\partial R_I}{\partial a_i} R_I\, dV + \int_{\partial D_f} \frac{\partial R_f}{\partial a_i} R_f\, ds = 0$$

$$D \qquad \underbrace{\qquad}_{\omega_i} \qquad\qquad \partial D_f \qquad \underbrace{\qquad}_{\omega_{f_i}}$$

$$\frac{\partial R_I}{\partial a_i} = \frac{\partial}{\partial a_i}\left[L_M(a_j\phi_j+\phi_p) - r\right] =$$

$$L_M(\phi_j)\left(\frac{\partial a_j}{\partial a_i}\right) + \underbrace{\left(\frac{\partial L_M(\phi_p) - r}{\partial a_j}\right)}_{0}$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$= L_M(\phi_j)\,\delta_{ij} = L_M(\phi_i)$$

$$\frac{\partial R}{\partial a_i} = L_M(\phi_i)$$

Similarly $$\frac{\partial R_f}{\partial a_i} = \frac{\partial\left(\bar{f} - L_f(a_j\phi_j+\phi_p)\right)}{\partial a_i}$$

$$= -L_f(\phi_i)$$

$$\int_D \underbrace{\frac{\partial R_I}{\partial a_i}}_{\omega_i = L_M(\phi_i)} R_I \, dV + \int_{\partial D_f} \underbrace{\frac{\partial R_f}{\partial a_i}}_{\omega_{f_i} = -L_f(\phi_i)} R_f \, ds = 0$$

- The minimum condition for the solution $[\mathrm{a}](\mathbf{u}^h = [\phi]^{\mathrm{T}}[\mathrm{a}] + \phi_p)$ in (155) can be expressed as,

$$[\mathrm{a}] \text{ is minimizer } \Rightarrow \frac{\partial R^2}{\partial a_i} = 0 \Rightarrow$$

$$\int_{\mathcal{D}} 2\frac{\partial L_M(\mathbf{u}^h)}{\partial a_i}(L_M(\mathbf{u}^h) - \mathrm{r})\,\mathrm{dv} + \int_{\partial\mathcal{D}_f}(-2)\frac{\partial L_f(\tilde{\mathbf{u}}^h)}{\partial a_i}(\bar{\mathbf{f}} - L_f(\tilde{\mathbf{u}}^h))\,\mathrm{ds} = 0 \tag{156}$$

- Noting the linearity of L_M and L_f and $[\mathrm{a}](\mathbf{u}^h = [\phi]^{\mathrm{T}}[\mathrm{a}] + \phi_p)$ we observe,

$$L_M(\mathbf{u}^h) = a_i L_M(\phi_i) + \phi_p \Rightarrow \qquad \frac{\partial L_M(\mathbf{u}^h)}{\partial a_i} = L_M(\phi_i) \tag{157a}$$

$$L_f(\mathbf{u}^h) = a_i L_f(\phi_i) + \phi_p \Rightarrow \qquad \frac{\partial L_f(\mathbf{u}^h)}{\partial a_i} = L_f(\phi_i) \tag{157b}$$

- Equations (156) and (157) yield,

$$\forall \phi_i \in \mathcal{V}^h : \int_{\mathcal{D}} L_M(\phi_i) \underbrace{(L_M(\mathbf{u}^h) - \mathbf{r})}_{\mathcal{R}_i} \,\mathrm{dv} + \int_{\partial \mathcal{D}_f} (-L_f(\phi_i)) \underbrace{(\bar{\mathbf{f}} - L_f(\hat{\mathbf{u}}^h))}_{\mathcal{R}_f} \,\mathrm{ds} = 0 \tag{158}$$

- In comparison to (126) for the general statement of weighted residual methods we observe,

$$\forall \mathrm{w} \in \mathcal{W}^h : \int_{\mathcal{D}} \mathrm{w}.\mathcal{R}_i(\mathbf{u}^h) \,\mathrm{dv} + \int_{\partial \mathcal{D}_f} \mathrm{w}^f.\mathcal{R}_f(\mathbf{u}^h) \,\mathrm{ds} = 0$$

- Discrete Least Square problem for linear differential operators L_M and L_f is equivalent to a discrete weighted residual statement with the weight functions:

Weight functions corresponding to Least Square Method

$$\mathrm{w} = L_M(\phi) \tag{159a}$$

$$\mathrm{w}^f = (-L_f(\phi)) \tag{159b}$$

What are the terms of stiffness matrix?

- According to (159), discrete least square method corresponds to a weighted residual method with particular weight functions given therein. Accordingly, (132) and (133) take the form:

$$\mathbf{K} = \int_{\mathcal{D}} [L_M(\phi)].[L_M(\phi)]^{\mathrm{T}} \,\mathrm{dv} + \int_{\partial \mathcal{D}_f} [L_f(\phi)].[L_f(\phi)]^{\mathrm{T}} \,\mathrm{ds} \tag{160a}$$

$$\mathbf{F} = \int_{\mathcal{D}} [L_M(\phi)].(\mathbf{r} - L_M(\phi_p)) \,\mathrm{dv} - \int_{\partial \mathcal{D}_f} [L_f(\phi)].(L_f(\phi_p) - \bar{\mathbf{f}}) \,\mathrm{ds} \tag{160b}$$

or alternatively the individual components are given,

$$K_{ij} = \int_{\mathcal{D}} L_M(\phi_i) L_M(\phi_j) \,\mathrm{dv} + \int_{\partial \mathcal{D}_f} L_f(\phi_i) L_f(\phi_j) \,\mathrm{ds} \tag{161a}$$

$$F_i = \int_{\mathcal{D}} L_M(\phi_i)(\mathbf{r} - L_M(\phi_p)) \,\mathrm{dv} - \int_{\partial \mathcal{D}_f} L_f(\phi_i)(L_f(\phi_p) - \bar{\mathbf{f}}) \,\mathrm{ds} \tag{161b}$$