What will be covered today:

1. Weighted residual statement for a bar problem.
2. Abstract form of weighted residual statement.

PDE (ODE here)

$$\frac{d}{dx}(F) + q = 0$$

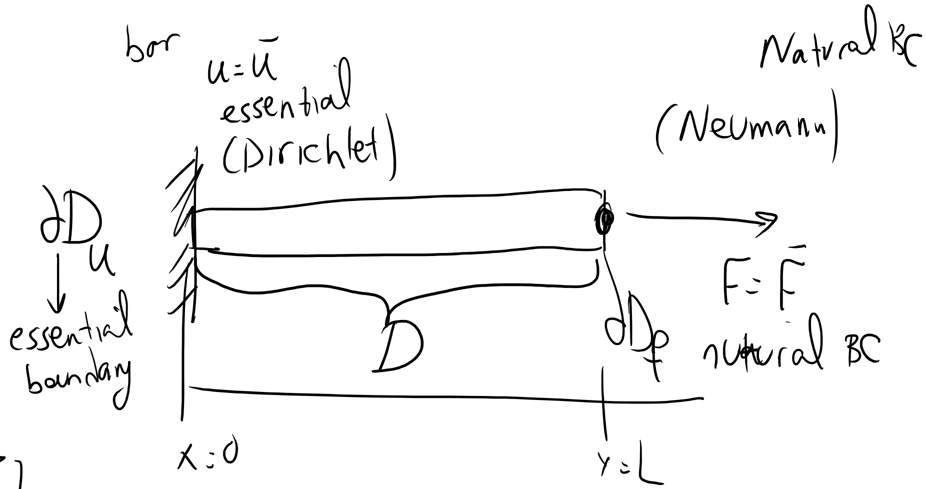
Constitutive eqn

$$F = A\sigma = A(E\,\varepsilon)$$

strain

Compatibility

$$\varepsilon = u_{,x}$$



$$\mathcal{L}_M(u) + q = (EAu')' + q = 0$$ DL in D

Differential operator on u

$$\mathcal{L}_M(u) = (EAu')'$$ order of differention $M = 2$

$m = 1$

BCs

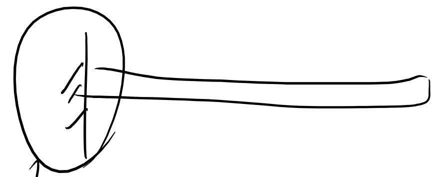
Left side $$u(x=0) = \bar{u}$$ Essential BC

$$\mathcal{L}_u(u) = \bar{u} \quad \text{on } \partial D_u$$ on essential

Here $L_u(V(x)) = v(x)$

order of differentiation 0 boundary

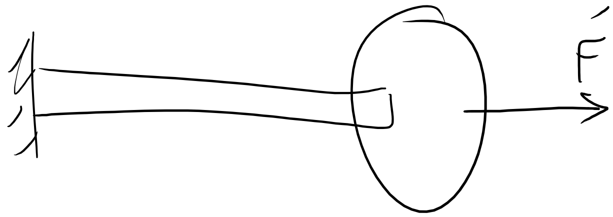
$$\partial D_u = \{0\}$$



$x = 0$

Natural BC:

$x = L$



$\bar{F}$

$$F = \bar{F} \quad , \; F = A\sigma = AE\varepsilon = AEu'$$

$$AEu' = \bar{F} \quad \text{at} \quad x = L$$

Abstract way of writting this equation

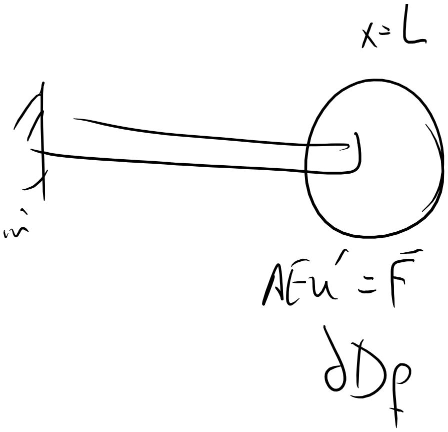
$$L_f(u) = \bar{F} \quad \text{at} \quad \underbrace{\partial D_f}_{\text{natural boundary of the domain}}$$

natural BC differential operator

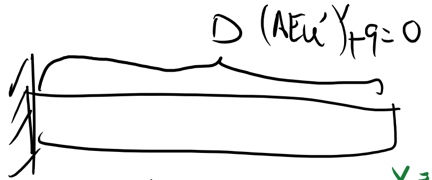
$$L_f(u) = AEu' \qquad \text{order of } L_f \quad 1$$

∂D $\{L\}$ $x = L$

$$\underbrace{\partial D_f}_{\text{natural boundary of the domain}} = \{L\}$$



Complete set of equations



1) • $\underbrace{L_M(u) - r}_{}$ $= 0$ (source term r) : $\underbrace{(AEu')'}_{L_M(u)} + \underbrace{q}_{= -r} = 0$ (DE) over D

$x = 0$: ∂D_u, $u = \bar{u}$ — Essential BC

$x = L$: $F(u): AEu' = \bar{F}$ — Natural BC, ∂D_f

2) • $L_u(u) = \bar{u}$ $\qquad \underbrace{u}_{L_u(u)} = \bar{u}$ on ∂D_u

3) • $L_f(u) = \bar{F}$ $\qquad \underbrace{AEu'}_{L_f(u)} = \bar{F}$ on ∂D_f

Exact solution satisfies all these equations (DE, Essential BC, and natural BC) in a strong form:

Strong form: anything written for all points:
either inside the domain or on the boundaries (essential or natural)

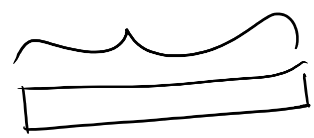
Weak form: anything in an integral form (example balance law).

Pathway to solving this numerically:
We define

RESIDUALS (= ERRORS)

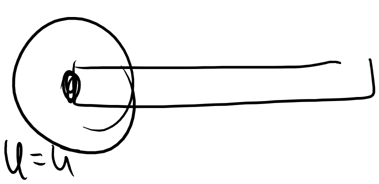
$$\underbrace{R_i}_{\text{inside residual (error)}}(u) = \overbrace{L_M(u) - r}^{\text{Differential eqn}} = (AEu')' + q$$

defined over D

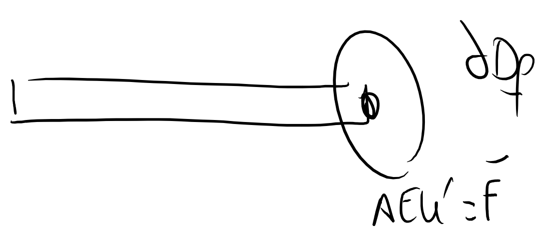


$$R_u(u) = \bar{u} - L_u(u)$$

defined on ∂D_u



$$R_f(u) = \bar{F} - L_f(u)$$



We can allow having residuals (errors) in satisfying PDE (Ri) and BCs (Ru and Rf)

Weighted Residual Statement (WRS):

$w(x)$ — Arbitrary function

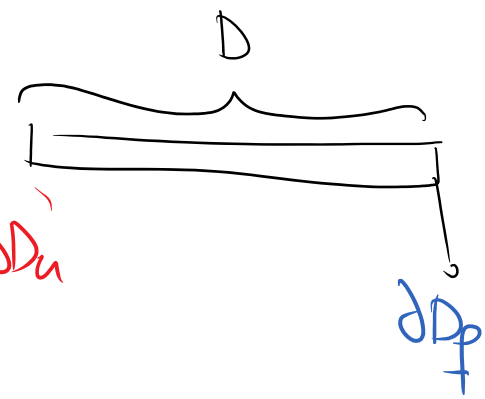
$R_i(u)$

$R_u(u)$

$R_f(u)$

$$\int_{D} w(x)\, R_i(u(x))\, dx + \int_{\partial D_u} w(x)\, R_u(u(x))$$

$$\int_{\partial D_f} w(x)\, R_f(u(x))$$



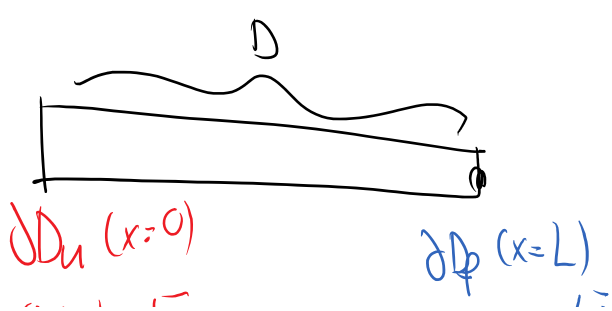
in 1D $\int_{\partial D_u}, \int_{\partial D_f} \rightarrow$ Point values

$G_u(u) = (AEu')' + q$

$R_i = (AEu')' + q$

$R_u = \bar{u} - u$

$R_f = \bar{F} - AEu'$



$R_f = F - AEu$

$L_u(u): u = \bar{u}$

@ (x=L)

$L_f(u) = AEu' = \bar{F}$

WRS:

$$\int_0^L \omega(x)\, R_i(u(x))\, dx + \omega'(x)\underbrace{(\bar{u} - u')}_{R_u}\Big|_{\underbrace{x=0}_{\partial D_u}} + \omega(x)\left(\bar{F} - AEu'\right) = 0$$

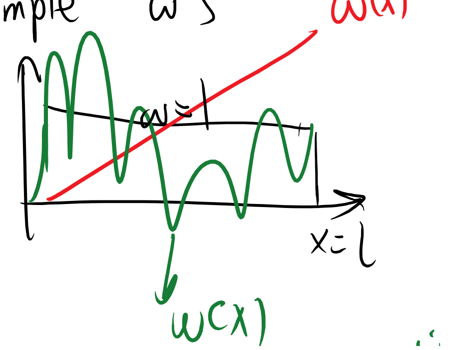
sum of orders $M-1 = 1$

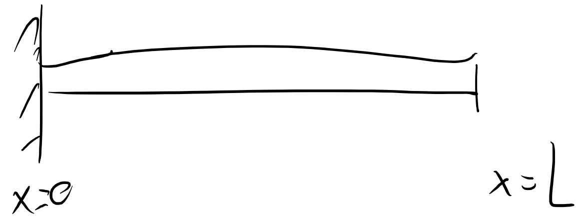
sum of orders $= M-1 = 2-1 = 1$

$$\forall \omega \quad \boxed{\int_0^L \omega(x)\overbrace{\left((AEu')' + q\right)}^{R_i} dx + \omega'(x)\overbrace{(\bar{u} - u)}^{R_u}\Big|_{x=0} + \omega(x)\underbrace{\left(\bar{F} - AEu'(x)\right)}_{R_f}\Big|_{x=L} = 0}$$

ω is arbitrary

sample ω's





any thing you like

Since the weight function is arbitrary (and we can choose anything we like) we can prove that the only way WRS = 0 for all weights is that

- All residuals should be strongly (meaning that at all points) be equal to zero!

$\forall w$

Balance law

$\boxed{\sum F = 0}$

w_1

$(\sum F)_{w_1} = 0$

w_2

$(\sum F)_{w_2} = 0$

w_3

$(\sum F)_{w_3} = 0$

$\forall x$

Exact sln

$R_i = (AEu')' + q = 0 \; \big|_{D=[0,L]}$

$R_u = \bar{u} - u = 0 \; \big|_{x=0}$

$R_p = \bar{F} - AEu' = 0$

WRS

$\int_0^L w(x) R_i(u)\, dx$

$+ w(x) \; R_u(x) \big|_{x=0}$

$+ w(x) \; R_p(x) \big|_{x=L} = 0$

$\forall w$

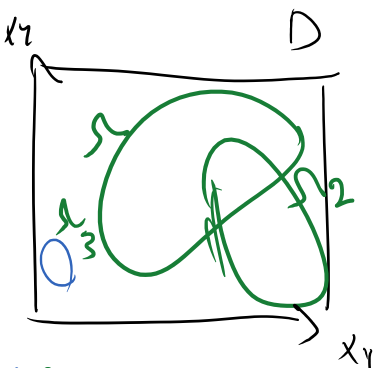
trivial

true because w is arbitrary!

Why did we start with weak form (balance laws integrated over arbitrary domains) ->
Strong form (PDEs - augmented with BCs)
AND finally got to WRS which again is an integral statement (involves integrals)

Elasto statics in 2D/3D (generalization of bar problem) read it from course notes

Balance law



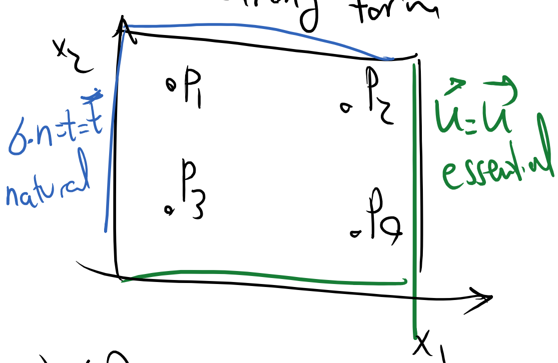
$\forall \Omega$:

$$\int_{\partial\Omega} f_x n \, ds - \int_{\Omega} r \, dV = 0$$

$$\int_{\Omega} (\sigma) \cdot n \, ds - \int_{\partial\Omega} \rho b \, dV = 0$$

$$\sigma = C \varepsilon$$

Strong form



$\forall P$ $\quad \nabla \cdot f_x - r = 0$

$$-\nabla \cdot \sigma - r = 0$$

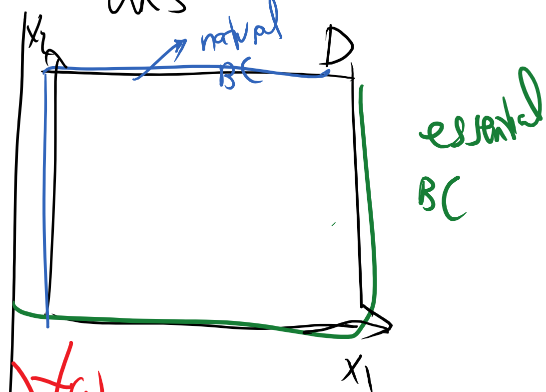
when augmented with

$\sigma = C \varepsilon$ const. eqn

$\varepsilon = \dfrac{\nabla u + \nabla u^T}{2}$ compatibility

$\forall P$: $L_M(u) = C_{ijkl} u_{k,lj} + \rho b_i = 0$

WRS



$\forall w$

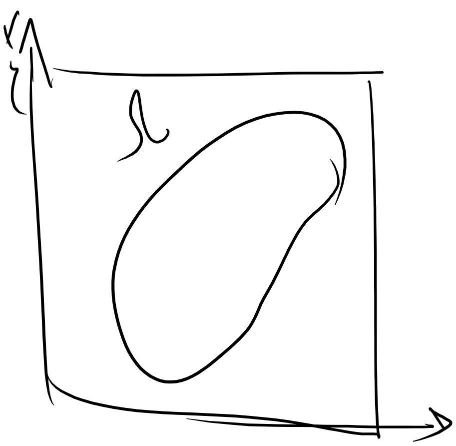
$$\int_D w_i(x) \overbrace{\left(C_{ijkl} u_{k,lj} + \rho b_i\right)}^{R_i(u(x))} dV + \int_{\partial D_u} w_i \overbrace{\left(U_i - u_i(x)\right)}^{R_u(u(x))} ds + \int_{\partial D_p} w_i(x) \overbrace{\left(t_i - C_{ijkl} u_{k,l} n_j\right)}^{R_p(u)} ds$$

∂D_f

$= c$

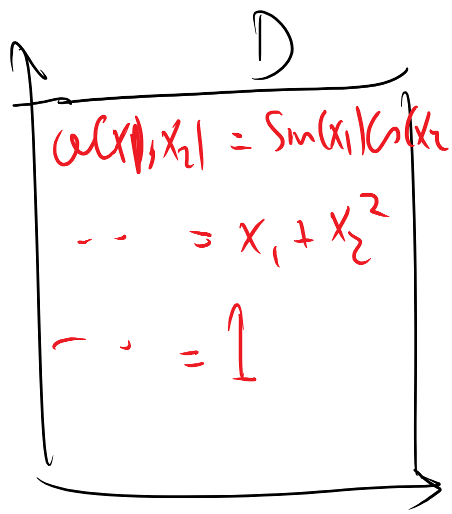
In balance law get to exact solution by having ALL possible domains of integration for balance law (Ω')

In WRS: domain of integration is fixed. We multiply errors = residuals (inside, and on boundaries) by ARBITRARY weight functions -> prove we recover the exact solution this way.



x_1

OR



Balance law

$$\int_{\partial\Omega} t_x\, ds - \int_{\Omega} r\, dV = 0$$

$\forall w$

$$\int_D w\, (R_i(u(x))\, dV$$

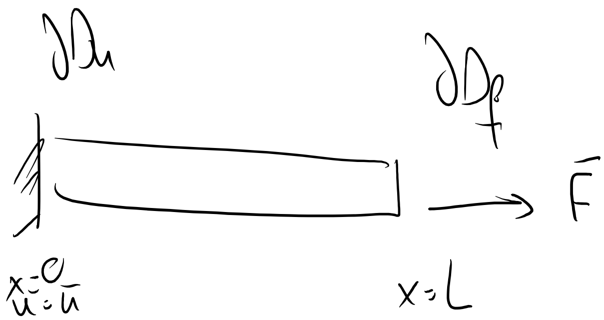
$$\int_{\partial D_u} w\, R_u(u(x))\, dS$$

$$\int_{\partial D_f} w\, R_f(u(x))\, dS$$

The motivation of WRS is that the domains of integrals are fixed, we just change the weight functions, as the choice of weight functions increases -> we approach the exact solution.

This is much easier than arbitrary domain integrations with balance laws.

Going back to our bar problem:



$$\int_0^L w(x)\, R_i(x) + w'(x)\, R_u(u(x))\Big|_{x=0} + w(x)\, R_f(u(x))\Big|_{x=L} = 0$$

$$\forall w(x) \int_0^L w(x)\left((AEu')' + q\right)dx + w'(0)\left(\bar{u} - u(0)\right) + w(L)\left(\bar{F} - AEu'(L)\right) = 0$$

All weight functions

order of $w = 0$

order of $u = 2$

order of $w = 1$

order of $u = 0$

order of $w = 0$

order of $u = 1$

order of $\omega = 0$ — 2 order of $\omega = 1$ order of $\omega = 0$

Maximum order of $u = 2$

" " " $\omega = 1$ (on ∂D_u)

Next time we will have variants of WRS wherein

1. Essential and Natural BCs are exactly satisfied (so they don't appear in WRS)
2. Essential BC is exactly satisfied.
3. Natural BC is exactly satisfied.