2017/09/28

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Last example was on Galerkin method where the weights are equal to trial functions

Last time we used this set of trial functions:

$\phi = \{x, x^2\}$ —Galerkin→ $\omega = \{x, x^2\}$

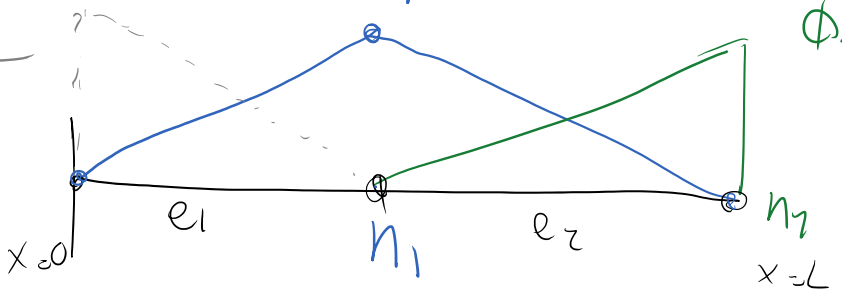
This time we want to choose FE type trial functions:

FE terminology "Shape Function"

$\phi_1(x) = N_1(x)$

$\phi_2(x) = N_2(x)$

shape functions take the value 1 at 1 node elsewhere



$u(x=0) = \bar{u} = 1$

fixed / prescribed

$$\phi_1(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \end{cases}$$

$$\phi_2(x) = \begin{cases} 0 & 0 < x < 1 \\ x - 1 & 1 < x < 2 \end{cases}$$

It's easier to use weak form:

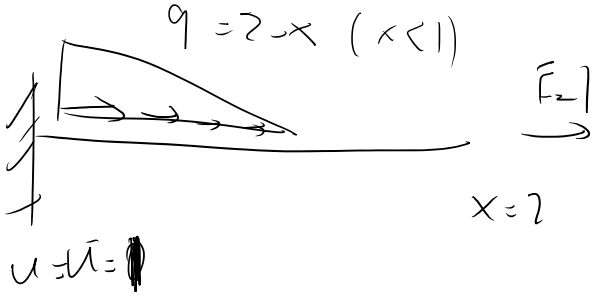
$$u^h = \phi_p + \begin{bmatrix} \phi_1(x) & \phi_2(x) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$\phi_p = 1$ we still use this

$$w = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix}$$ because it's a Galerkin method

Weak statement for this problem

$$\textcircled{*} \quad \int_0^2 w' \underbrace{AE u'}_{1} dx = \int_0^1 w q(x) + w(2) \bar{F}$$

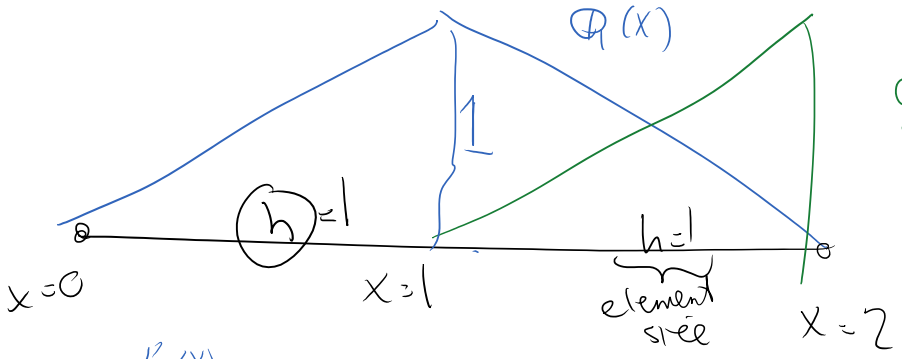


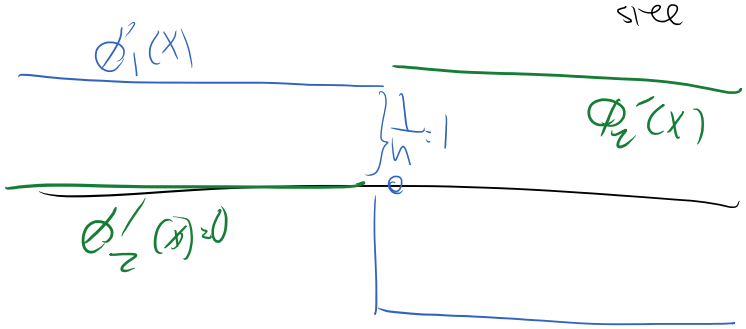
plugging u^h & $w = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$ in $\textcircled{*}$ we obtain:

$$\underbrace{\left(\int_0^2 \begin{bmatrix} \phi_1'(x) \\ \phi_2'(x) \end{bmatrix} \begin{bmatrix} \phi_1'(x) & \phi_2'(x) \end{bmatrix} dx \right)}_{\text{stiffness matrix}} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underbrace{\int_0^1 \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \underbrace{q}_{(2-2x)} dx + \begin{bmatrix} \phi_1(2) \\ \phi_2(2) \end{bmatrix}}_{\text{force vector}}$$

Stiffness matrix

Force vector





$$\text{Stiffness Matrix} = \int_0^2 \begin{bmatrix} \phi_1'\phi_1' & \phi_1'\phi_2' \\ \phi_2'\phi_1 & \phi_2'\phi_2' \end{bmatrix} dx$$

$$\int_0^2 \phi_1'\phi_1'\,dx = \int_0^1 \phi_1'\phi_1'\,dx + \int_1^2 \phi_1'\phi_1'\,dx = \int_0^1 (1)(1)\,dx + \int_1^2 (-1)(-1)\,dx$$

$$= 2$$

$$K_{12} = \int \phi_1'\phi_2'\,dx = \int_1^2 (1)(-1)\,dx = -1$$

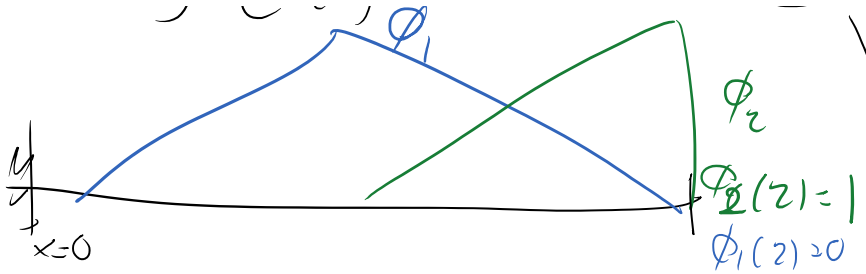
$$K_{22} = \int \phi_2'\phi_2'\,dx = \int_1^2 (1)(1)\,dx = 1$$

$$K = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

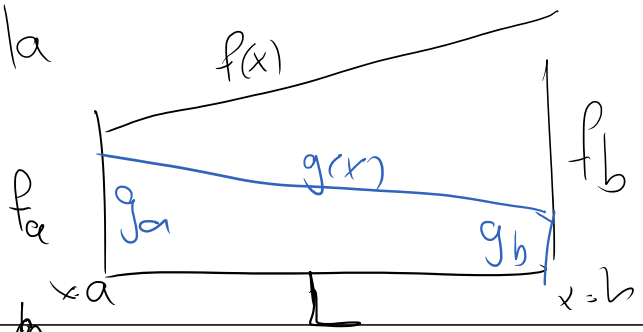
$$\text{RHS}: \quad F = \int \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} (2-2x)\,dx + \begin{bmatrix} \phi_1(2) \\ \phi_2(2) \end{bmatrix}$$

$\phi_1(2) = 0$

$\phi_2(2) = 1$



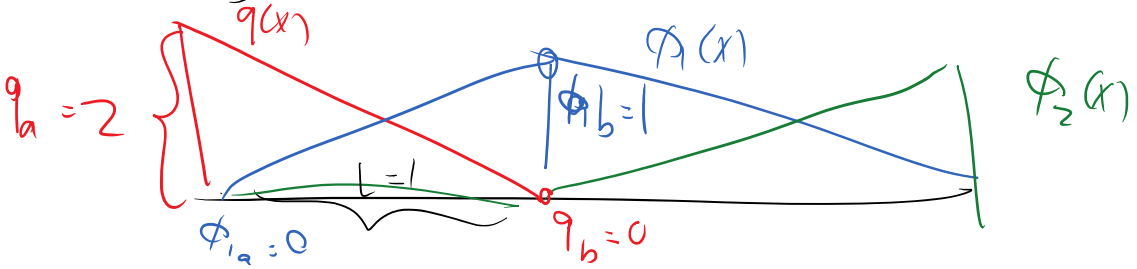
Useful formula



f & g are linear

$$\int_a^b f(x)g(x)\,dx = \frac{L}{6}\left(2f_a g_a + 2f_b g_b + f_a g_b + f_b g_a\right)$$

$$F = \int_0^1 \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} q(x)\,dx + \begin{bmatrix}0\\ 1\end{bmatrix}$$



$$\int_0^1 \phi_1 q\,dx = \frac{1}{6}\left(2\times2\times0 + 2\times0\times1 + 2\times1 + 0\times0\times1\right) = \frac{1}{3}$$

$$\int_0^1 \phi_2 q\,dx = 0$$

$$\int_0^6 \phi_2 \, q \, dx = 0$$

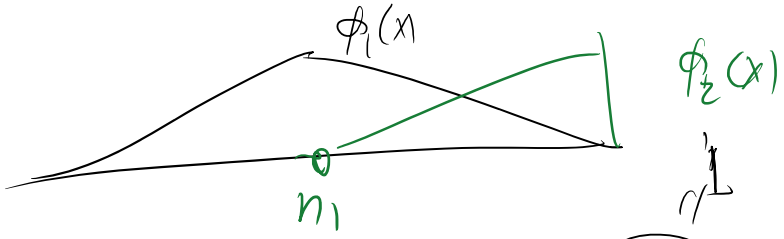
$$F = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$Ka = F \Rightarrow \boxed{a = \begin{bmatrix} 4/3 \\ 7/3 \end{bmatrix}}$$

$$u^h(x) = \phi_p + a_1 \phi_1(x) + a_2 \phi_2(x)$$

$$\boxed{u^h(x) = 1 + 4/3 \, \phi_1(x) + 7/3 \, \phi_2(x)}$$



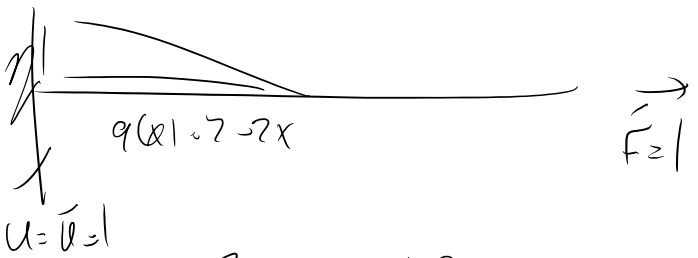
$$u^h(n_1) = u^h(x=1) = 1 + 4/3 \, \phi_1(1) = 7/3$$

This and one from last time where both Galerkin methods. Last time we used $\{x, x^2\}$. This time $\phi_1(x), \phi_2(x)$

Last method:
Energy method in discrete setting:
We write the energy statement for the approximate solution space ->
Energy will be a function of n unknowns ->
Then we minimize this function of n variable to get the numerical solution (basically numerical solution MINIMIZES the energy in the space of functions sampled)

This approach is called Ritz method.

Example for the same problem we did last time



$$\Pi = V - W = \int_0^2 \frac{1}{2} EA u'^2 dx - \left(\int_0^2 q u \, dx + u(2)\bar{F} \right)$$

Let's plug approximate solution

$$u^h = \phi_p + \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{in} \quad \Pi$$

$\phi_p = 1$

$\phi_1 = x$, $\phi_2 = x^2$ (we use this basis function)

$$\boxed{u^h = 1 + a_1 x + a_2 x^2}$$ plug it in

$u = 1 + a_1 x + a_2 x^2$ ply " "

$$\Pi(u^h) = \int_0^2 \frac{1}{2} (u_h')^2 \, dx - \int_0^1 u^h(x) q(x) \, dx - u^h(2) F$$

$u^{h'} = a_1 + 2a_2 x$ (Recall)

$$\Rightarrow \Pi(u^h) = \int_0^2 \frac{1}{2}(a_1 + 2a_2 x)^2 dx - \int_0^1 (1 + a_1 x + a_2 x^2)(2 - 2x) \, dx$$

$$- (1 + 2a_1 + 4a_2) \times 1$$

$$\Pi(a_1, a_2) = a_1^2 + 4a_1 a_2 + \frac{16}{3} a_2^2 - \frac{7}{3} a_1 - \frac{25}{6} a_2 - 2$$

Minimize Π

$$\nabla \Pi = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{\partial \Pi}{\partial a_1} \\ \frac{\partial \Pi}{\partial a_2} \end{bmatrix} = \begin{bmatrix} 2a_1 + 4a_2 - \frac{7}{3} \\ 4a_1 + \frac{32}{3} a_2 - \frac{25}{6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow K a = F$$

$$K = \begin{bmatrix} 2 & 4 \\ 4 & 32/3 \end{bmatrix} \quad F = \begin{bmatrix} 7/3 \\ 25/6 \end{bmatrix} \rightarrow a = \begin{bmatrix} 37/24 \\ -3/16 \end{bmatrix}$$

$$u^h = \phi^p + a_1 x + a_2 x^2 \Rightarrow u^h_{FE} = 1 + \frac{37}{24} x - \frac{3}{16} x^2$$

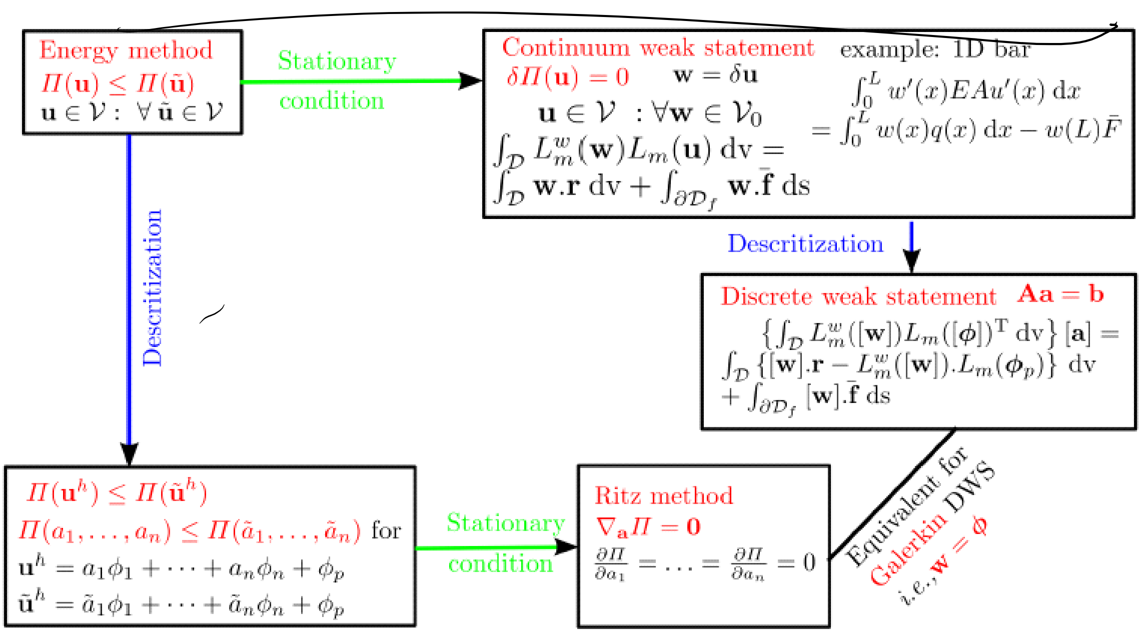
$u^h = \phi^p + a_1 x + a_2 x^2 \Rightarrow u^{h}_{E_2} = 1 + \frac{57}{24}x - \frac{3}{16}x^2$

energy, 2 unknowns

We got exactly the same solution with $\phi = \{x, x^2\}$ Galerkin $w = \{x, x^2\}$

& weak statement

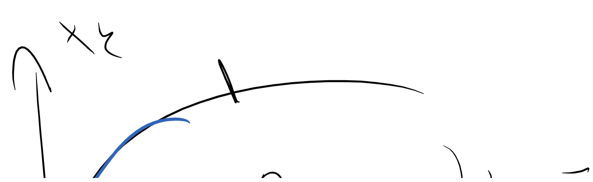
Relation between Energy Method and Weak Statement



This means that Galerkin method gives us the best solution in terms of having the minimum energy!

General weighted residual statement:

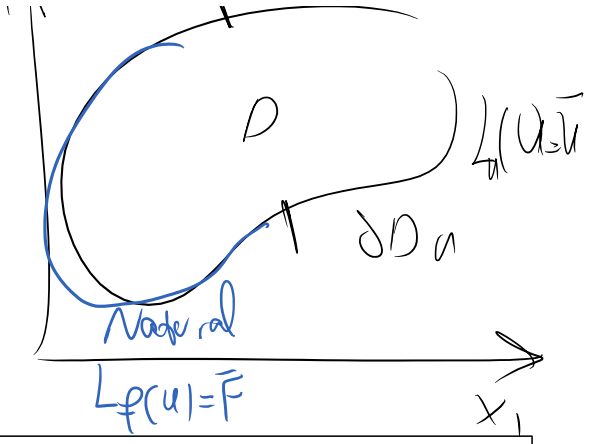
$R_i = L_M(u) - r$



$$R_i = L_M(u) - r$$

$$R_f = \bar{F} - F = \bar{F} - L_f(u)$$

$$R_u = \bar{u} - u$$



unknowns

II.a $u^h = \phi_p + \sum_{i=1} a_i \phi_i(x)$

II.b WR: $\int_D w\, R_i(u^h) + \int_{\partial D_f} w^f R_f(u)\, ds = 0$

w^f: can be distinct from inside weights

ϕ_p: satisfies essential BC

ϕ_i's: Homogeneous essential BC

plug solution from II.a into WR (II.b)

$$\int_D \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \underbrace{\left\{ L_M\left(\phi_p + [\phi_1 \cdots \phi_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) - r \right\}}_{R_i} dV$$

u^h

$$\int \begin{bmatrix} w_1^f \\ w_2^f \\ \vdots \\ w_n^f \end{bmatrix} \underbrace{\left(\bar{F} - L_f\left(\phi_p + [\phi_1 \cdots \phi_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) \right)}_{R_f} ds = 0$$

$$K = \int \begin{bmatrix} w_1 \\ w_2 \\ \vdots \end{bmatrix} \begin{bmatrix} L_M(\phi_p) & \cdots & L_M(\phi_n) \end{bmatrix} - \int \begin{bmatrix} w_1^f \\ w_2^f \\ \vdots \end{bmatrix} \begin{bmatrix} L_f(\phi_1) & \cdots & L_f(\phi_n) \end{bmatrix} ds$$

$$K = \int_D \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \left[L_M(\phi_1) \cdots L_M(\phi_n) \right] - \int_{\partial D_f} \begin{bmatrix} w_1^f \\ \vdots \\ w_n^f \end{bmatrix} \left[L_f(\phi_1) \cdots L_f(\phi_n) \right] ds$$

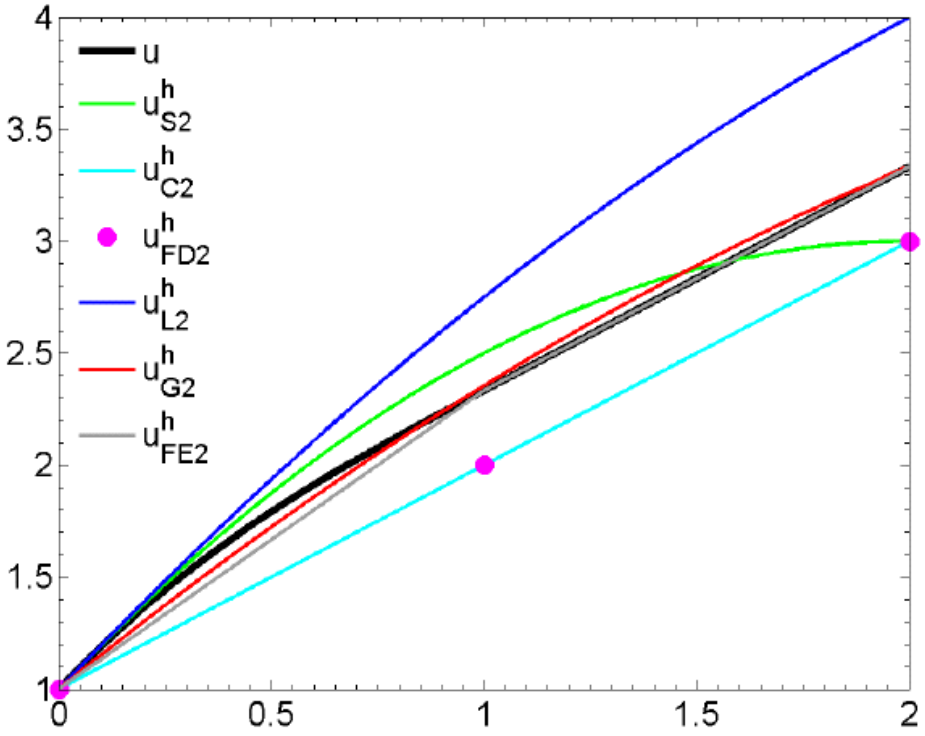
$$F = -\int_D \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \left(L_M(\phi_p) - r \right) dv - \int_{\partial D_f} \begin{bmatrix} w_1^f \\ \vdots \\ w_n^f \end{bmatrix} \bar{F} \, ds$$

$$Ka = F$$

General K, F statements for ANY weighted residual method

Going back to comparison of all solutions we did for two unknowns:

Bar example, $n = 2$, Comparison of solutions



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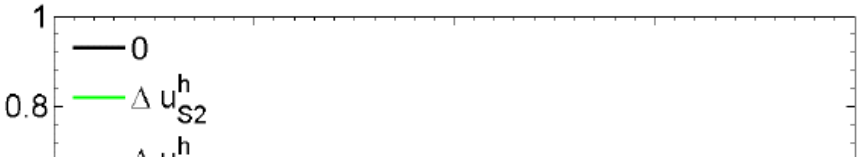
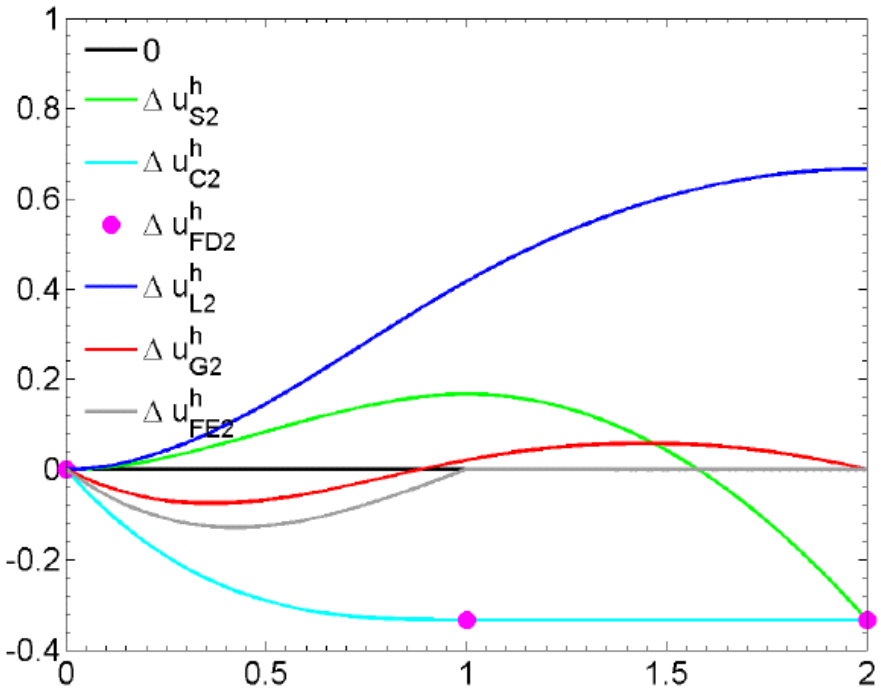


Bar example, $n = 2$, Comparison of solutions



Notes:

1. Least square method is not very accurate. Even though that method has the smallest error in residual (since we minimized R2), it's solution if farther than some other methods that use the same interpolation function space of {x, x2} (subdomain, Galerkin--the one with {x, x2}, not FEM bases--collocation). The reason is that we minimize R2 which does not necessarily imply that the error of solution versus the exact solution is minimum. We cannot minimize the error w.r.t exact solution because if had the exact solution, we wouldn't bother solving for it!
2. Collocation and FD are not accurate either because they only enforce PDE (ODE here) (and natural boundary conditions) at a few points. As in the problem solved, they may completely miss important information such as source term in some regions. Finite difference is even worse, because the derivative values are only approximate.
   The idea of collocation method (that is satisfying something at few points and hoping that the solution approaches exact solution as the number of sample point increases) is very useful.

For example, in stochastic finite element methods, one method is to sample the problem in random space at a few points that are carefully chosen and solve the sampled problem (for example at certain temperature, elastic modulus, etc., i.e. things that are random variables) with a conventional FE package such as Ansys. That's where this idea becomes very useful.

3. Galerkin methods have the smallest energy within the space of interpolated solution functions ({x, x^2} for the first one and FEM hat functions for the FEM solution). The reason for this is that Galerkin method is equivalent to Ritz method that minimizes the energy with the space of sampled solutions. One can also show that they have the smallest error in terms of energy (That is computing the energy induced by the error between the approximate and exact solutions). In fact, having the smallest energy error as can be seen is a much better condition as having the smallest residual error (as can be seen in the solution here).
4. Between the two Galerkin solutions represented, FEM method recovers the exact solution at finite element nodes. Unfortunately this amazing property that almost always holds for 1D problems does not extend to 2D and 3D problems.