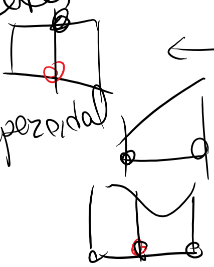
	# Quad pts	p (polynomial order)
Last time	n	
Gauss pts	n	$\longrightarrow p = 2n-1$
Newton-Cotes	n	$\longrightarrow p = \begin{cases} n-1 & n \text{ even} \\ n & n \text{ odd} \end{cases}$

Newton-Cotes

rectangle

trapezoidal

simpson



n	p
1	$0+1=1$
2	1
3	$2+1=3$
4	3
5	$4+1=5$
6	5

p	n
1	1
2	3
3	3

From order # pts

Gauss $\quad p = 2n-1 \rightarrow n = \text{ceil}\left(\frac{p+1}{2}\right)$

Newton-Cotes $\quad n = \begin{cases} p & p \text{ odd} \\ \boxed{p+1 \quad p \text{ even}} \end{cases}$

$\boxed{p+1 \quad p \text{ even}}$

Short formula for you (in FEM order p is almost always $\boxed{\text{EVEN}}$ ⟹)
simplified case

$$p \text{ given} \begin{cases} \rightarrow n = \text{ceiling}\left(\frac{p+1}{2}\right) & \text{GPs} \\ \rightarrow n = p+1 & \text{Newton-Cotes} \end{cases}$$

From last time we had the stiffness matrix integral

$$k^e = \underbrace{\int_{-1}^{1} \frac{1}{\underbrace{\frac{L}{2}(4\alpha\xi+1)}_{J}} \begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \underbrace{AE(\xi)}_{\text{Composite FGM... changes}} \left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi\right] d\xi}_{I(\xi)}$$

Full integration order $J = \text{cte}$ (constant for $\alpha = 0$)

AE is constant (homog. material)

$P = \text{order} = 2$

$\rightarrow$ $n = \text{ceiling}\left(\frac{?+1}{2}\right) = 2$ GPs

$w_1 = 1 \quad w_2 = 1$

ξ_i's $\quad -\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}}$

$\rightarrow$ Newton-Cotes $\quad n = p + 1 = 3$

$w_1 = \frac{1}{6} \quad w_2 = \frac{4}{6} \quad w_3 = \frac{1}{6}$

$\xi = -1 \quad \xi = 0 \quad \xi = 1$

Example

Gauss pts $\quad k^e = w_1 I(\xi_1) + w_2 I(\xi_2)$

$$\xi_1 = \frac{-1}{\sqrt{3}} \qquad \xi_2 = \frac{1}{\sqrt{3}}$$

$$w_1 = 1 \qquad w_2 = 1$$

Newton Cotes $\quad k^e = (9) \; (w_1 I(\xi_1) + w_2 I(\xi_2) + w_3 I(\xi_3))$

Newton Cotes $k^e = \underbrace{(2)}_{\text{length of domain}} \left(w_1 I(\xi_1) + w_2 I(\xi_2) + w_3 I(\xi_3) \right)$

$w_1 \to \frac{1}{6}$, $\xi_1 \to -1$, $w_2 \to \frac{4}{6}$, $\xi_2 \to 0$, $w_3 \to \frac{1}{6}$, $\xi_3 \to 1$

if $\alpha = 0$ we exactly integrate k^e

$$\boxed{k^e = \frac{AE}{L} \begin{bmatrix} 7/3 & -8/3 & 1/3 \\ & 16/3 & -8/3 \\ & & 7/3 \end{bmatrix}} \quad I$$

If (alpha != 0, skewed element) we never can integrate it exactly, no matter how many points we use (results may appear to be converged in finite precision calculations).

Rank of stiffness matrix, and rigid modes

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{AE}{L} \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{stiffness}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

n_1 $\qquad$ n_2

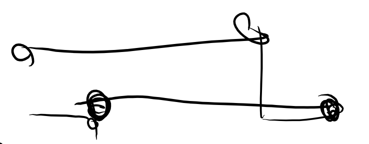
displacements $\rightarrow$ u_1 $\qquad$ $\vec{u_2}$

forces $\rightarrow$ f_1 $\qquad$ $\rightarrow f_2$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

are there $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq$ such that $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = 0$

$$u_1 = u_2 = u \longrightarrow f = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

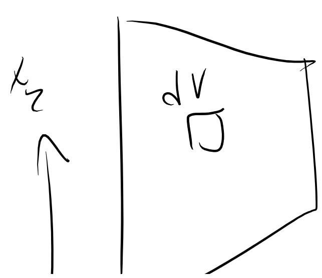


$u_1 = u$ $\qquad$ $u_2 = u$

For this 1 D element there is only 1 zero mode

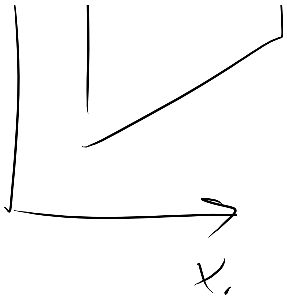
$$\int w' EA u' \, dx$$

$$u' = 0 \longrightarrow u = \text{constant}$$



x_2 $\qquad$ dV

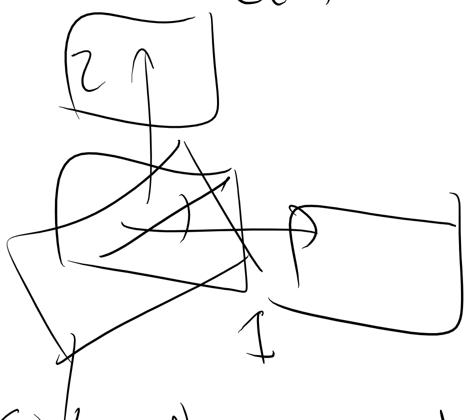
$$\int \varepsilon(w)\, C\, \varepsilon(u) \, dV$$



$$\int_D \varepsilon(w)\, C\, \varepsilon(u)\, dV$$

$\varepsilon(u) = 0$ $\qquad \varepsilon = \dfrac{\nabla u + \nabla u^T}{2}$

in 2D



3 small angle rotation (3 zero modes)

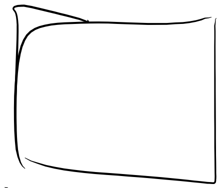
Thermal equation

weak form → $\int \nabla w \; k \; \nabla T \, dV$

k ↓ conductivity

T is scalar $\qquad \nabla T$

T is constant $\qquad T = cte \rightarrow q = -k\nabla T = 0$



T = cte everywhere

1 zero mode

Physically zero modes are primary field functions (T for thermal, displacement vector for elasticity, …) for which

SPATIAL FLUX fx = 0

$$\begin{cases} \text{solid} & \sigma = E\varepsilon = 0 \\ \text{thermal} & q = -k\nabla T = 0 \end{cases}$$

cases that we get

Zero "force"

with nonzero unknowns.

$$K\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}u_1\\u_2\end{bmatrix}$$

eigenvector

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq 0$$

$$k = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

eigenvector

$$\boxed{k\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}$$

eigenvalue

$$Ku = \lambda u$$

$u \neq 0$ → eigenvector

λ → eigenvalue

Zero modes ->

Eigenvectors of stiffness matrix for 0 eigenvalue

$$k = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Only 1 zero eigenvalue for which $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

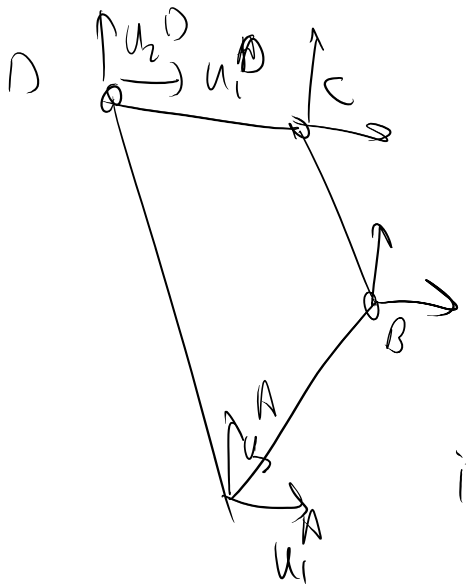
$$\underbrace{k\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{u} = \underbrace{0}_{\lambda}\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{u}$$

$$\underline{\mathrm{Rank}(k)} = \underbrace{\dim(k)}_{2} - \underbrace{\dim(\ker(k))}$$

def of rank   size of matrix   # zero eigen values

$$k = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times 2}$$

$$\text{rank}(k) = 2 - 1 = 1$$



$k^{e}_{8\times 8}$ # zero modes = 3

if we form stiffness for this

$$\text{rank}(K) = \text{dim} - \#\text{zero modes}$$
$$= 8 - 3 = 5$$

Good news:
With full integration order, although we may not integrate stiffness exactly but we preserve the rank of the stiffness, meaning that we don't add nonphysical zero modes.

If we integration that 8x8 stiffness and get rank = 2 ->

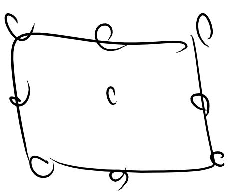
Number of zero modes = dim (k) - rank(k) = 8 - 2 = 6

3 of these modes are physically correct (x & y translations and z axis small deformation rotation), but unfortunately the remaining 3 are not physical:(

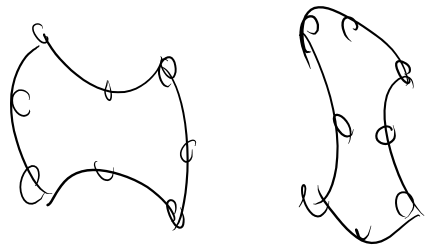
What happens with these nonphysical modes?

The element can take up ARBITRARILY LARGE DISPLACEMENTS without showing any resistance (forces)

p=2 2D element

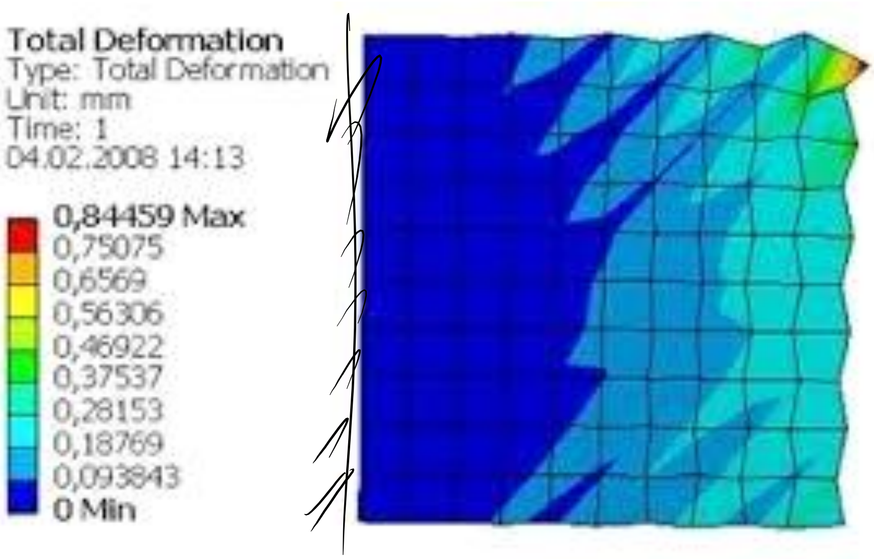


hour glass modes

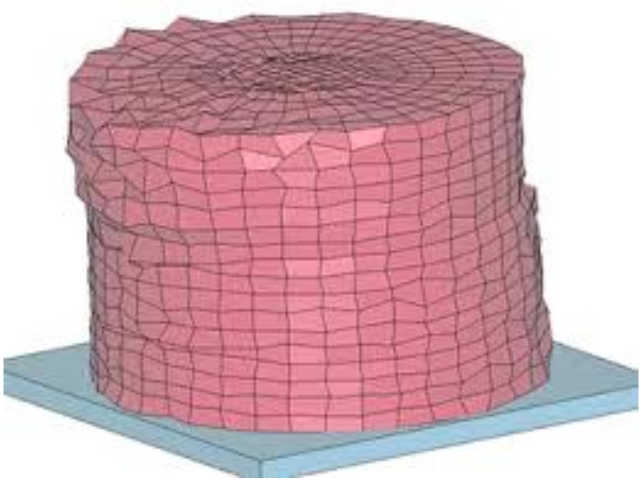


zero modes (force = 0)

These are not physical zero modes (rigid motion)



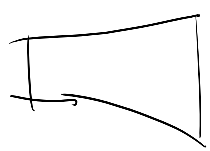
elements with nonphysical zero modes



If we under-integration (reduced integration order), meaning that we use fewer quadrature points than full integration order -> there is the possibility of getting nonphysical zero modes and in practice getting solutions like above.

Why do want to even go below full-integration number of points?

1. Reduce computation cost (e.g. use 3 points rather than 6 needed in a full integration scheme).
2. U -> F FEMs (stiffness-based) give stiffer results (stiffness is higher than exact). Under-integration often counter-acts this (make structure less stiff). So, if we under-integrate we may cancel these two effects. But this is dangerous because we may introduce nonphysical zero modes!



$$k_{FEM} = 1.5\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \qquad k_{exact} = \frac{1}{\ln 2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 1.44\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

In practice for some elements people have found under-integration rules that do not introduce nonphysical zero modes. They are frequently used in FEM analysis.

From Bathe's book:

Using this integration order for a geometrically distorted element will not yield the exactly integrated element matrices. The analysis is, however, reliable because the numerical integration errors are acceptably small assuming of course reasonable geometric distortions. Indeed, as shown by P. G. Ciarlet [A], if the geometric distortions are not excessive and are such that in exact integration the full order of convergence is still obtained (with the provisions discussed in Section 5.3.3), then that same order of convergence is also obtained using the full numerical integration recommended here. Hence, in that case, the order of numerical integration recommended in Table 5.9 does not result in a reduction of the order of convergence. On the other hand, if the element geometric distortions are very large, and in nonlinear analysis of course, a higher integration order may be appropriate (see Section 6.8.4).

Example from equation I: dim 3 stiffness for 1D elasticity -> 1 zero mode -> rank must be 2.

2D and 3D problems

We first do thermal problem, because it's a scalar problem (simpler than solid mechanics)

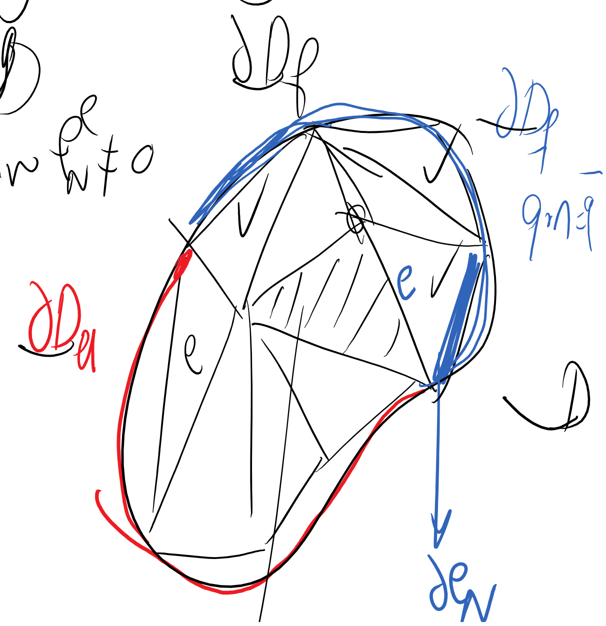
Weak form: $\int_D \underbrace{\nabla w}_{L_m} \underbrace{k}_{D} \underbrace{\nabla T}_{L_m} dV = \int_D w Q\, dV \quad \int_{\partial D} w \bar{q}\, dS$

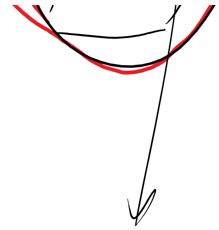
∇'s how fit to ∂D^e

$K^e = \int_e B^{e^T} D^e B^e \, dV$

material section

$B^e = L_m N^e, \quad D^e = k$ conductivity





$$f_r^e = \int_e N^{e^T} a \, dv$$

$$f_N^e = -\int_{\partial e_N} N^{e^T} \bar{q} \, ds$$

Neumann

∂e_N

$\partial e_N = \phi$ (empty set)

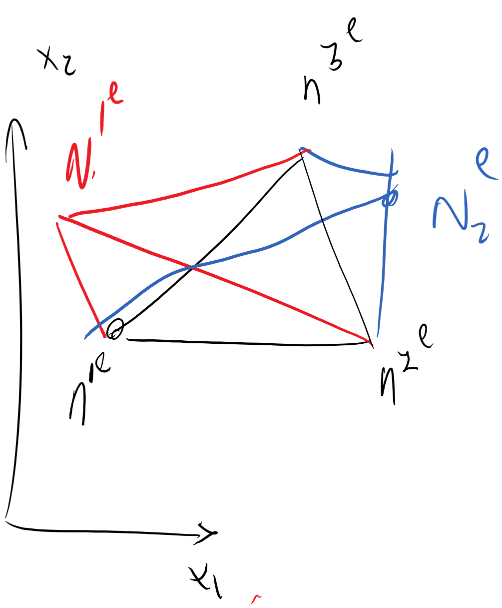
$$f_D^e = K^e a^e$$

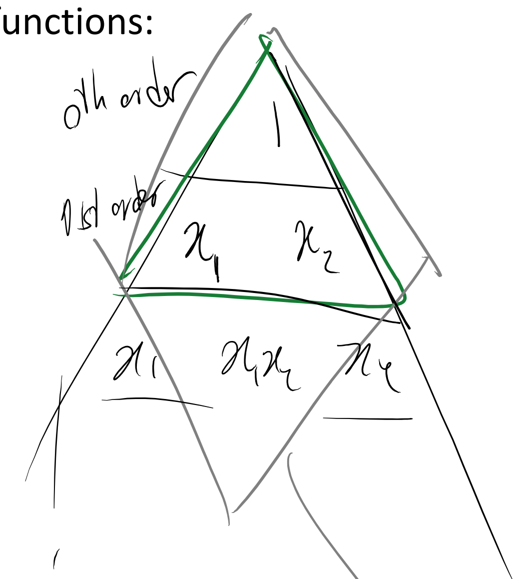
Dirichlet

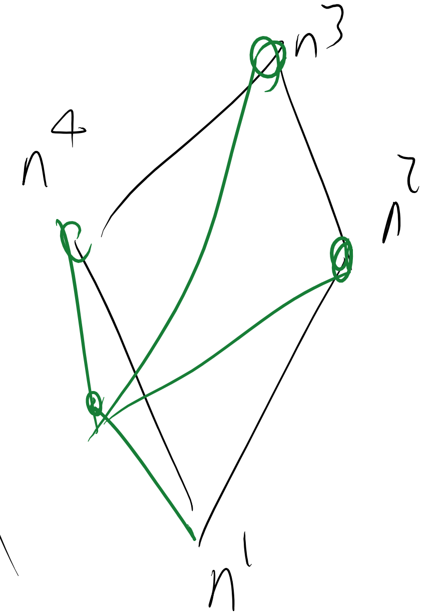
$$\underline{f^e = f_r^e + f_N^e - f_D^e}$$

Let's compute ke in class:

First we need to find shape functions:







$$N^i(n^j) = \delta^{ij}$$

$N^1(n^1) = 1, \; N^1(n^2) = 0 \; N^1(n^3) = 0 \rightarrow$ 3 eqns

$N^1(x_1, x_2) = \alpha_0 + \alpha_1 x_1 +$

$N^1(n^1)=1, N^1(n^2)=0\ N^1(n^3)=0 \rightarrow$ 3 eqns

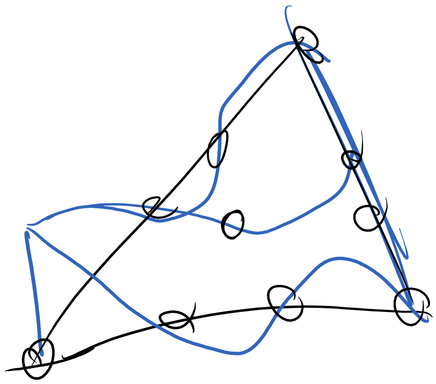
$N^1(x_1,x_2)=\alpha_0+\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_1 x_2$

$N^1(x_1,x_2) = \alpha_0 1 + \alpha_1 x_1 + \alpha_2 x_2$

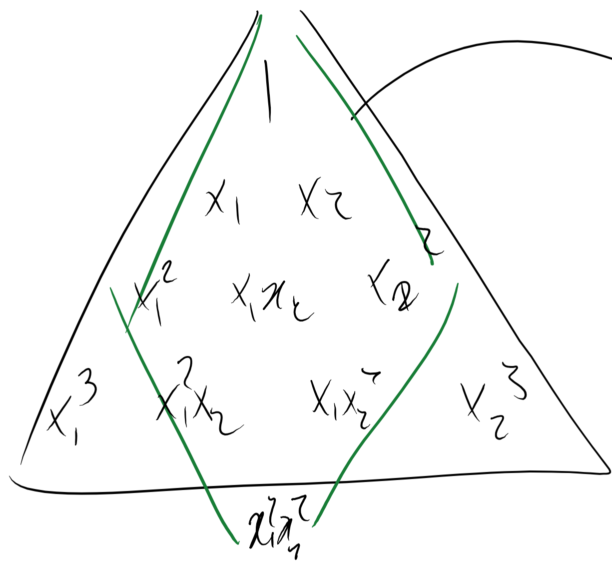
find $\alpha_0, \alpha_1, \alpha_2$ are found by

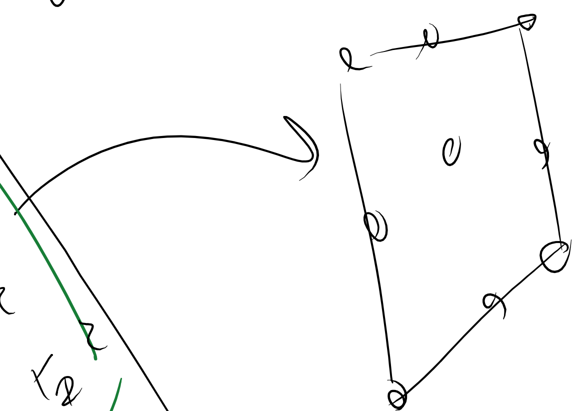
$N^1(n^1)=1, N^1(n^2)=0\ N^1(n^3)=0$

p=2



p=3 element





Now let's for example find shape functions for a p = 1, quad element:

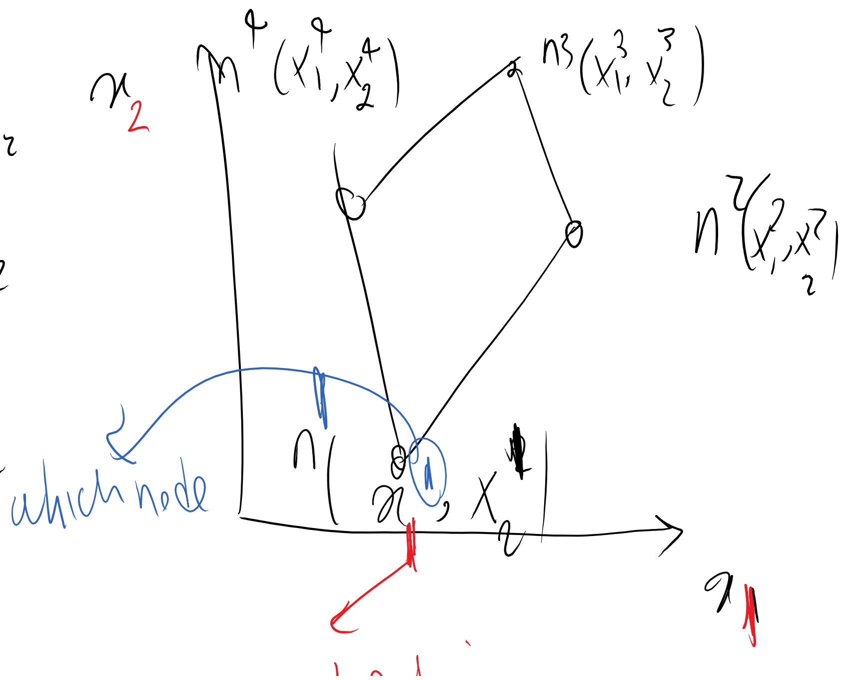
$N^1(x_1,x_2)=\alpha_0+\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_1 x_2$

$N^2(x_1,x_2)=\beta_0+\beta_1 x_1+\beta_2 x_2+\beta_3 x_1 x_2$

$N^3 \quad \gamma$

$N^4 = \delta_1 \quad \delta_1 \quad \delta_2 \quad \delta_3 x_1 x_2$

find $\alpha_i, \beta_i, \gamma_i, \delta_i$



Find $\alpha_i, \beta_i, \gamma_i, \delta_i$

direction

$N^1(n^1)=1 \quad N^1(n^2)=0 \quad N^1(n^3)=0 \quad N^1(n^4)=0$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & x_1^1 & x_2^1 & x_1^1 x_2^1 \\ 1 & x_1^2 & x_2^2 & x_1^2 x_2^2 \\ 1 & x_1^3 & x_2^3 & x_1^3 x_2^3 \\ 1 & x_1^4 & x_2^4 & x_1^4 x_2^4 \end{bmatrix}}_{C} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

matrix

$$C\left[\alpha \,|\, \beta \,|\, \gamma \,|\, \delta\right] = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I}$$

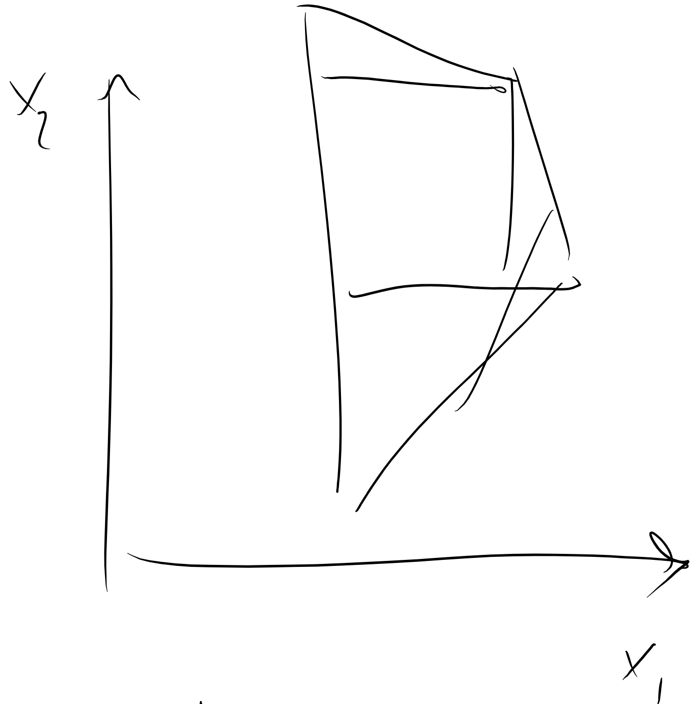
$$\boxed{\left[\alpha \,|\, \beta \,|\, \gamma \,|\, \delta\right] = C^{-1}}$$

Problem 1: A bit difficult to get shape functions but doable

Problem 2:

....

$$k^e = \int \underbrace{\nabla \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}}_{B^T} k \; \nabla [N_1 \; N_2 \; N_3 \; N_4] \, dV$$



We map all the elements to the same square parent element