2016/10/25

Tuesday, October 25, 2016

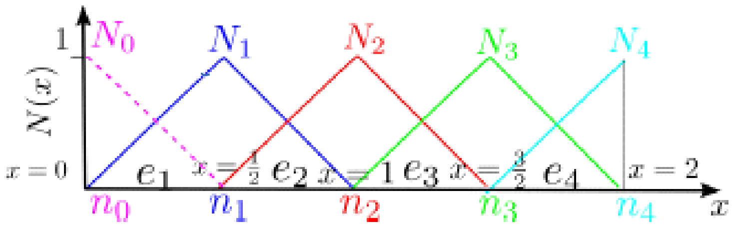
10:04 AM

Difference between the global and local approaches

Global approach is what we have covered so far

Finite Element Method: Global

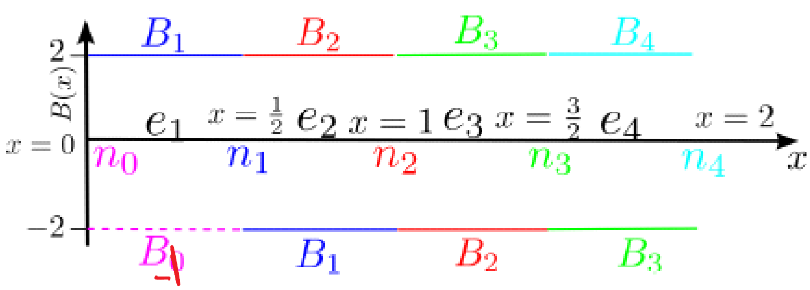
Global shape functions:



$$K = \int B^T D B \, dV$$

$$\mathbf{K} = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} \, \mathrm{d}x$$

$$B = \frac{d}{dx} N$$



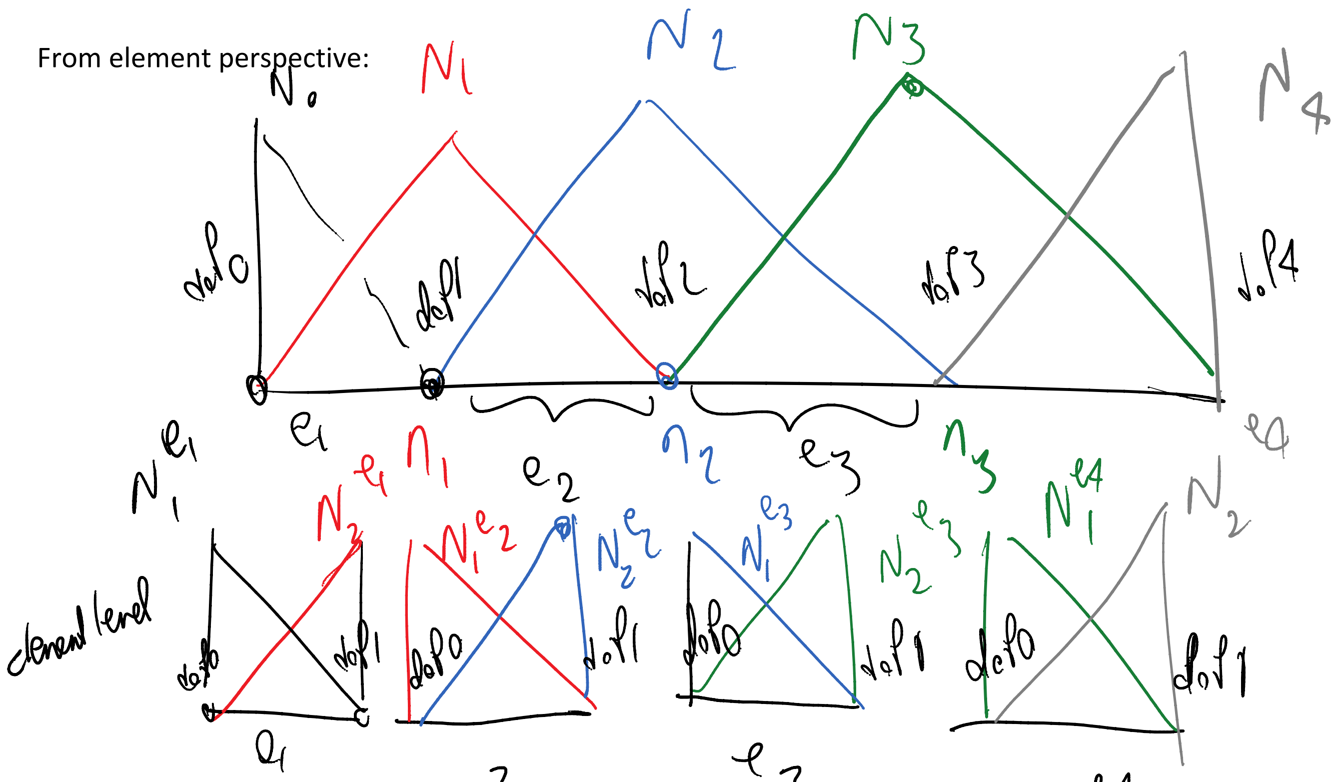
$$\mathbf{K} = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} \, \mathrm{d}x = \begin{bmatrix} \int_0^2 B_1 B_1 \, \mathrm{d}x & \int_0^2 B_1 B_2 \, \mathrm{d}x & \int_0^2 B_1 B_3 \, \mathrm{d}x & \int_0^2 B_1 B_4 \, \mathrm{d}x \\ & \int_0^2 B_2 B_2 \, \mathrm{d}x & \int_0^2 B_2 B_3 \, \mathrm{d}x & \int_0^2 B_2 B_4 \, \mathrm{d}x \\ \text{sym.} & & \int_0^2 B_3 B_3 \, \mathrm{d}x & \int_0^2 B_3 B_4 \, \mathrm{d}x \\ & & & \int_0^2 B_4 B_4 \, \mathrm{d}x \end{bmatrix}$$

$$= \begin{bmatrix} \int_{e_1} B_1 B_1 \, \mathrm{d}x + \int_{e_2} B_1 B_1 \, \mathrm{d}x & \int_{e_2} B_1 B_2 \, \mathrm{d}x & 0 & 0 \\ & \int_{e_2} B_2 B_2 \, \mathrm{d}x + \int_{e_3} B_2 B_2 \, \mathrm{d}x & \int_{e_3} B_2 B_3 \, \mathrm{d}x & 0 \\ \text{sym.} & & \int_{e_3} B_3 B_3 \, \mathrm{d}x + \int_{e_4} B_3 B_3 \, \mathrm{d}x & \int_{e_4} B_3 B_4 \, \mathrm{d}x \\ & & & \int_{e_4} B_4 B_4 \, \mathrm{d}x \end{bmatrix}$$

$$
= \begin{bmatrix} \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 & 0 \\ & \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 \\ \text{sym.} & & \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) \\ & & & \frac{1}{2}.(2).(2) \end{bmatrix} \Rightarrow
$$

$$
\mathbf{K} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ \text{sym.} & & 4 & -2 \\ & & & 2 \end{bmatrix}
$$

From element perspective:



$M^{e_1} = [0 \quad 1]$; $M^{e_2} = [1 \ 2]$ $\quad M^{e_3} = [2 \ 3]$ $\quad M^{e_4} = [3 \quad 4]$

Map of local to global dof

All the elements locally look the same

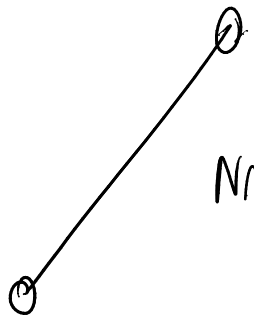
ELEMENT DOF MAP

Map from **LOCAL ELEMENT <u>DOFs</u> to GLOBAL <u>DOFs</u>**

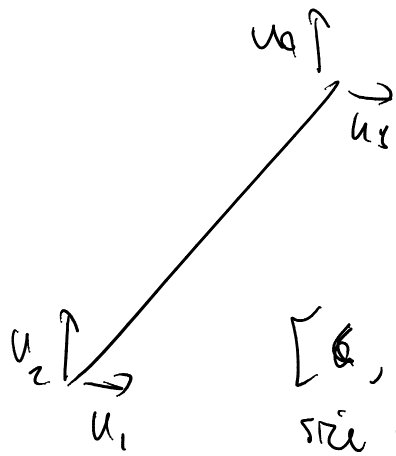
ELEMENT NODAL MAP

Map from **LOCAL ELEMENT <u>NODES</u> to GLOBAL <u>NODES</u>** NMAP

$NMAP^{e_1}: [0 \;\; 1]$ $\quad NMAP^{e_2}: [1 \;\; 2]$ $\quad NMAP^{e_3}: [2 \;\; 3]$ $\quad NMAP^{e_4}: [3 \;\; 4]$



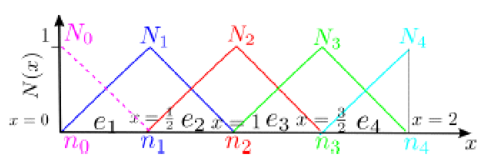
NMAP $: [a, b]$



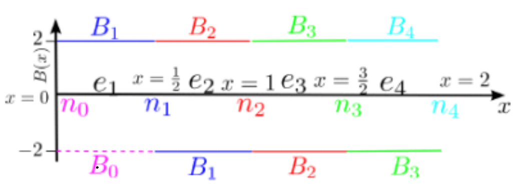
M:
$[c, d, e, f]$
size 4

- NMAP is used to create the geometry of the element
- M is used to transfer local to global stiffness matrix and force vector

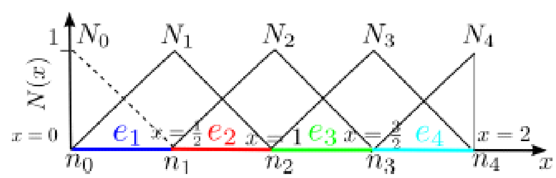
How the local (element-centered approach works)



$$\mathbf{K} = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} \, \mathrm{d}x$$



global (shape function centered)



$$\mathbf{K} = \int_0^2 \mathbf{B}^T EA\mathbf{B} \, \mathrm{d}x = \mathbf{K}^{e1} + \mathbf{K}^{e2} + \mathbf{K}^{e3} + \mathbf{K}^{e4} \quad \text{where}$$

$$\mathbf{K}^{e1} = \int_{e_1} \mathbf{B}^{\mathrm{T}} EA\mathbf{B} \, \mathrm{d}x \qquad \mathbf{K}^{e2} = \int_{e_2} \mathbf{B}^{\mathrm{T}} EA\mathbf{B} \, \mathrm{d}x$$

$$\mathbf{K}^{e3} = \int_{e_3} \mathbf{B}^{\mathrm{T}} EA\mathbf{B} \, \mathrm{d}x \qquad \mathbf{K}^{e4} = \int_{e_4} \mathbf{B}^{\mathrm{T}} EA\mathbf{B} \, \mathrm{d}x$$

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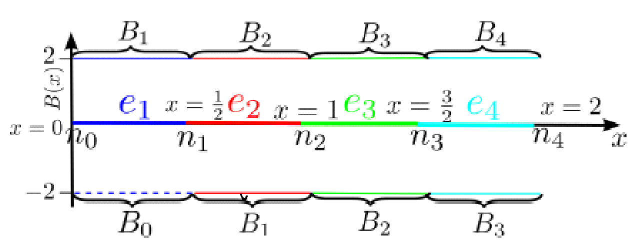


Continue with local approach:

$$\mathbf{K} = \int_0^2 \mathbf{B}^T EA\mathbf{B}\, dx = \overbrace{\int_{e_1} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e1}} + \overbrace{\int_{e_2} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e2}} + \overbrace{\int_{e_3} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e3}} + \overbrace{\int_{e_4} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e4}} = \tag{327}$$

$$K^{e_1} = \int \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} dx$$

$$= \int_{e_1} \begin{bmatrix} B_1 B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} dx$$

$$K^{e_2} = \int_{e_2} \begin{bmatrix} B_1 B_1 & B_1 B_2 & 0 & 0 \\ & B_2 B_2 & 0 & 0 \\ \text{sym} & & 0 & 0 \\ & & & 0 \end{bmatrix} dx$$

$$\mathbf{K} = \int_0^2 \mathbf{B}^T EA\mathbf{B}\, dx = \overbrace{\int_{e_1} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e1}} + \overbrace{\int_{e_2} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e2}} + \overbrace{\int_{e_3} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e3}} + \overbrace{\int_{e_4} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e4}} =$$

$$= \begin{bmatrix} \int_{e_1} B_1 B_1\, dx & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} \int_{e_2} B_1 B_1\, dx & \int_{e_2} B_1 B_2\, dx & 0 & 0 \\ & \int_{e_2} B_2 B_2\, dx & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \int_{e_3} B_2 B_2\, dx & \int_{e_3} B_2 B_3\, dx & 0 \\ \text{sym.} & & \int_{e_3} B_3 B_3\, dx & 0 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & \int_{e_4} B_3 B_3\, dx & \int_{e_4} B_3 B_4\, dx \\ & & & \int_{e_4} B_4 B_4\, dx \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}.(2).(2) & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 & 0 \\ & \frac{1}{2}.(2).(2) & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 \\ \text{sym.} & & \frac{1}{2}.(2).(2) & 0 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) \\ & & & \frac{1}{2}.(2).(2) \end{bmatrix} \Rightarrow$$

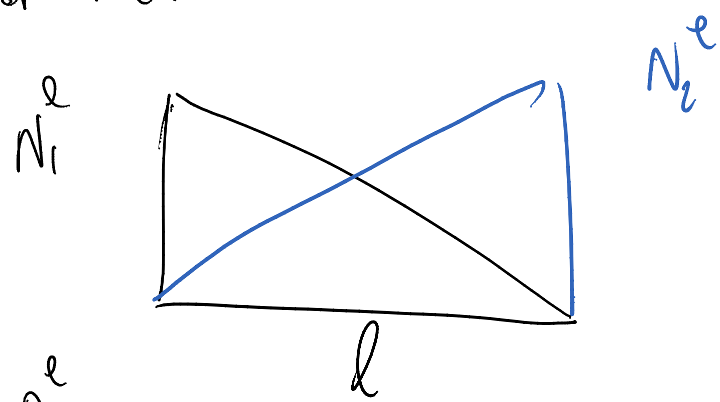
$$\mathbf{K}^{e1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix} \qquad \mathbf{K}^{e2} = \begin{bmatrix} 2 & -2 & 0 & 0 \\ & 2 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

$$\mathbf{K}^{e3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ \text{sym.} & & 2 & 0 \\ & & & 0 \end{bmatrix} \qquad \mathbf{K}^{e4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 2 & -2 \\ & & & 2 \end{bmatrix}$$

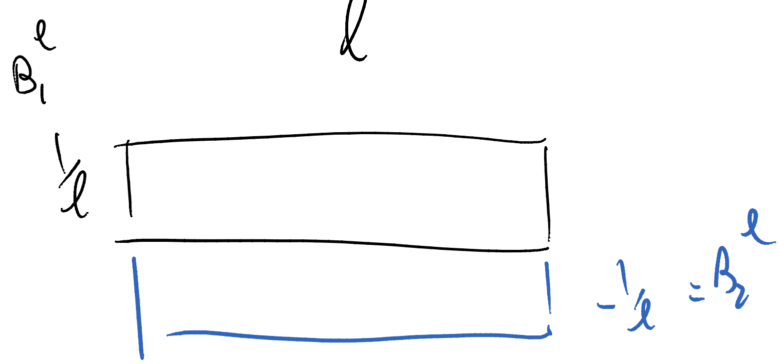
$$\mathbf{K} = \mathbf{K}^{e1} + \mathbf{K}^{e2} + \mathbf{K}^{e3} + \mathbf{K}^{e4} \quad \Rightarrow$$

$$\mathbf{K} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ \text{sym.} & & 4 & -2 \\ & & & 2 \end{bmatrix}$$

stiffness matrix for 1 element



$B = \frac{dN}{dx}$



$$k^e = \int_e \begin{bmatrix} B_1^e \\ B_2^e \end{bmatrix} EA \begin{bmatrix} B_1^e & B_2^e \end{bmatrix} dx$$

$$= \int \begin{bmatrix} \frac{1}{l} \\ -\frac{1}{l} \end{bmatrix} EA \begin{bmatrix} \frac{1}{l} & -\frac{1}{l} \end{bmatrix} dx$$

EA constant

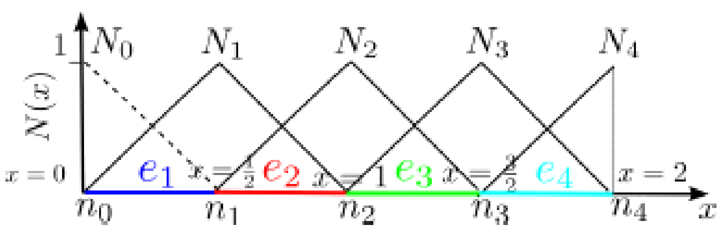
$$= \int_e \begin{bmatrix} \frac{1}{\ell} \\ -\frac{1}{\ell} \end{bmatrix} EA \begin{bmatrix} \frac{1}{\ell} & -\frac{1}{\ell} \end{bmatrix} dx$$

EA constant

$$= \frac{EA}{\ell^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left(\overbrace{\int_e 1\, dx}^{\ell} \right)$$

$$\boxed{k^e = \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

2x2 local stiffness matrix



For all elements $\ell^e = \frac{2}{4} = \frac{1}{2}$

$AE = 1$

$$k^e = \frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{1/2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\mathbf{K}^{e_1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

k^{e_2}

$$\mathbf{K}^{e_2} = \begin{bmatrix} 2 & -2 & 0 & 0 \\ & 2 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

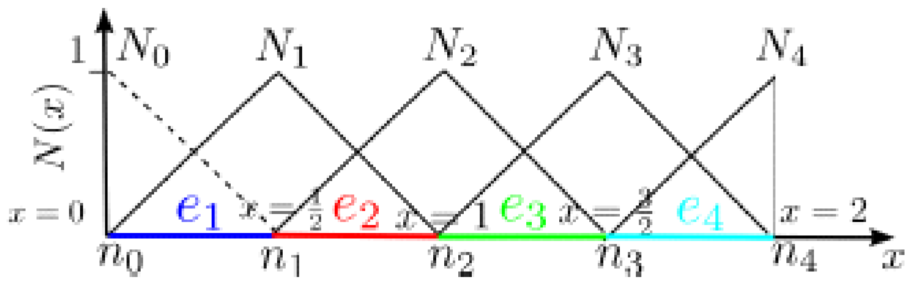
$$\mathbf{K}^{e1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix} \qquad \mathbf{K}^{e2} = \begin{bmatrix} \boxed{\begin{matrix} 2 & -2 \\ & 2 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \text{sym.} & \begin{matrix} 0 & 0 \\ & 0 \end{matrix} \end{bmatrix}$$

$$\mathbf{K}^{e3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \boxed{2} & \boxed{-2} & 0 \\ \text{sym.} & & \boxed{2} & 0 \\ & & & 0 \end{bmatrix} \quad K^{e_3} \qquad \mathbf{K}^{e4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & \boxed{2} & \boxed{-2} \\ & & & \boxed{2} \end{bmatrix}$$

K^{e_4}

$$\mathbf{K} = \mathbf{K}^{e1} + \mathbf{K}^{e2} + \mathbf{K}^{e3} + \mathbf{K}^{e4} \quad \Rightarrow$$

$$\mathbf{K} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ \text{sym.} & & 4 & -2 \\ & & & 2 \end{bmatrix}$$



$M^{e_1}:[0\ 1]$ $\quad M^{e_2}:[1\ 2]$ $\quad M^{e_3}:[2\ 3]$ $\quad M^{e_4}:[3\ 4]$

$$K^{e_1} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 2 & -2 \\ 1 & -2 & 2 \end{array}$$

$\cancel{K(0,1)}$

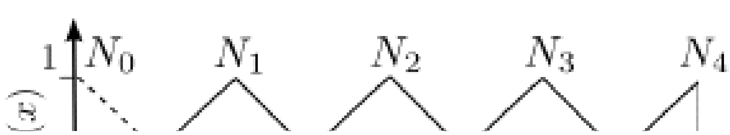
$\cancel{K(0,0)}$

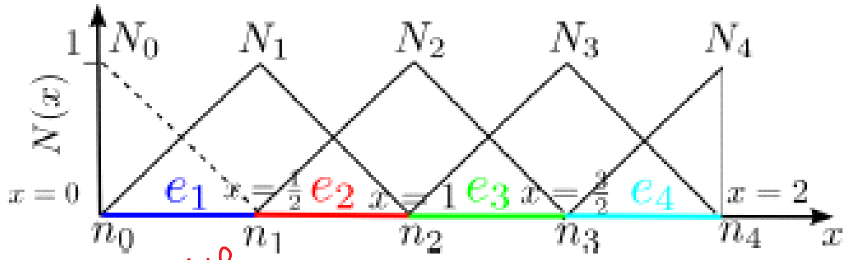
$K(1,1)$

$\cancel{K(1,0)}$

$$K^{e_1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ \text{sym} & & & 0 \end{bmatrix}$$

Element 2



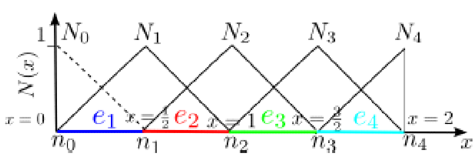


$M^{e_2} = [1 \quad 2]$

$$K^{e_2} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 2 & -2 \\ 2 & -2 & 2 \end{array}$$

$$K^{e_2} = \left[\begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & \\ \hline 0 & & 0 \end{array}\right]$$

Same with the other elements

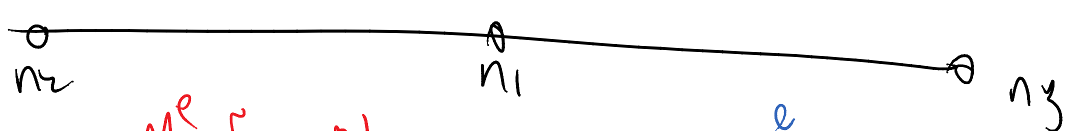


$M^{e_3} = [2 \quad 3]$

$$K^{e_3} = \begin{array}{c|cc} & 2 & 3 \\ \hline 2 & 2 & -2 \\ 3 & -2 & 2 \end{array}$$

$$\mathbf{K}^{e3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 2 & -2 & 0 \\ \text{sym.} & & & 2 & 0 \\ & & & & 0 \end{bmatrix}$$

Using element nodal dof maps to assemble
Example from class on 2016/09/20



n_2 n_1 n_3

$M^{e_1} = [2 \;\; 1]$

$M^{e_2} = [1 \;\; 3]$

$$K^{e_1} = \begin{array}{c|cc} & 2 & 1 \\ \hline 2 & \multicolumn{2}{c}{\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}} \\ 1 & & \end{array}$$

$$K^{e_2} = \begin{array}{c|cc} & 1 & 3 \\ \hline 1 & \multicolumn{2}{c}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \\ 3 & & \end{array}$$

$$K = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \multicolumn{3}{c}{\begin{bmatrix} 1+2 & -2 & -1 \\ -2 & 2 & \\ -1 & & 1 \end{bmatrix}} \\ 2 & & & \\ 3 & & & \end{array}$$

$$K = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 2 & \\ -1 & & 1 \end{bmatrix}$$

Comparison of local and global stiffness matrices and force vectors

Weak statement for finite element formulation

- Similar to our development for solid bar, stiffness matrix can be expressed as,

$$\mathbf{K} = \int_D \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \, \mathrm{dv} \tag{341}$$

- Recalling (312) forces are given by:

$$\mathbf{F}_r = \left(\mathbf{N}^{\mathrm{T}}, r\right)_r = \int_D \mathbf{N}^{\mathrm{T}} r \, \mathrm{dv} \quad \text{source term force} \tag{342a}$$

$$\mathbf{F}_N = \left(\mathbf{N}^{\mathrm{T}}, \bar{\mathbf{F}}\right)_N = \int_{\partial \mathcal{D}_f} \mathbf{N}^{\mathrm{T}} \bar{\mathbf{F}}.\mathbf{N} \, \mathrm{ds} \quad \text{essential BC force} \tag{342b}$$

$$\mathbf{F}_D = \mathcal{A}\left(\mathbf{N}^{\mathrm{T}}, \phi_p\right) = \int_{\mathcal{D}} L_m(\mathbf{N})^{\mathrm{T}} \mathbf{D} L_m(\phi_p) \, \mathrm{dv} = \left\{ \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} \mathbf{D} \bar{\mathbf{B}} \, \mathrm{dv} \right\} \bar{\mathrm{a}}$$

$$= \mathbf{K}_{fp} \bar{\mathrm{a}} \quad \text{essential BC force} \tag{342c}$$

$$\mathbf{F}_n = \begin{bmatrix} F_{n1} \\ \vdots \\ F_{n n_f} \end{bmatrix} \quad \text{nodal forces} \tag{342d}$$

and the total force is given by (*cf.* (311)):

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_N + \mathbf{F}_n - \mathbf{F}_D \tag{343}$$

- Next, we related element-level stiffness and force to global ones.

local level

$$K^e = \int_e B^{e^T} D B^e \, dV$$

$$f_r^e = \int_e N^e r \, dV$$

$$f_N^e = \int_{\partial e_N} N^e \bar{f} \, ds$$

∂e_N

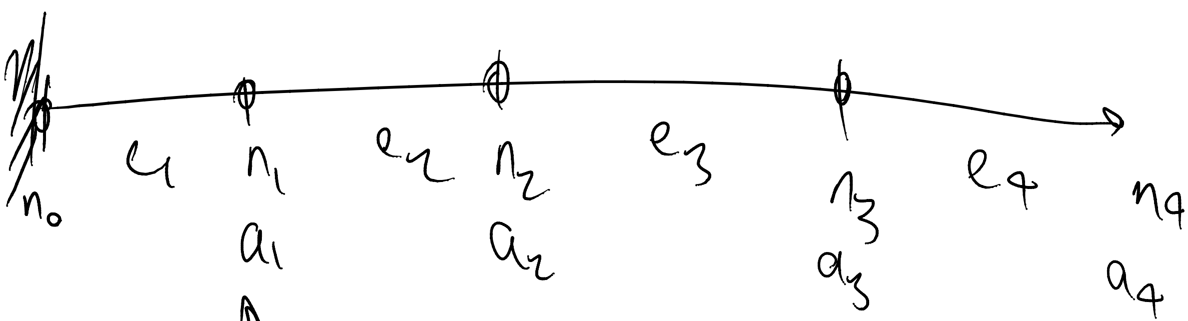
f_n^e ~~X~~

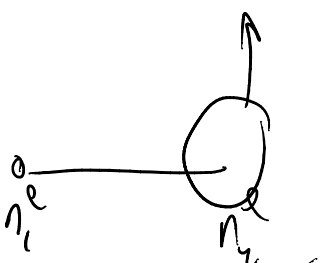
nodal forces

Global Nodal forces

$$\boxed{f_D^e = \underbrace{k^e}_{\text{element stiffness}} \underbrace{a^e}_{\text{element unknowns}}}$$

$\bar{u} = 1$





$a^{e_1} = \begin{bmatrix} 1 \\ a_1 \end{bmatrix}$?

$f_D^e = k^e a^e$

set unknown a's to zero

$$f_D^e = \left(\frac{AE}{\ell}\right)^{e_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$f_D^e = \left(\frac{AE}{\ell}\right)^e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} f \\ 0 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For each element

we calculate $\underline{2 \text{ things}}$:

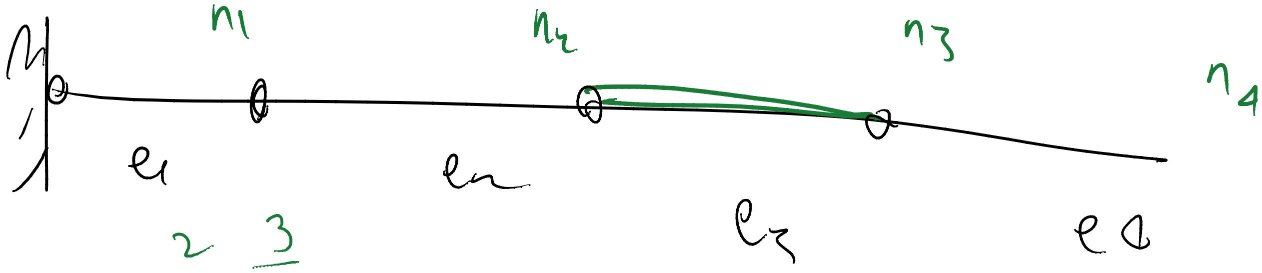
$$K^e = \int_e B^{e^T} D B^e \, dV$$

not at element level

force $\leftarrow$ $f^e = f_r^e + f_N^e + \cancel{f_n^e} - f_D^e$

How are these Assembled in the

global system?



$$K^{e_2} = \begin{array}{c} \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}$$

$$M^{e_3} = [2 \quad 3]$$

$$f^{e_2} = f_r^{e_2} + f_N^{e_2} - f_D^{e_2}$$

$$M^{e_3} = \begin{array}{c} 2 \\ 3 \end{array}\begin{bmatrix} 15 \\ -2 \end{bmatrix}$$

$$K = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} & & & \\ & 2 & -2 & \\ & -2 & 2 & \\ & & & \end{bmatrix} \end{array}$$

$$F = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} \\ 15 \\ -2 \\ \\ \end{bmatrix}$$