2016/11/22B

Tuesday, November 22, 2016
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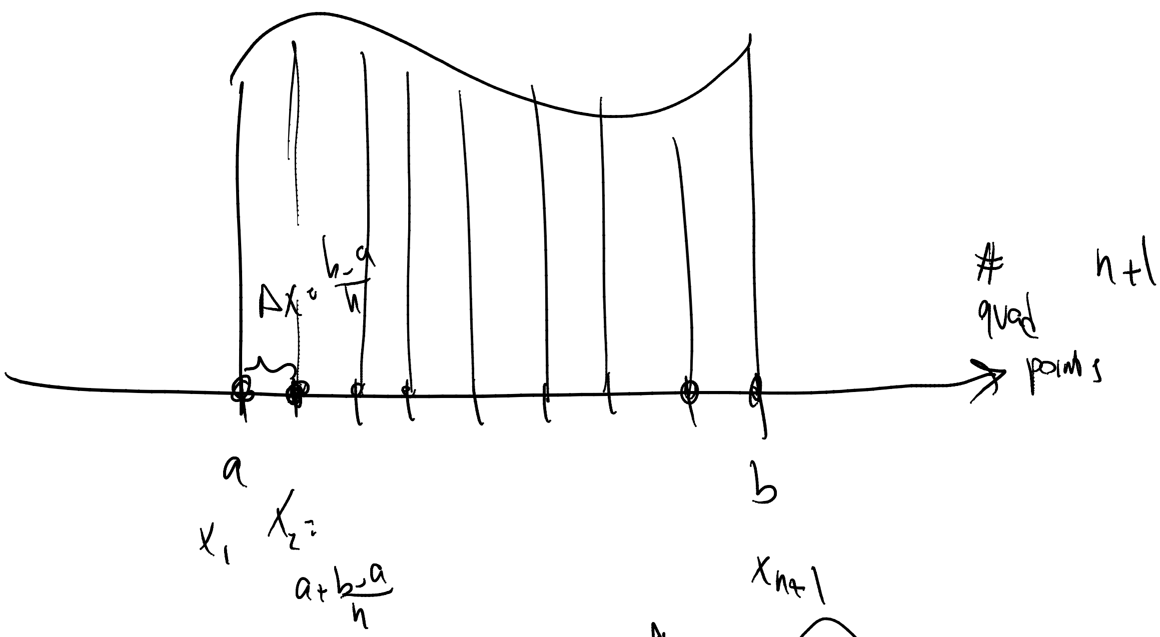
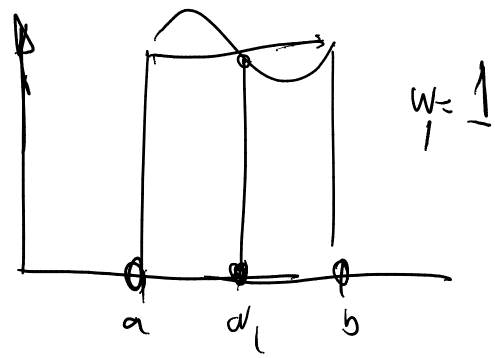
Newton-Cotes integration method:



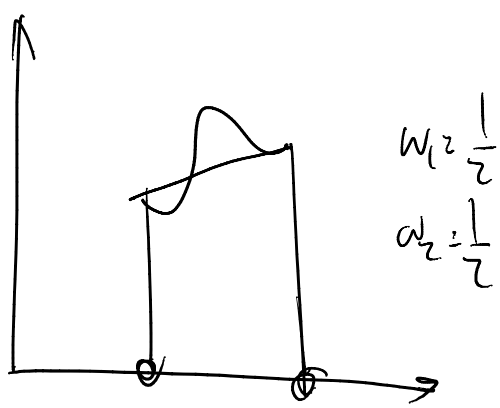
$w_i = ?$

n = 1



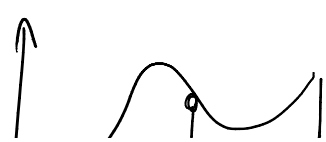
n = 2

Trapezoidal rule



n = 3

g



$n = 3$

Simpson's rule

d

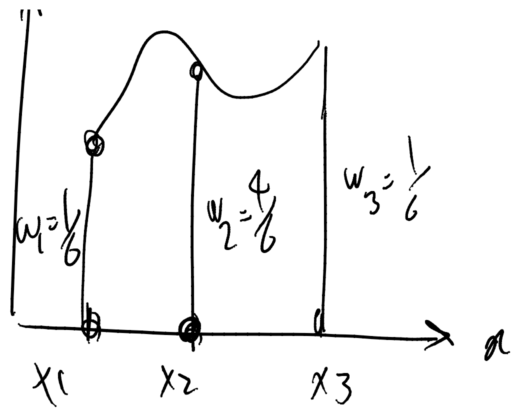


TABLE 5.5 *Newton-Cotes numbers and error estimates*

weights

n_g	Number of intervals n	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n	Upper bound on error R_n as a function of the derivative of F
2	1	$\frac{1}{2}$	$\frac{1}{2}$						$10^{-1}(b-a)^3F^{\text{II}}(r)$
3	2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$					$10^{-3}(b-a)^5F^{\text{IV}}(r)$
4	3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				$10^{-3}(b-a)^5F^{\text{IV}}(r)$
5	4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$			$10^{-6}(b-a)^7F^{\text{VI}}(r)$
6	5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$		$10^{-6}(b-a)^7F^{\text{VI}}(r)$
7	6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9F^{\text{VIII}}(r)$

Simpler way to calculate weights for Newton-Cotes method (or any quadrature rule once we have quadrature points)

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} w_i\, f(x_i)$$

n pts $\longrightarrow$ n w_i's

we can integrate $n-1$ polynomials exactly

A direct method to obtain each wi individually

Is there a function

$$\int_a^b f(x)\,dx = \sum_{j=1}^{n} w_j f(x_j)$$

s.t.

$$f(x_j) = \begin{cases} 1 & x_j = x_i \quad (j=i) \\ 0 & \text{otherwise} \end{cases}$$

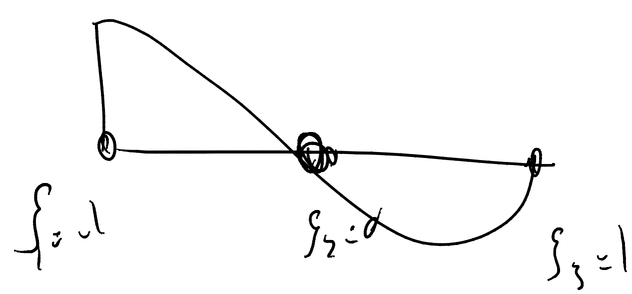
assume G_i does that

$$\int_a^b G_i(x) = \sum_{j=1}^{<i} w_j \cancel{G_i(x_j)}^{0} + w_i G_i(x_i) + \sum_{j>i}^{n} w_j \cancel{G_i(x_j)}^{0} = w_i G_i(x_i)^{1}$$

$$\Longrightarrow \quad \boxed{w_i = \int_a^b G_i(x)}$$

Lagrange function just do that for us:

Example



$$L_1(\xi) = \frac{\xi(\xi-1)}{2}$$

$$w_1 = \int_{-1}^{1} \frac{\xi(\xi-1)}{2}\, d\xi = \frac{1}{6}$$

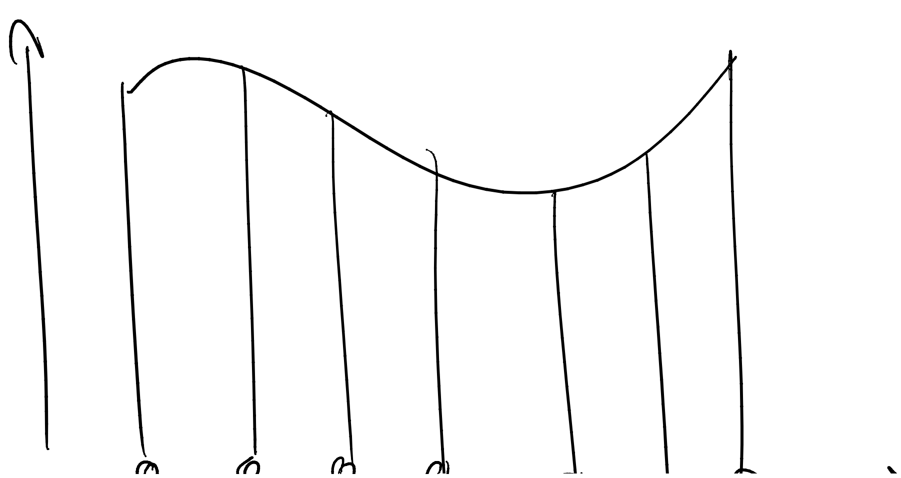
$$w_2 = \int_{-1}^{1} L_2(\xi)\, d\xi = \int_{-1}^{1} 1-\xi^2 = \frac{4}{6}$$

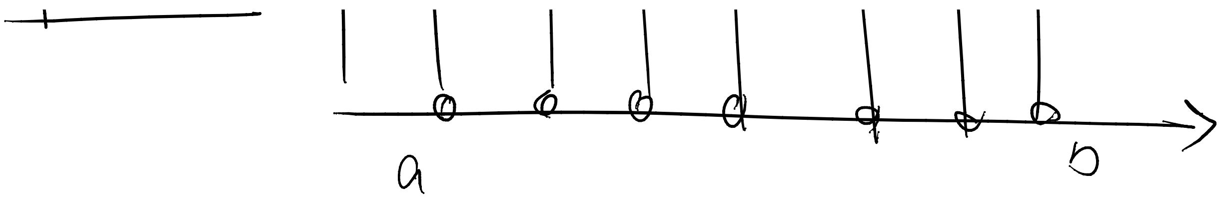
So we don't need to form an n x n system of equations
Each time find one of the w's

Not useful for FEM but in general useful for integration

approach 1

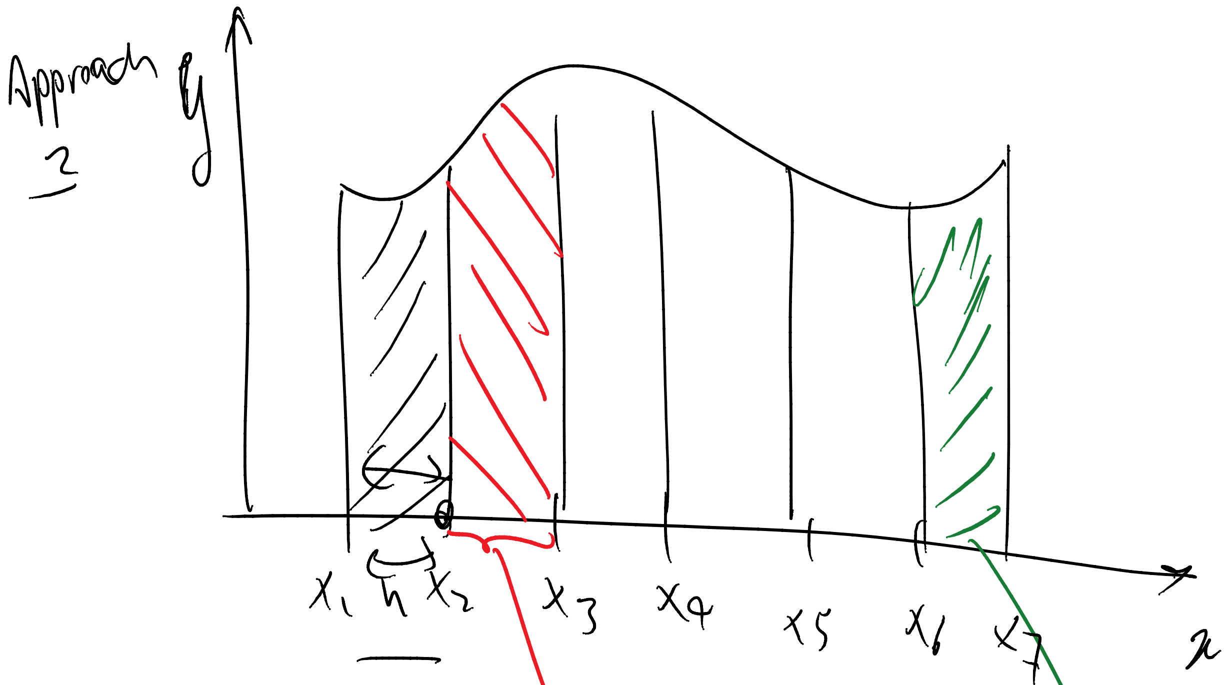
p increases





use table above

Approach 2



trapezoidal rule

$$h\left(\frac{f(x_1)+f(x_2)}{2}\right)$$

$$\frac{h}{2}\left(\frac{f(x_2)+f(x_3)}{2}\right)$$

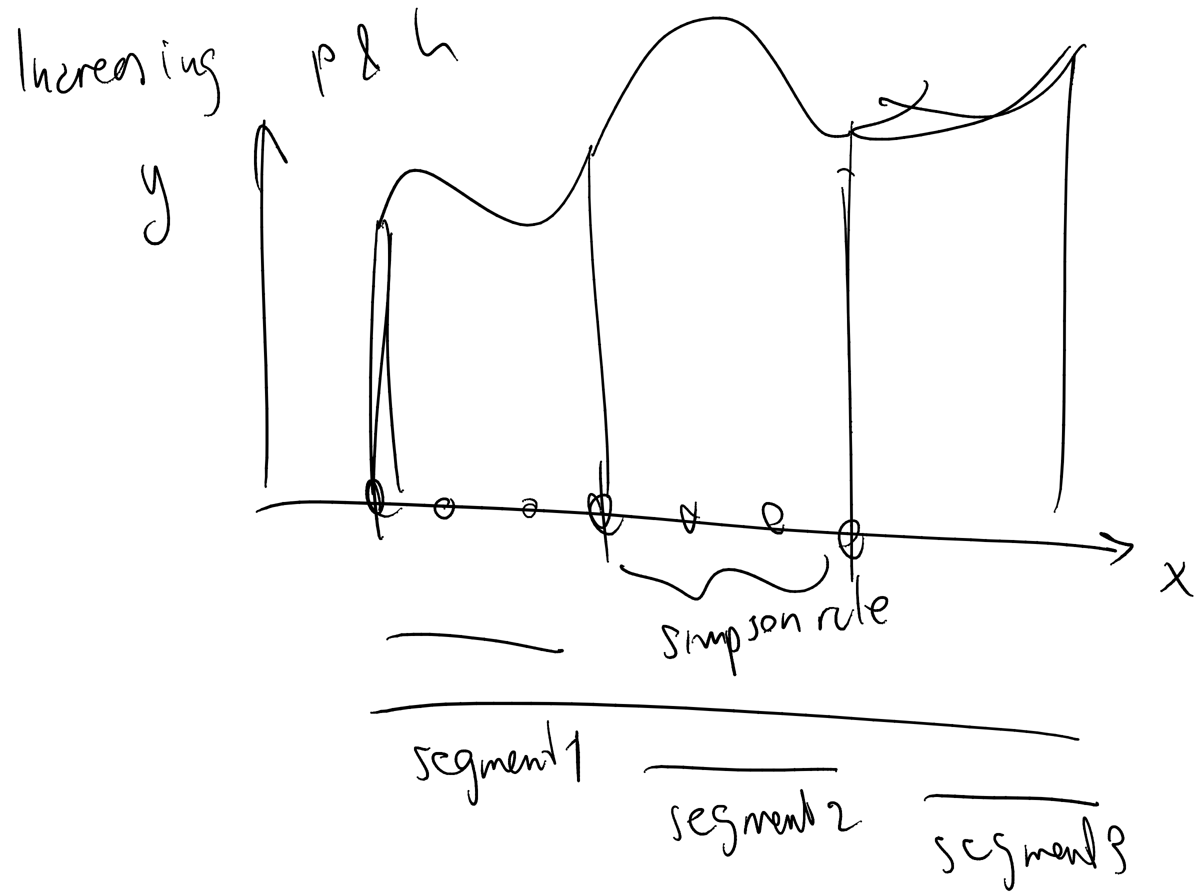
$$\frac{h}{2}\left(f(x_6)+f(x_7)\right)$$

$$\int f(x)\,dx = \frac{h}{2}\left(f(x_1) + 2f(x_2) + 2f(x_3) \ldots + 2f(x_6) + f(x_7)\right)$$

Trapezoidal rule applied to an h-refined grid

$$\int f(x)dx = \frac{h}{2}\left(f(x_1) + 2f(x_2) + 2f(x_3) \cdots + 2f(x_6) + f(x_7)\right)$$

Trapezoidal rule applied to an h-refined grid



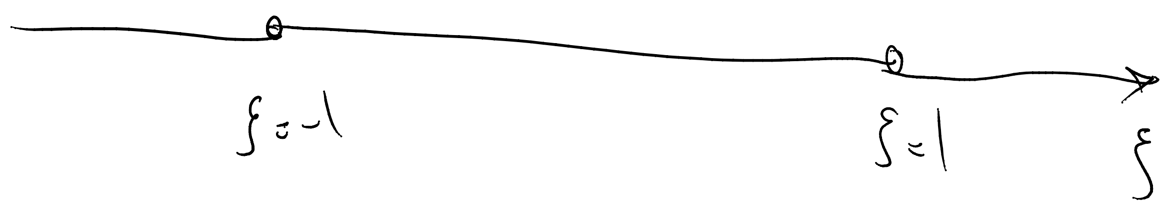
We can use a hybrid method to integrate an interval:

- We divide it to intervals and inside each interval use a higher order Newton-Cotes approach (so that each interval has subintervals and interior interval quadrature points)
- We add the integrals of intervals to get the total integral value

- Division to intervals is similar to h-refinement in FEM analysis -> higher accuracy is achieved by having smaller segments
- Higher Newton-Cotes schemes is similar to p-enrichment in FEM as each interval integrates higher order polynomials exactly
- Combination of the two is similar to FEM hp-adaptive schemes that both higher order interpolation and smaller segment size features are used to obtain better accuracy

Gauss quadrature

The idea is that if the position of quadrature points is also unknown -> the number of unknowns double so instead of integrating polynomial order n - 1 exactly (n quad points) we integrate polynomial order 2n - 1 exactly



Two quad points ξ_1, ξ_2

with weights w_1, w_2

$$\int_{-1}^{1} f(\xi)\, d\xi = \underbrace{(b-a)}\left(w_1 f(\xi_1) + w_2 f(w_2)\right)$$

$\int_{-1}$

$1-(-1)$

in Gauss quadrature because the interval is always from -1 to 1 we don't premultiply RHS by $b-a=2$

$$\boxed{\text{Gauss Quadrature rule in general} \quad \int_{-1}^{1} f(\xi)\, d\xi = \sum_{i=1}^{n} w_i f(\xi_i)}$$

Let's obtain a 2 point Gauss quadrature rule:

(7) $$\int f(\xi)\, d\xi = w_1 f(\xi_1) + w_2 f(\xi_2)$$

w_1, w_2, ξ_1, ξ_2 are are unknowns

4 unknowns $\Longrightarrow$

What polynomial order can we integrate exactly

$$f(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3$$

$$\int_{-1}^{1} f(\xi)\,d\xi = w_1 f(\xi_1) + w_2 f(\xi_2)$$

$f(\xi) = 1$ $\quad \int_{-1}^{1} 1\,d\xi = 2 = w_1(\xi_1)^0 + w_2(\xi_2)^0$

$f(\xi) = \xi$ $\quad \int_{-1}^{1} \xi\,d\xi = 0 = w_1(\xi_1)^1 + w_2(\xi_2)^1$

$f(\xi) = \xi^2$ $\quad \int_{-1}^{1} \xi^2\,d\xi = \frac{2}{3} = w_1(\xi_1)^2 + w_1 \xi_2^2$

$f(\xi) = \xi^3$ $\quad \int \xi^3\,d\xi = 0 = w_1 \xi_1^3 + w_2 \xi_2^3$

$$\longrightarrow \begin{cases} w_1 + w_2 = 2 & (1) \\ w_1 \xi_1 + w_2 \xi_2 = 0 & (2) \\ w_1 \xi_1^2 + w_2 \xi_2^2 = 2/3 & (3) \\ w_1 \xi_1^3 + w_2 \xi_2^3 = 0 & (4) \end{cases}$$

(2)–(4) $\times \ \xi_1^2$

$$w_1 \xi_1^3 + w_2 \xi_2^3 = 0 \quad (4)$$

$$1 \times \xi_1^2 \qquad w_1 \xi_1^3 + w_2 \xi_2 \xi_1^2 = 0$$

substract $\Rightarrow$

$$w_2 \left(\xi_2 \xi_1^2 - \xi_2^3 \right) = 0 \Rightarrow$$

$$w_2 \xi_2 \left(\xi_1 - \xi_2 \right)\left(\xi_1 + \xi_2 \right) = 0$$

$\Rightarrow$

$w_2 = 0$ OR

$\xi_2 = 0$ OR

$\xi_1 = \xi_2$ OR

$\boxed{\xi_1 = -\xi_2}$

$\rightarrow$ we will not be able to satisfy all equations!

$\boxed{\xi_1 = -\xi_2}$ going back to original equations:

$w_1 + w_2 = 2$ | $\qquad w_1 + w_2 = 0$ | ..

$$\left.\begin{aligned} w_1 + w_2 &= 2 \\ w_1 \xi_1 - w_2 \xi_1 &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} w_1 + w_2 &= 0 \\ \xi_1 (w_1 - w_2) &= 0 \end{aligned}\right\} \text{ if}$$

$$w_1 \xi_1^2 + w_2 \xi_1^2 = 2/3 \quad (3) \qquad \left(\begin{aligned} &\xi_1 = 0 \\ &\text{no solution} \end{aligned} \right)$$

$$w_1 \xi_1^3 - w_2 \xi_1^3 = 0 \quad (4)$$

$$\left.\begin{aligned} w_1 + w_2 &= 2 \\ w_1 - w_2 &= 0 \end{aligned}\right\} \longrightarrow \boxed{w_1 = w_2 = 1}$$

$$w_1 \xi_1^2 + w_2 \xi_1^2 = 2/3$$

$$\longrightarrow 1 \xi_1^2 + 1 \xi_1^2 = 2/3 \Longrightarrow \boxed{\xi_1 = \mp\sqrt{\frac{1}{3}}}$$

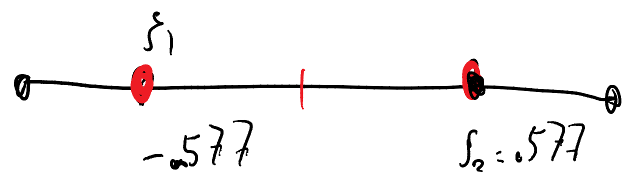
$$\xi_2 = -\xi_1 \qquad \qquad \xi_1 < \xi_2$$

$$\boxed{\begin{array}{ll} \xi_1 = -\dfrac{1}{\sqrt{3}} & \xi_2 = \dfrac{1}{\sqrt{3}} \\ w_1 = 1 & w_2 = 1 \end{array}}$$

$$\int_{-1}^{1} f(\xi)\, d\xi = \underbrace{1}_{w_1}\, f\Big(\underbrace{-\tfrac{1}{\sqrt{3}}}_{\xi_1}\Big) + \underbrace{1}_{w_2}\, f\Big(\underbrace{\tfrac{1}{\sqrt{3}}}_{\xi_2}\Big)$$

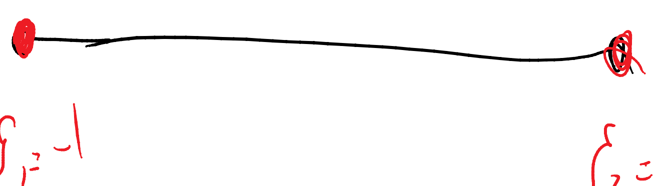
2 point Gauss quadrature

Gauss 2pts



order of polynomial integrated exactly = 3

Newton-Cotes 2pts



order of

$\xi_1 = -1$ $\xi_2 = 1$

order of polynomial integrated exactly: 1

Order p integrand:

$$n_g : \begin{cases} p+1 & \text{Newton-Cotes} \\ \text{ceiling}\left(\frac{p+1}{2}\right) & \text{Gauss Method} \end{cases}$$

Gauss method provides the same order of accuracy with almost half the number of quadrature points!

That is why in FEM and many other methods if numerical integration is needed, Gauss quadrature is used.

HW:

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
$\mp$ 0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
− 0.77459 66692 41483	0.55555 55555 55555

$\xi_1 = -.577$ $w_1 = 1$
$\xi_2 = .577$ $w_2 = 1$

$\xi_1 = -.7745$ $w_1 = .5\bar{5}$
$\xi_2 = 0$ $w_2 = .\bar{8}8$

0.00000 00000 00000	0.88888 88888 88888
∓ 0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189

$\xi_2 = 0$ $\quad w_2 = .8\bar{8}$

$\xi_3 = \pm .7745$ $\quad w_3 = .55$

HW5: problem #1

$$I = \int_{-1}^{2} \frac{dx}{1+x^2}$$

If we want to integrate the integrate as a 2nd order polynomial ->

Use Simpson's rule:

$$\int_{-1}^{2} \frac{1}{1+x^2}\,dx$$

$h = 3/2$

$x_1 = -1$ $\qquad x_2 = \frac{2+(-1)}{2} = .5$ $\qquad x_3 = 2$

n	C_0	W_1	W_2	W_3	W_4	W_5	C_1	k	Name
1	1	1					1/2	1	Rectangle
2	1/2	1	1				–1/12	2	Trapezium
3	1/3	1	4	1			–1/90	4	Simpson
4	3/8	1	3	3	1		–3/80	4	4-point
5	2/45	7	32	12	32	7	–8/945	6	5-point

$$\int_a^b f(x)\,dx \simeq C_0 h \sum_{i=1}^{n} W_i f(x_i) + C_1 h^{k+1} f^{(k)}(\xi)$$

error do NOT add it

$$= \frac{1}{3}\cdot\frac{3}{2}\left(1 f(\underset{-1}{x_1}) + 4 f(\underset{.5}{x_2}) + 1 f(\underset{2}{x_3})\right)$$

$$f(x) = \frac{1}{1+x^2}$$

You are going to use 5-point quadrature rule

5	2/45	7	32	12	32	7

How to use Gauss method:

$$\int_{-1}^{2} \frac{1}{1+x^2}\,dx$$

How to use Gauss Table?

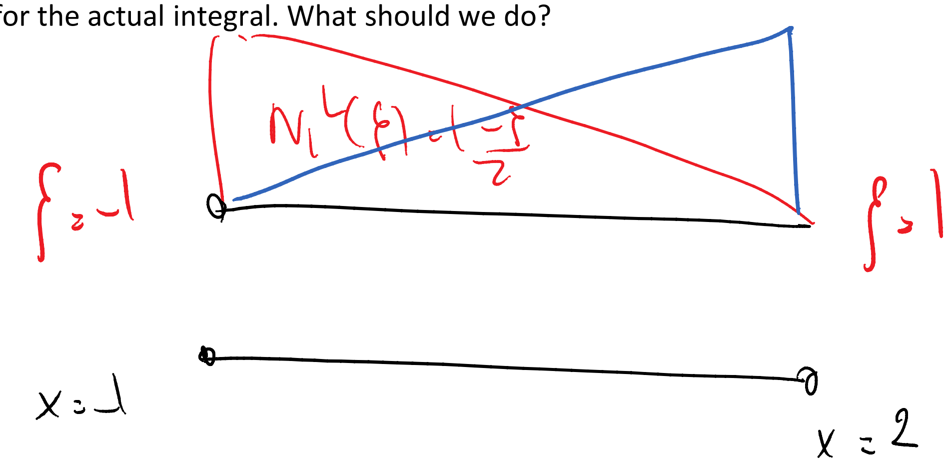
$$\int_{-1}^{2} \frac{1}{1+x^2}\, dx$$

How to use Gauss Table?

$$\int_{-1}^{1} f(\xi)\, d\xi = \sum_{i=1}^{ng} w_i\, f(\xi_i)$$

$N_2^L(\xi) = \frac{1+\xi}{2}$

Not the same limits for the actual integral. What should we do?



$$x(\xi) = x_1 N_1^L(\xi) + x_2 N_2^L(\xi)$$

$$= (-1)\left(\frac{1-\xi}{2}\right) + 2\left(\frac{1+\xi}{2}\right)$$

$$\Rightarrow \boxed{x(\xi) = \frac{3}{2}\xi + \frac{1}{2}}$$

$\xi = -1 \rightarrow x = -1$

$\xi = 1 \rightarrow x = 2$

$\int^{2} 1 \ldots \quad \int^{\xi=1} \quad \underbrace{\qquad}_{}$ dx

$$\int_{-1}^{2} \frac{1}{1+x^2}\,dx = \int_{\xi=-1}^{\xi=1} \frac{1}{1+\left(\frac{3}{2}\xi+1\right)^2} \left(\frac{3}{2}d\xi\right)$$

$$= \int_{-1}^{1} \underbrace{\left\{ \frac{3}{2} \frac{1}{1+\left(\frac{3}{2}\xi+1\right)^2} \right\}}_{f(\xi)} d\xi$$

$$= \sum_{i=1}^{nq} w_i \, f(\xi_i)$$

Imagine if I asked to integrate the integrand as a second order polynomial. $p=2$

How many Gauss points are needed?

$$nq = \text{Ceiling}\left(\frac{p+1}{2}\right) = \text{Ceiling}(1.5) = 2$$

$$w_1 = 1 \quad \xi_1 = \frac{-1}{\sqrt{3}}$$

$w_2 = 1 \qquad \xi_2 = \frac{1}{\sqrt{3}}$

Now that we know how to integrate numerically we go back to stiffness equation for the second order element from the last session:

$$k^e = \underbrace{\overbrace{\int_{\xi=-1}^{\xi=1} \underbrace{\frac{EA(\xi)}{\underbrace{\frac{L}{2} + 2\alpha L\xi}_{J}}}_{\text{part that we have trouble with}} \underbrace{\begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \begin{bmatrix} \xi - \frac{1}{2} & \xi + \frac{1}{2} & -2\xi \end{bmatrix}}_{\text{2nd order polynomial}}}^{f(\xi)} d\xi}_{\text{What is the order of integrand?}} \qquad \text{(I)}$$

What if : (1) Element is homogeneous EA constant

(2) Element is not skewed

$\downarrow$

$J =$ constant

From Bathe: element full-integration order is used if element is not highly distorted or is not highly nonlinear

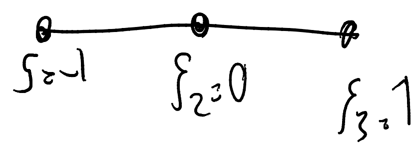
Using this integration order for a geometrically distorted element will not yield the exactly integrated element matrices. The analysis is, however, reliable because the numerical integration errors are acceptably small assuming of course reasonable geometric distortions. Indeed, as shown by P. G. Ciarlet [A], if the geometric distortions are not excessive and are such that in exact integration the full order of convergence is still obtained (with the provisions discussed in Section 5.3.3), then that same order of convergence is also obtained using the full numerical integration recommended here. Hence, in that case, the order of numerical integration recommended in Table 5.9 does not result in a reduction of the order of convergence. On the other hand, if the element geometric distortions are very large, and in nonlinear analysis of course, a higher integration order may be appropriate (see Section 6.8.4).

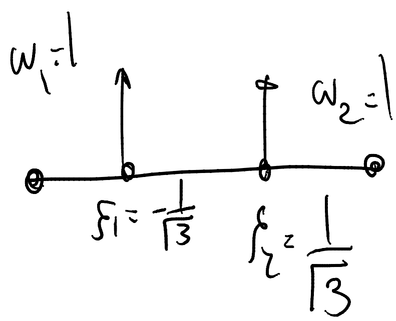
So -> full integration order (integrand order for homogeneous and un-skewed element) -> p = 2

Simpson rule $w_1 = \frac{2}{6}$ $w_2 = \frac{8}{6}$ $w_3 = \frac{2}{6}$

Number of quad points:

1. Newton-Cotes: nq = p + 1 = 3
2. Gauss quadrature: nq = ceiling((p + 1)/2) = ceiling (1.5) = 2





$$K^e = \frac{2}{6} f(\xi = -1) + \frac{8}{6} f(\xi = 0) + \frac{2}{6} f(\xi = 1)$$

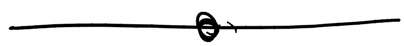
Newton-Cotes

$$k = 1\, f\left(-\frac{1}{\sqrt{3}}\right) + 1\, f\left(\frac{1}{\sqrt{3}}\right)$$

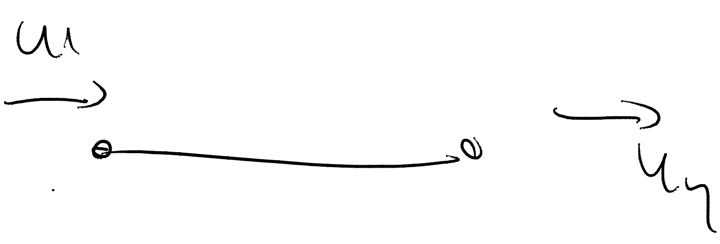
Gauss - Quadrature

What happens if we use fewer points that full-integration order?

$$k^e = 2\, f(0)$$



We may get nonphysical zero modes



$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\underbrace{\qquad\qquad}_{K}$$

$$\det K = 0$$

zero eigenvalue $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{f} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{u}$$

rigid body motion

"no force"

$\vec{u}_1 = 1$ ——— $\vec{u}_2 = 1$

$\vec{f}_1 = 0$ ——— $\vec{f}_2 = 0$

Element stiffness always has singular modes

\# 0 eigenvalues $\equiv$ # physical zero modes

$$\text{rank}(K) = \dim(K) - \text{zero}(K)$$

$$= 2 \begin{bmatrix} & \\ & \end{bmatrix}_{2\times 2} - 1 = 1$$

$$K = \begin{bmatrix} 7/3 & -8/3 & 1/3 \\ & 16/3 & -8/3 \\ \text{sym} & & 7/3 \end{bmatrix}_{3\times 3}$$

$$\left\{ \begin{array}{l} \dim = 3 \\ \#\,\text{zero} = 1 \end{array} \right.$$

$$\text{rank} = 3 - 1 = 2$$