2016/10/20A

Thursday, October 20, 2016
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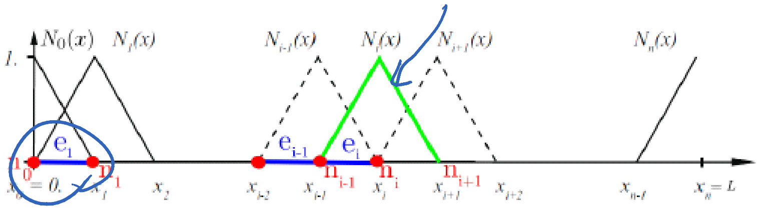
Our approach right now is global or shape function approach

We work with global shape functions Ni(x)



not directly working with elements

From last time

$$K = \int_{\text{Domain}} B^T D B \, dV$$

$D = EA$

$$B = L_m(N)$$

Solid bar $\quad L_m = \frac{d}{dx}$

$$B = \frac{d}{dx}\left[N_1 \cdots N_n\right] = \left[\underset{B_1}{N_1'} \cdots \underset{B_n}{N_n'}\right]$$

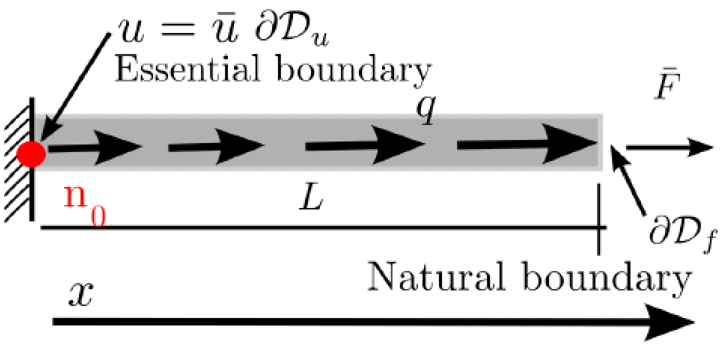
$B \quad \left[q_1 \right.$

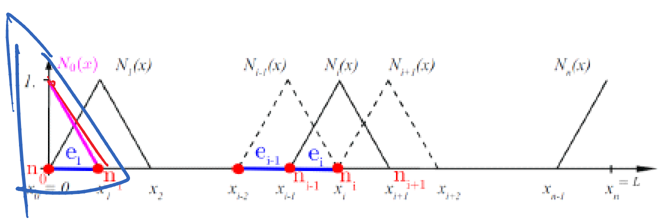
Remember

$$\underbrace{\varepsilon(x)}_{\text{strain}} = \overbrace{\underbrace{\begin{bmatrix} B_1 & \cdots & B_n \end{bmatrix}}_{\text{displacement to strain matrix}}}^{B} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{\text{nodal displacements}}$$

Forces from essential BC, natural BC, and source term (and concentrated forces)

B. Essential Boundary Conditions

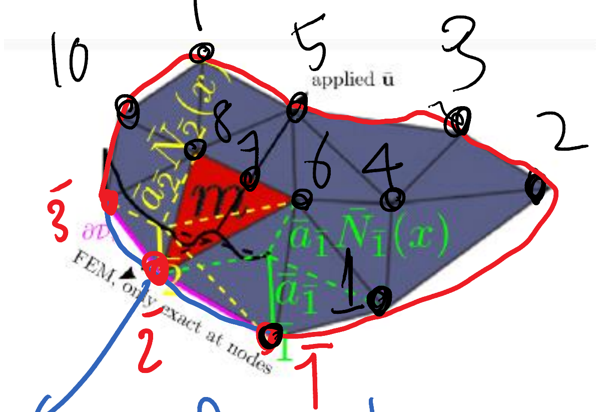




- We let, $\quad \phi_p = \bar{u}N_0 \quad \Rightarrow \phi_p(0) = \phi_p(x_0) = \bar{u}N_0(x_0) = \bar{u}\underbrace{\delta_{11}}_{1} = \bar{u}$ (307)
- It is clear that for all the trial functions (*i.e.*, shape functions corresponding to unknowns - $I \in \{N_1, \ldots, N_{n_f}\}$, $N_I(0) = N_I(x_0) = \delta_{I0} = 0$. That is, trial functions satisfy homogeneous essential boundary condition.

$$\phi_p = \bar{u} N_0 \qquad \phi_p(0) = \bar{u} N_0(x_0) = \bar{u} \cdot 1 = \bar{u}$$

Higher dimensions:



essential boundary

$\forall x \in \partial D_u \quad T(x) = \bar{T}(x)$

Thermal problem:

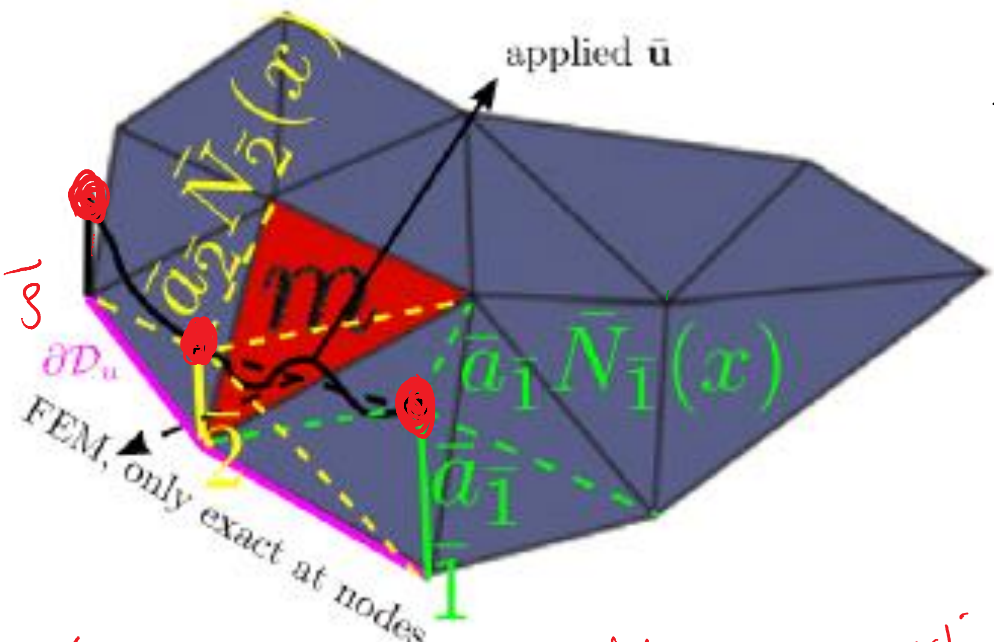
at each node has 1 temperature

$n_f = 10$: free dofs

corresponds to

$T_1, \dots T_{10}$

$$T^h(x) = \sum_{i=1}^{n_f} N_i(\vec{x})\, a_i + \phi_p(\vec{x})$$



$\forall x \in \partial D_u$

$\phi_p(x) = \bar{T}(x)$

in FEM ϕ_p matches prescribed values

ONLY at FE prescribed nodes:

$$N_{\bar{i}}(\bar{x}) = \begin{cases} 1 & \text{at node } \bar{1} \\ 0 & \text{at all the other nodes} \end{cases}$$

$$\phi_p(\vec{x}) = \sum_{i=1}^{n_p} a_{\bar{i}} N_{\bar{i}}(\bar{x})$$

n_p: # of prescribed dofs

$\bar{i}$: #'s with $(\bar{\ })$ correspond to prescribed dofs

$a_{\bar{i}}$: prescribed value at prescribed node # $\bar{i}$

$$\phi_p(n_{\bar{j}}) = \sum_{i=1}^{n_p} a_{\bar{i}} \underbrace{N_{\bar{i}}(n_{\bar{j}})}_{= \delta_{\bar{i}\bar{j}} = \begin{cases} 1 & \bar{i}=\bar{j} \\ 0 & \text{otherwise} \end{cases}}$$

n

$$= \delta_{ij} = \begin{cases} \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{i=1}^{n_p} a_i \delta_{ij} = a_j$$

becomes j

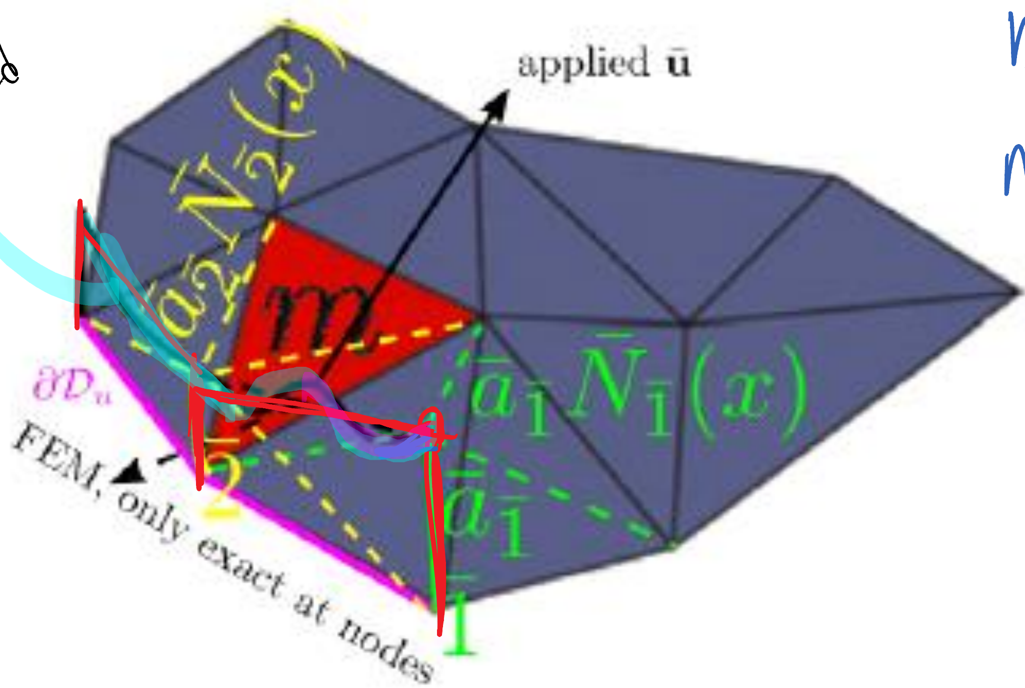
$$\boxed{\phi_p(n_j) = a_j}$$

the particular function matches the prescribed BC at ALL prescribed nodes.

- It may not match prescribed values between nodes exact prescribed boundary

$n_f = 10$

$n_p = 3$



Although the essential BC is not satisfied exactly (given that it only matches the prescribed values at FEM nodes) the error induced by this approximation is of the same order of magnitude of FEM discretization error and all the other errors that we will cover ->

This type of error is acceptable!

$$T(\vec{x}) = \underbrace{\sum_{i=1}^{n_f} a_i \overbrace{N_i(x)}^{\text{shape fun. @ free dofs}}} + \phi_p(x)$$

unknowns of the problem

n_p

$$\phi_p(x) = \sum_{\bar{i}=1}^{n_p} a_{\bar{i}} N_{\bar{i}}(\vec{x})$$

prescribed value at prescribed node # $\bar{i}$

$$T^h(x) = \sum_{i=1}^{n_f} a_i N_i(x) + \sum_{\bar{i}=1}^{n_p} a_{\bar{i}} N_{\bar{i}}(x)$$

$$= [N_1 \cdots N_{n_f}] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_f} \end{bmatrix} + [N_{\bar{1}} \cdots N_{n_p}] \begin{bmatrix} a_{\bar{1}} \\ \vdots \\ a_{\bar{n}_p} \end{bmatrix}$$

$$= N a + \underbrace{\bar{N} \bar{a}}_{\phi_p}$$

Form of F_D force from essential BC

n_p

BC

$I = 1, \dots n_f$

$$F_{D_I} = \mathcal{A}(N_I, \phi_p)$$

force from essential BC

$$= \mathcal{A}\left(N_I, \sum_{\bar{J}=1}^{n_p} a_{\bar{J}} N_{\bar{J}}(x)\right)$$

$$= \sum_{\bar{J}=1}^{n_p} \mathcal{A}(N_I, N_{\bar{J}})\, a_{\bar{J}}$$

$$\left(F_D\right)_{n_f \times 1} = \left(K_{fp}\right)_{n_f \times n_p} \bar{a}_{n_p \times 1}$$

$$K_{fp\, I\bar{J}} = \mathcal{A}(N_I, N_{\bar{J}})$$

$I = 1, \dots n_f$

N_I: free dofs

$N_{\bar{J}}$: prescribed

$I = 1 \ldots n_f$

$\bar{J} = 1 \ldots n_p$

free dofs

prescribed dofs

Remember

$K_{n_f \times n_f}^{in} \quad K_{ff} a = F$

was $\quad K_{IJ} = \mathcal{A}(N_I, N_J)$

I, J both go from 1 to n_f

Summary:

$$\underset{n_f \times n_f}{K} \; a_{n_f \times 1} = F_{n_f \times 1}$$

$$F = F_r + F_N - \underbrace{F_D}_{\text{essential BC}}$$

$$K_{n_f \times n_f} = \mathcal{A}\left(N^T_{n_f \times 1}, N_{1 \times n_f}\right) = \int_V B^T D B \, dV$$

$$F_D = K_{fp}(a_p)_{n_p}$$

$$\left(K_{fp}\right)_{n_f \times n_p} = \mathcal{A}\left(N^T_{n_f \times 1}, \bar{N}_{1 \times n_p}\right) = \int_V B^T D \bar{B} \, dV$$

Second force

$$\left(F_r\right)_I = \int_0^L N_I(x) \, q \, dx$$

q: source term

for the bar

for the bar

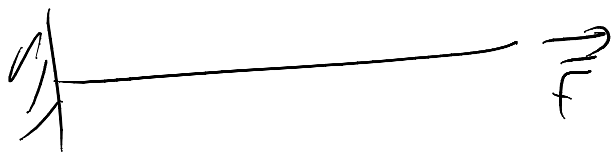
In general

$$\boxed{F_r = \int_V N^T r \, dV}$$

general expression for source term force

Force from natural BC

1D bar



natural/Neumann

$$(F_N)_I = \bar{F} \, N_I(x=L)$$

shape function

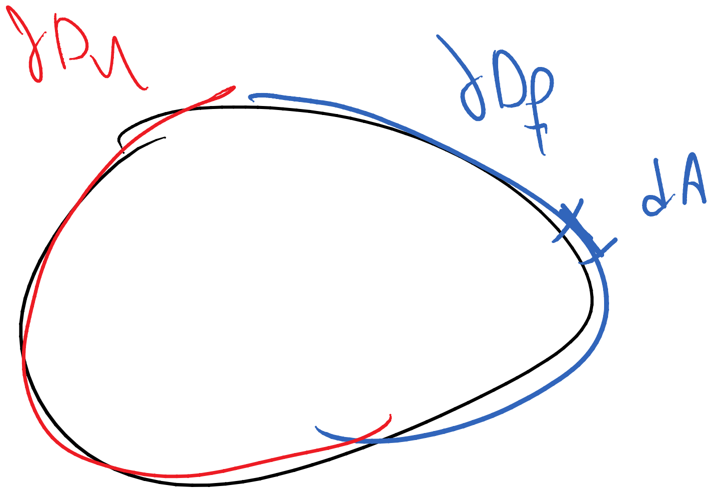
general expression

$$F_N = \int_{\partial\Omega_f} N^T \bar{f} \, dA$$

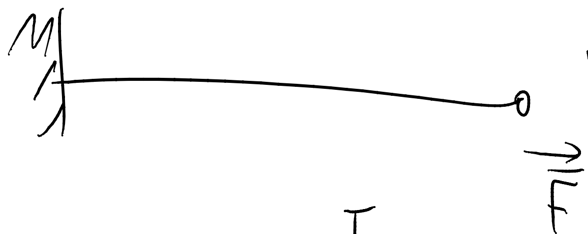
letter N

prescribed boundary condition

natural



1D



$\partial D_f = \{x = L\}$

$$\int_{\partial D_f} N^T \bar{F}\, dA = \underline{N^T(x=L)\,\bar{F}}$$

on boundary

$$Ka = F \qquad K = \int_V B^T D B \, dV$$

$$= \int_0^L \frac{dN^T}{dx} EA \frac{dN}{dx}\, dx \quad \text{1D bar}$$

$$F = F_r + F_N - F_D + F_n$$

nodal forces

source term

$$F_r = \int_V N^T r \, dV = \int_0^L N^T q \, dx \quad \text{1D bar}$$

Neumann

essential

Neumann natural BC

essential BC

$$F_N = \int_{\partial D_f} N^T \bar{f}\, dA = N^T(x=L)\,\bar{F}$$

1D bar

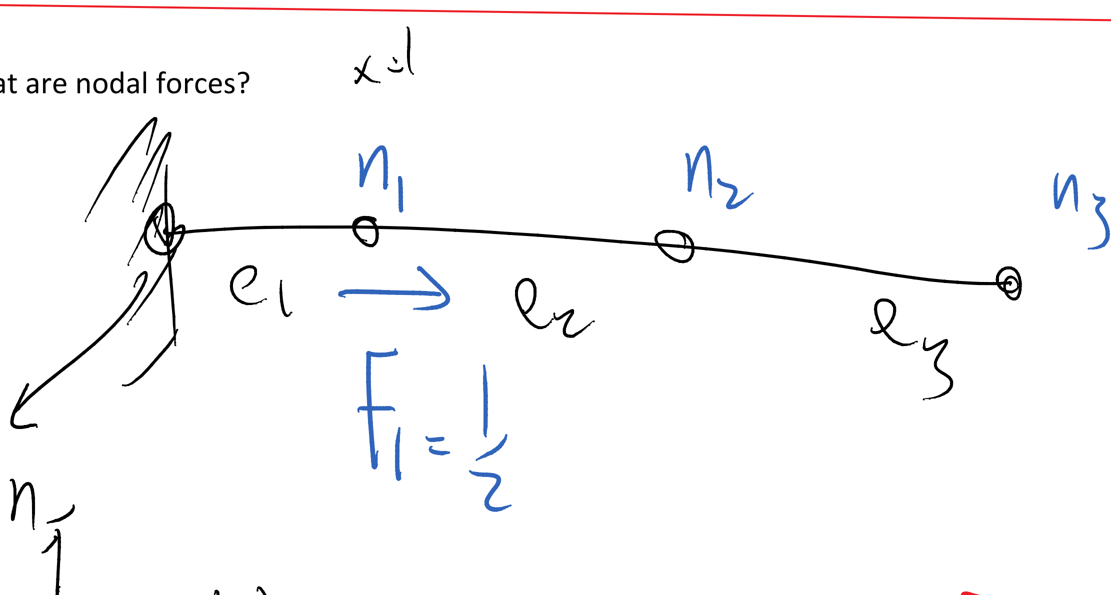
$$F_D = K_{fp}\,\bar{a}$$

$$K_{fp} = \int_V B^T D \bar{B}\, dV =$$

$$\int \left(\frac{d}{dx}N\right) EA \left(\frac{d}{dx}\bar{N}\right) dx$$

1D bar

What are nodal forces?



$$(K)_{3\times3}\, a_{3\times1} = F_r + F_N - F_D + F_{n\,3\times1}$$

3x1

$$\begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$$

How do we show it?

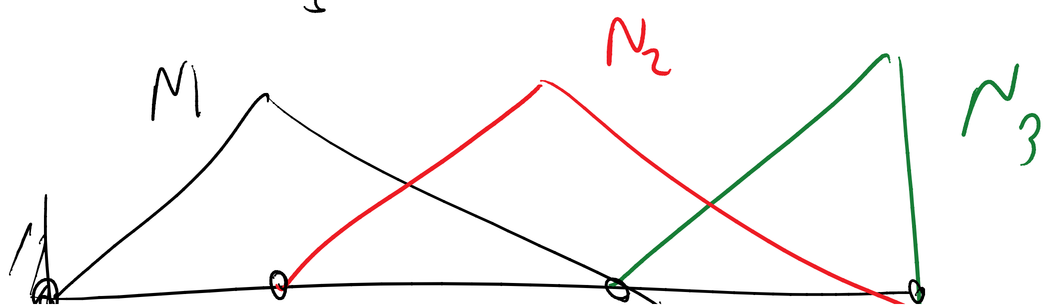
concentrated load if a source term in δ function form

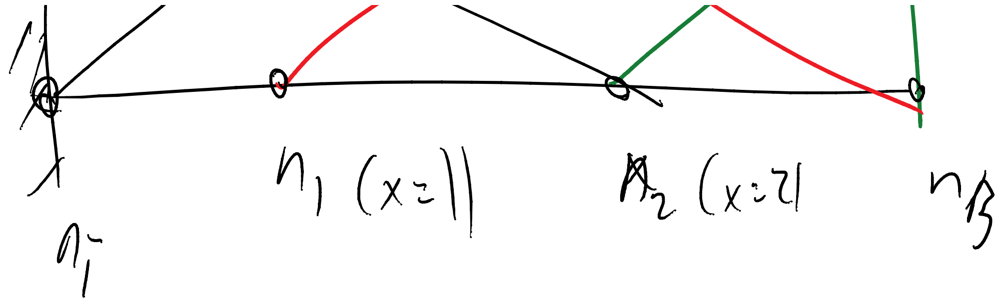
$$q = \delta(x-1)\,\frac{1}{2}$$

$$(F_r)_I = \int_0^L N_I(x)\, q \, dx$$

$$= \int_0^L N_I(x)\, \delta(x-1) \left(\frac{1}{2}\right) dx$$

$$N_I(\cancel{x}1)\,\frac{1}{2} = N_I(\text{node}1)\,\frac{1}{2} = \frac{1}{2}\,\delta_{1I}$$





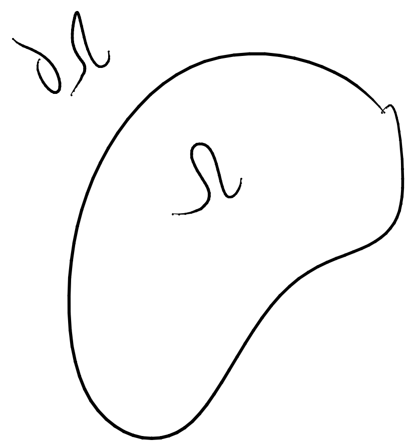
This way you show

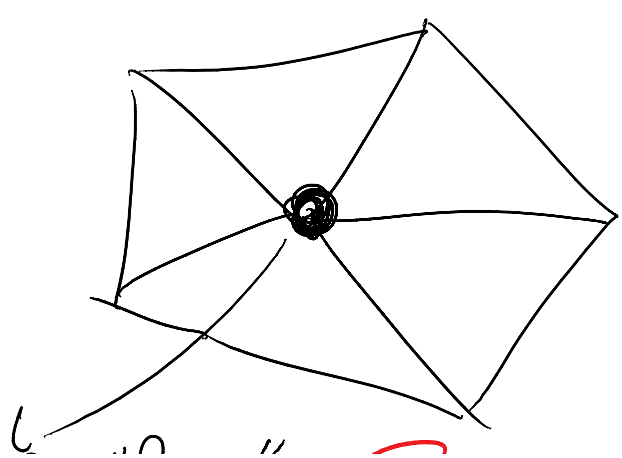
$$F_n = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

Thermal problem balance law

balance of energy

$$\int_{\partial\Omega} q \cdot ds - \int_{\Omega} Q \, dV = 0$$



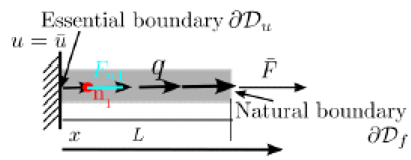


nodal "force" Concentrated version of Q " point-like input energy"

Summary: Force vectors

- Force vector is given by:

$$\boxed{\mathbf{F} = \mathbf{F}_r + \mathbf{F}_N + \mathbf{F}_n - \mathbf{F}_D} \qquad (311)$$



- $\mathbf{F}_r$, $\mathbf{F}_N$, $\mathbf{F}_n$ and $\mathbf{F}_D$ are given by (*cf.* (301) and (310))

$$\mathbf{F}_r = \left(\mathbf{N}^{\mathrm{T}}, q\right)_r = \int_{\mathcal{D}} \mathbf{N}^{\mathrm{T}} q \,\mathrm{dv} = \int_0^L \begin{bmatrix} N_1 \\ \vdots \\ N_{n_{\mathrm{f}}} \end{bmatrix} q \,\mathrm{d}x \qquad (312a)$$

$$\mathbf{F}_N = \left(\mathbf{N}^{\mathrm{T}}, \bar{F}\right)_N = \int_{\partial\mathcal{D}_f} \mathbf{N}^{\mathrm{T}} \bar{\mathbf{F}}.\cancel{\mathbf{N}} \,\mathrm{ds} = \left(\begin{bmatrix} N_1 \\ \vdots \\ N_{n_{\mathrm{f}}} \end{bmatrix} \bar{F} \right)_{x=L} \qquad (312b)$$

$$\mathbf{F}_D = \mathcal{A}\left(\mathbf{N}^{\mathrm{T}}, \phi_p\right) = \int_{\mathcal{D}} \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}^{\mathrm{T}} EA \frac{\mathrm{d}}{\mathrm{d}x}\phi_p \,\mathrm{dv} \qquad (312c)$$

$$= \left\{ \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} EA \bar{\mathbf{B}} \,\mathrm{dv} \right\} \bar{\mathbf{a}} = \left\{ \int_0^L \begin{bmatrix} B_1 \\ \vdots \\ B_{n_{\mathrm{f}}} \end{bmatrix} EA \left[\bar{B}_{\bar{1}} \quad \cdots \quad \bar{B}_{n_{\bar{\mathrm{p}}}}\right] \mathrm{d}x \right\} \begin{bmatrix} \bar{a}_{\bar{1}} \\ \vdots \\ \bar{a}_{n_{\bar{\mathrm{p}}}} \end{bmatrix} = \mathbf{K}_{fp}\bar{\mathbf{a}}$$

$$\mathbf{F}_n = \begin{bmatrix} F_{n\,1} \\ \vdots \\ F_{n\,n_{\mathrm{f}}} \end{bmatrix} \qquad (312d)$$

Force Essential Boundary Condition

- We have used (309) in (312c) to write,

$$\boxed{\mathbf{F}_D = \mathcal{A}\left(\mathbf{N}^{\mathrm{T}}, \phi_p\right) = \mathbf{K}_{fp}\bar{\mathbf{a}}} \tag{313}$$

- The prescribed to free stiffness matrix $\mathbf{K}_{fp}$ is an $n_{\mathrm{f}} \times n_{\mathrm{p}}$ matrix given by,

$$\mathbf{K}_{fp} = \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} EA\bar{\mathbf{B}}\,\mathrm{dv} = \int_0^L \begin{bmatrix} B_1 \\ \vdots \\ B_{n_{\mathrm{f}}} \end{bmatrix} EA \begin{bmatrix} \bar{B}_{\bar{1}} & \cdots & \bar{B}_{n_{\bar{\mathrm{p}}}} \end{bmatrix} \mathrm{d}x \tag{314}$$

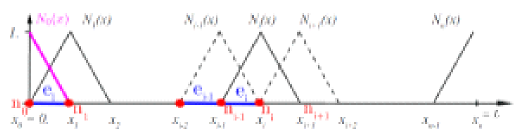
- From (306) we had,

$$\mathbf{K} = \mathcal{A}\left(\phi^{\mathrm{T}}, \phi\right) = \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} EA\mathbf{B}\,\mathrm{dv} = \int_0^L \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n_{\mathrm{f}}} \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & \cdots & B_{n_{\mathrm{f}}} \end{bmatrix} \mathrm{d}x$$

  where $\mathbf{K}$ was an $n_{\mathrm{f}} \times n_{\mathrm{f}}$ matrix.
- "Prescribed" dofs $\bar{i}$ do not go into $\mathbf{K}$ because their value $\bar{a}_{\bar{i}}$ are already known.
- This is opposite to dofs $I = 1, \ldots, n_{\mathrm{f}}$ which correspond to "free" dofs.

Force Essential Boundary Condition

For example for the problem in the figure $n_{\mathrm{p}} = 1$, $\bar{1} = 0$, $\bar{a}_{\bar{1}} = \bar{u} \Rightarrow$



$$\mathbf{F}_D = \bar{a}_{\bar{1}} \int_0^L \begin{bmatrix} B_1 \\ \vdots \\ B_{n_{\mathrm{f}}} \end{bmatrix} EA\bar{B}_{\bar{1}}(x)\,\mathrm{d}x = \bar{u}\int_0^L \begin{bmatrix} B_1 \\ \vdots \\ B_{n_{\mathrm{f}}} \end{bmatrix} EAB_0(x)\,\mathrm{d}x \tag{315}$$

$$F_D = \left(\int_0^L \begin{bmatrix} B_1 \\ \\ B_{n_f} \end{bmatrix} EA \begin{bmatrix} B_{\bar{1}} & B_{\bar{2}} & \cdots & B_{\bar{n}_p} \end{bmatrix} dx\right) \begin{bmatrix} a_{\bar{1}} \\ \vdots \\ a_{\bar{n}_p} \end{bmatrix}$$

$$\left.\begin{matrix} \\ 0 \end{matrix}\right. \left\lfloor B_{np} \right\rfloor \qquad \left\lfloor a_{np} \right\rfloor$$

$$f_D = \left(\int_0^L \begin{bmatrix} B_1 \\ \vdots \\ B_{np} \end{bmatrix} EA \, B_{\bar{I}} \right) a_{\bar{I}}$$

$$a_{\bar{I}} = \bar{u}$$

WR no boundary conditions on weight function

Weak form weight function satisfies homogeneous essential BC "