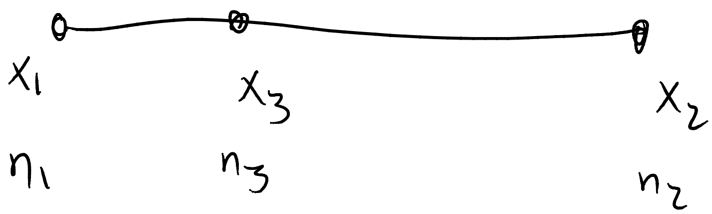
2016/11/22A

Tuesday, November 22, 2016
10:06 AM

2nd order bar element



$$N_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \qquad B_1 = \frac{dN_1}{dx} = \frac{2x-(x_2+x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$N_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \qquad B_2 = N_2' = \frac{2x-(x_1+x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$N_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \qquad B_3 = N_3'$$

From B_1, B_2, B_3 we can calculate k^e

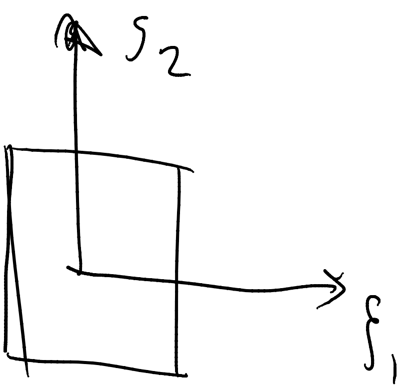
$$k^e = \int_e B^T D B \, dV =$$

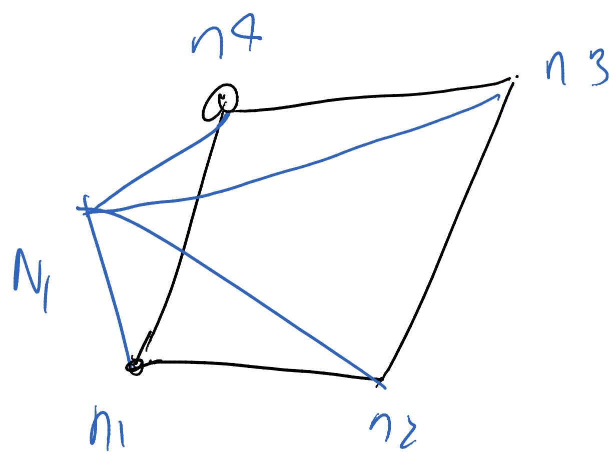
$$k^e = \int_{x_1}^{x_3} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \overbrace{EA}^{D} \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix} dx$$

plug B's from equation above to form k

This is not the approach that we calculate k!

We are going to use parent element geometry to form stiffness matrix:

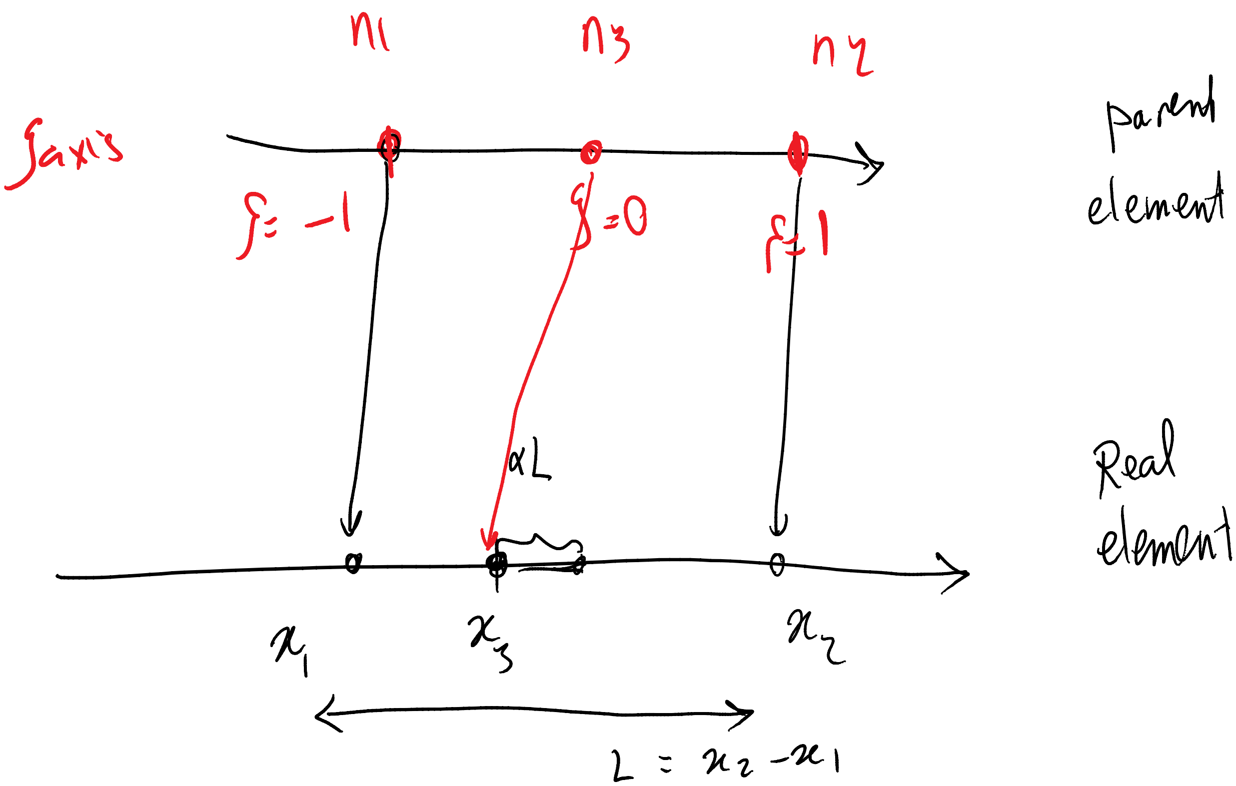




Even if we form shape functions in real geometry the integration of element stiffness matrix is very difficult in this geometry ->

If we map the geometry to a simple 2 x 2 square integration of k becomes much simpler

Using parent element geometry

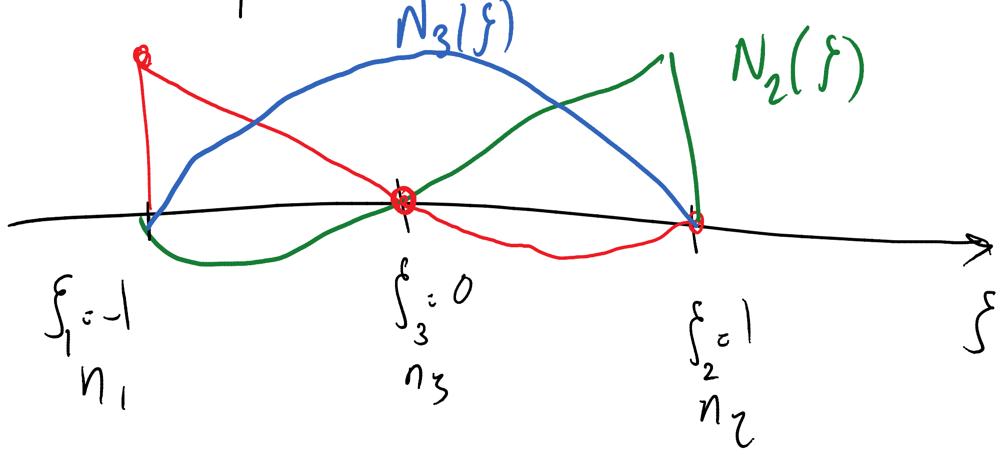


$$x_1 = 0$$

$$x_2 = L$$

$$x_3 = \frac{L}{2} - \alpha L = L\left(\frac{1}{2} - \alpha\right)$$

Step 1 shape functions.



$$N_i(\xi) = L_i(\xi) = \frac{\prod_{j \neq i} (\xi - \xi_j)}{\prod_{j \neq i} (\xi_i - \xi_j)}$$

$$N_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)} = \frac{\xi(\xi - 1)}{2}$$

$$N_2(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = \frac{\xi(\xi + 1)}{2}$$

$$N_3(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = 1 - \xi^2$$

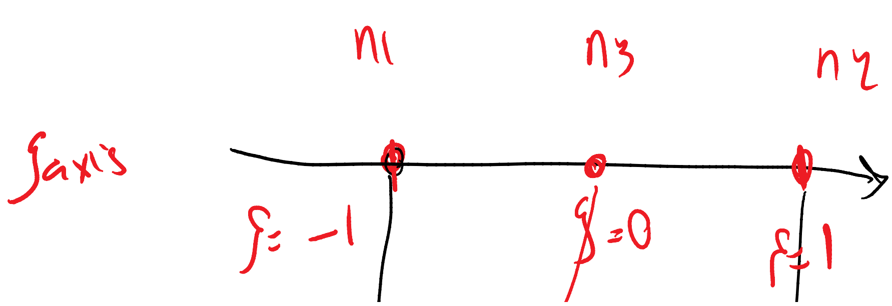
$$\boxed{\begin{aligned} N &= \begin{bmatrix} N_1(\xi) & N_2(\xi) & N_3(\xi) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\xi(\xi-1)}{2} & \frac{\xi(\xi+1)}{2} & (1-\xi^2) \end{bmatrix} \end{aligned}}$$

$$k^e = \int_e B^T D B \, dx$$

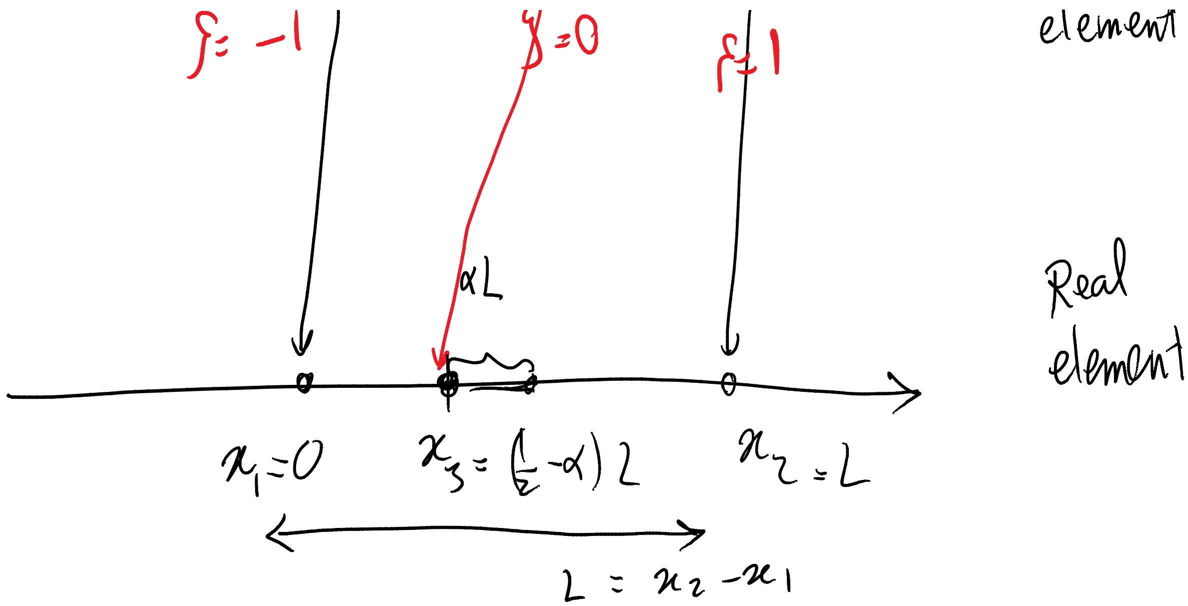
$$B = \frac{d}{dx} N \quad , D = EA$$

we need a relation between

X & ξ



parent element



$x = x(\xi)$

$$x(-1) = 0$$
$$x(1) = L$$
$$x(0) = \left(\frac{1}{2} - \alpha\right) L$$

(★) already know this

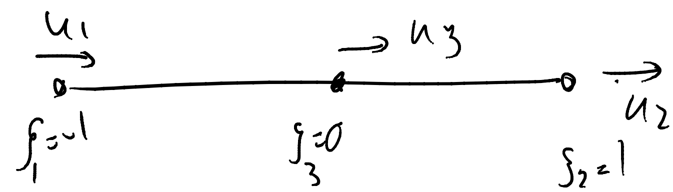
How can we find the function $x(\xi)$ that satisfies (★):

$$x(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 \quad \text{satisfy (★)}$$
$$\Rightarrow \text{get } \alpha_0, \alpha_1, \alpha_2$$

Not a good approach

FE solution

$$u(\xi) = u_1 N_1(\xi) + u_2 N_2(\xi) + u_3 N_3(\xi)$$



$$x(\xi) = \overset{?}{x_1} N_1(\xi) + \overset{?}{x_2} N_2(\xi) + \overset{?}{x_3} N_3(\xi)$$

check $x_i = x(\xi_i)$

$$x(\xi_1=-1) = x_1 \overset{1}{N_1(\xi_1)} + x_2 \overset{0}{N_2(\xi_1)} + x_3 \overset{0}{N_3(\xi_1)}$$

$$= x_1$$

Similarly $x(\xi_2) = x_2$

$x(\xi_3) = x_3$

basically $\xi_i \xrightarrow{\text{mapped to}} x_i$ (* is satisfied)

$$u(\xi) = u_1 N_1(\xi) + u_2 N_2(\xi) + u_3 N_3(\xi)$$

solution $\downarrow$

$$x(\xi) = x_1 N_1(\xi) + x_2 N_2(\xi) + x_3 N_3(\xi)$$

geometry

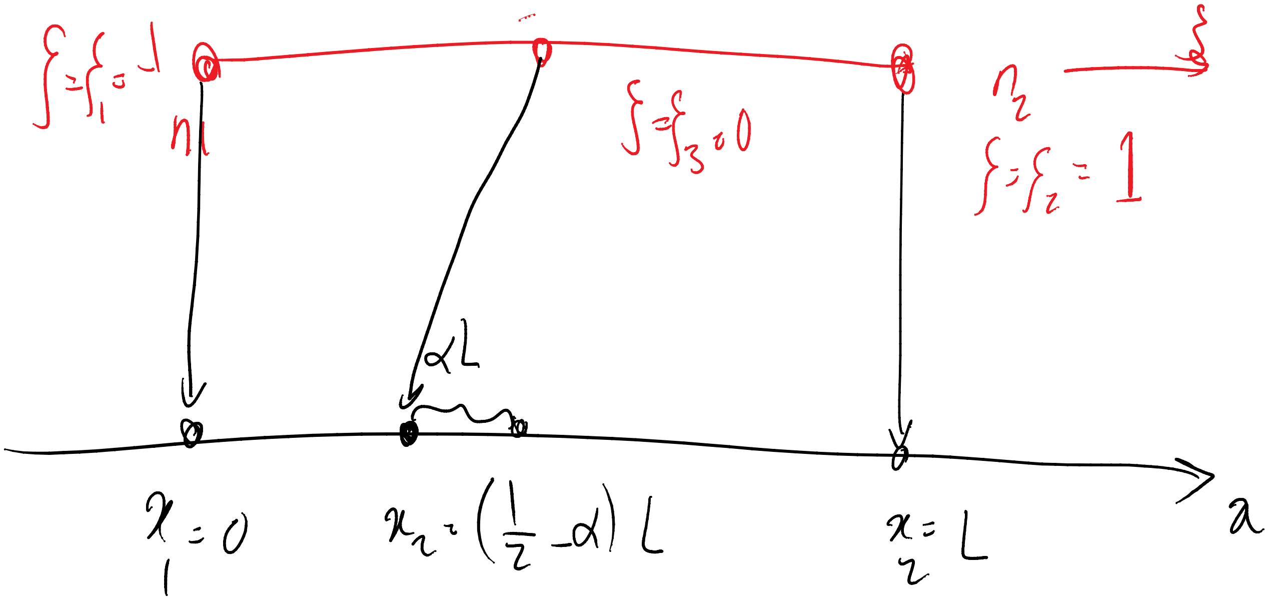
iso parametric

$\downarrow$

same (map for solution & geometry)

Now that we have the expression for $x(\xi)$

let's simplify it



$$x = x_1 N_1(\xi) + x_2 N_2(\xi) + x_3 N_3(\xi)$$

$$= 0 \, \frac{\xi(\xi-1)}{2} + L\left(\frac{\xi(\xi+1)}{2}\right) + \left(\frac{1}{2} - \alpha\right) L \left(1 - \xi^2\right)$$

$$\Rightarrow \quad \boxed{x = L(1+\xi)\left(\alpha \xi + \left(\frac{1}{2} - \alpha\right)\right)}$$

What happens when $\alpha = 0$ (element is not skewed)?

x is 2nd order in ξ in general

x " 1st order " " if $\alpha = 0$

x " 1st order " " if $\alpha = 0$

Summary:

$$\boxed{\begin{gathered} N = [N_1 \quad N_2 \quad N_3] = \left[\frac{\xi(\xi-1)}{2} \quad \frac{\xi(\xi+1)}{2} \quad 1-\xi^2\right] \\ x(\xi) = L\left((1+\xi)\left(\alpha\xi + \left(\tfrac{1}{2}-\alpha\right)\right)\right) \end{gathered}}$$

$$k^e = \int B^T D B \, dx \qquad B = \frac{d}{dx} N$$

$$D = EA$$

$$k^e = \int_{\xi=-1}^{\xi=1} \underbrace{\frac{d}{dx}\begin{bmatrix} N_1(\xi) \\ N_2(\xi) \\ N_3(\xi) \end{bmatrix}}_{B^T} EA(\xi) \underbrace{\frac{d}{dx}\left[N_1(\xi) \quad N_2(\xi) \quad N_3(\xi)\right]}_{\text{(1) how to evaluate this?}} \underbrace{dx}_{\text{(2) what is } dx}$$

$$\frac{dN_i(\xi)}{} = \frac{dN_i(\xi)}{d\xi}\,\frac{d\xi}{d}$$

$\xi(x)$

$$\frac{dN_i(\xi)}{dx} = \frac{dN_i(\xi)}{d\xi}\frac{d\xi}{dx} \qquad \xi(x)$$

$$= \frac{dN_i(\xi)}{d\xi}\frac{1}{\frac{dx}{d\xi}} \qquad J$$

$$dx = \left(\frac{dx}{d\xi}\right) d\xi$$

J Jacobian

$$B = \frac{dN}{dx} = \frac{1}{\frac{dx}{d\xi}}\overbrace{\left(\frac{d}{d\xi}N\right)}^{B_\xi} = \frac{1}{J} B_\xi$$

$$dx = \frac{dx}{d\xi}\, d\xi \Rightarrow J\, d\xi$$

plug in (☆☆)

$$k^e = \int_{\xi=-1}^{\xi=1} B^T E A B \, dx =$$

$$= \int_{\xi=-1}^{\xi=1} \left(\frac{1}{J} B_\xi^T\right) EA \left(\frac{1}{\cancel{J}} B_\xi\right) \left(\cancel{J}\, d\xi\right)$$

$\Longrightarrow$ ★★★

$$\boxed{k^e = \int_{\xi=-1}^{\xi=1} \frac{1}{J(\xi)} \begin{bmatrix} B_1(\xi) \\ B_2(\xi) \\ B_3(\xi) \end{bmatrix} EA(\xi) \begin{bmatrix} B_1(\xi) & B_2(\xi) & B_3(\xi) \end{bmatrix} d\xi}$$

$B_\xi = ?$

$J = ?$

$$B_\xi = \frac{d}{d\xi} N = \frac{d}{d\xi} \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} =$$

$$\frac{d}{d\xi}\left[\frac{\xi(\xi-1)}{2} \quad \frac{\xi(\xi+1)}{2} \quad 1-\xi^2\right]$$

$$\boxed{B_\xi = \left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi\right]} \quad (\circ)$$

$J = ?$

$$x = x(\xi) = (1+\xi)\, L\left(\alpha\xi + \left(\frac{1}{2} - \alpha\right)\right)$$

$$\boxed{J = \frac{dx}{d\xi} = \frac{L}{2}\left(4\alpha\xi + 1\right)} \quad (\circ\circ)$$

if $\alpha = 0$ (non-skewed element) $\longrightarrow J$ is constant!

Plug $(\circ)$ & $(\circ\circ)$ in $(\star\star\star)$ to get:

$$k^e = \int_{\xi=-1}^{\xi=1} \frac{EA(\xi)}{\frac{L}{2} + 2\alpha L \xi} \underbrace{\begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \begin{bmatrix} \xi - \frac{1}{2} & \xi + \frac{1}{2} & -2\xi \end{bmatrix}}_{\text{2nd order polynomial}} d\xi \qquad \text{(I)}$$

Under what conditions integrand becomes a polynomial:

1. EA is constant (or a polynomial)
2. $\alpha = 0$ (so the element is not-skewed)

In numerical integration (quadrature) we make TWO assumptions:

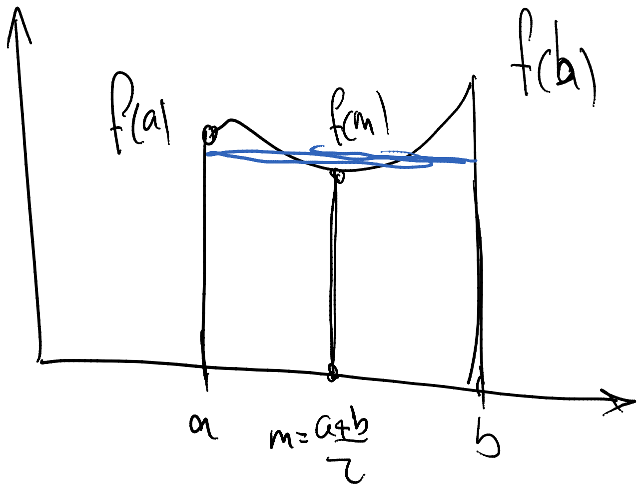
1. EA (or any section / material properties are constant)
2. The element is not-skewed

The integration scheme that integrates such element exactly is called a
full-integration scheme

-- What is the full-integration order of this example? 2nd order (we need to integrate a second order polynomial exactly)

Quadrature (Numerical integration)

Previous example shows the need for numerical integration:



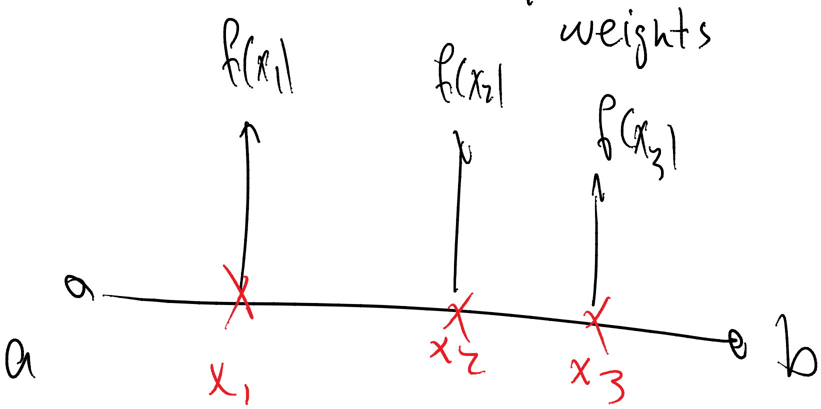
Quadrature rule:

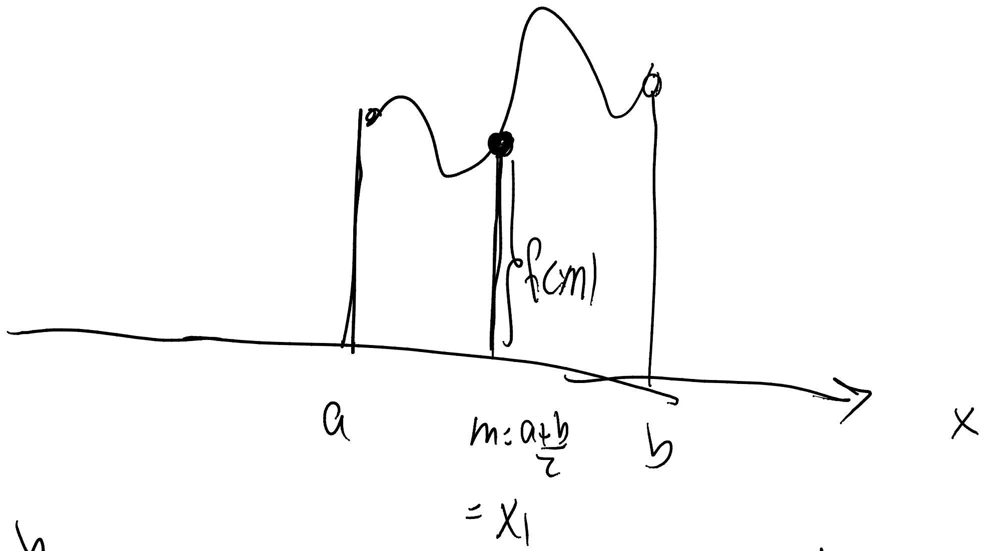
quadrature points ← nq

$$\int_a^b f(x)\,dx = (b-a)\sum_{i=1}^{nq} w_i\, f(x_i)$$

quadrature weights

quadrature points





$$\int_a^b f(x)\,dx = \underset{?}{w_1}\, f(x_1)$$

$(b-a)$

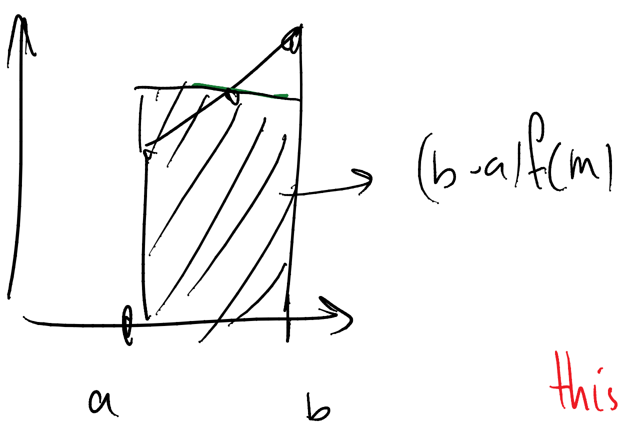
$$\int_a^b f(x)\,dx \approx (b-a)\, f(x_1)$$

w_1 $\quad$ $x_1 = \frac{a+b}{2}$

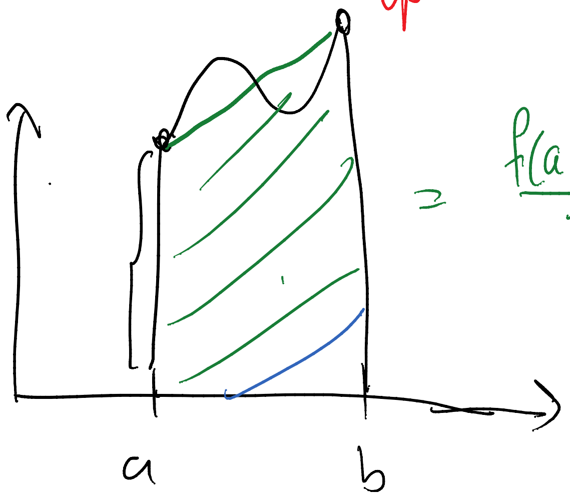
$nq = 1$

$w_1 = \underline{1}$

$x_1 = \frac{a+b}{2}$



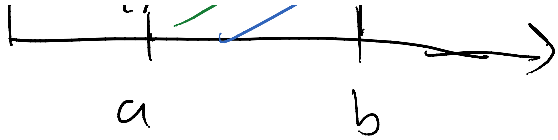
this quadrature rule integrates up to 1st order polynomials exactly



$$= \frac{f(a)+f(b)}{2}(b-a)$$

$x_1 = a$

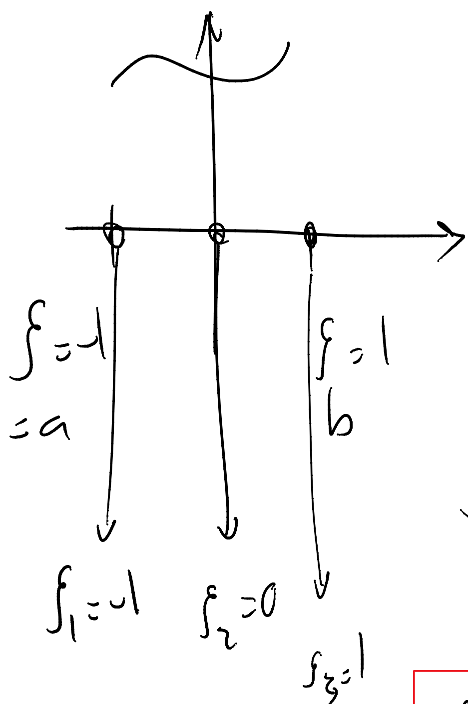
$x_2 = b$



$x_2 = b$

$$\int_a^b f(x)\,dx = (b-a)\left(\underbrace{w_1}_{w_1=\frac{1}{2}} f(x_1) + \underbrace{w_2}_{w_2=\frac{1}{2}} f(x_2)\right)$$

Derivation of 3 point quadrature rule:



$f(\xi) =$

$$\int_{-1}^{1} f(\xi)\,d\xi = \underbrace{(b-a)}_{\substack{(1-(-1)) \\ 2}}\left(w_1 f(\xi_1) + w_2 f(\xi_2) + w_3 f(\xi_3)\right)$$

$$\boxed{\int_{-1}^{1} f(\xi)\,d\xi = 2\left(w_1 f(-1) + w_2 f(0) + w_3 f(1)\right)}$$

unknowns 3

What polynomial can be integrated exactly what polyn...

$$f(\xi) = \underbrace{\alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2}\Big| + \ldots$$

we can integrate up to $p=2$ polynomials exactly (3 eqns)

$$\int_{-1}^{1} f(\xi)\, d\xi = 2\left(w_1 f(-1) + w_2 f(0) + w_3 f(1) \right)$$

$f(\xi) = 1$

$$\int_{-1}^{1} 1\, d\xi = 2 = 2\left(w_1 1 + w_2 1 + w_3 1 \right)$$

$$\Rightarrow \boxed{w_1 + w_2 + w_3 = 1} \quad (i)$$

$f(\xi) = \xi$

$$\int_{-1}^{1} \xi\, d\xi = 0 = 2\left(w_1(-1) + w_2(0) + w_3(1) \right)$$

$$\boxed{w_3 - w_1 = 0} \quad (ii)$$

$f(\xi) = \xi^2$

$$\int^{1}$$

$f(\xi) = \xi^2$

$$\int_{-1}^{1} \xi^2 \, d\xi = \frac{2}{3} = 2\left(w_1 (-1)^2 + w_2 (0)^2 + w_3 (1)^2 \right)$$

$$\Rightarrow \boxed{w_3 + w_1 = \frac{1}{3}} \quad iii)$$

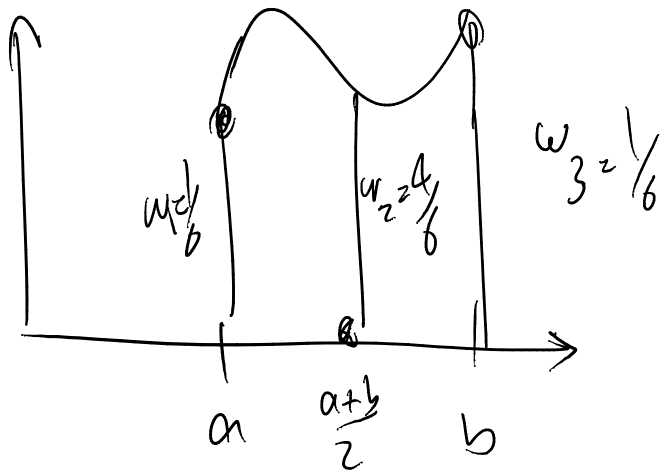
$$\left.\begin{array}{ll} ii & w_1 - w_3 = 0 \\ iii & w_1 + w_3 = \frac{1}{3} \end{array}\right\} \Rightarrow \boxed{w_1 = w_3 = \frac{1}{6}}$$

$$i: \quad w_1 + w_2 + w_3 = 1 \Rightarrow \boxed{w_2 = \frac{4}{6}}$$

$$\int_{\xi=-1}^{\xi=1} f(\xi) \, d\xi = 2\left(\overset{1/6}{w_1} f(\underset{-1}{\xi_1}) + \overset{4/6}{w_2} f(\underset{0}{\xi_2}) + \overset{1/6}{w_3} f(\underset{1}{\xi_3}) \right)$$

$$\boxed{\int_a^b f(x) \, dx \approx (b-a) \left(\frac{1}{6} f(a) + \frac{4}{6} f\left(\frac{a+b}{2}\right) + \frac{1}{6} f(b) \right)}$$

$$\int_a f(x)\,dx \approx (b-a)\left(\frac{1}{6}f(a) + \frac{4}{6}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right)$$



Simpson's rule