2022/09/06

Tuesday, September 6, 2022 11:16 AM

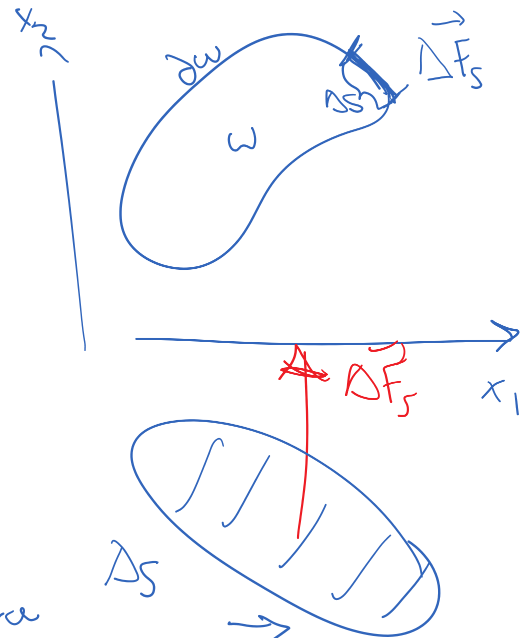
From last time

Surface force

$$\vec F_s = \sum \Delta \vec F_s = \sum \frac{\Delta \vec F_s}{\Delta S}\,\Delta S$$

let $\Delta S \to 0$

$$= \int_{\partial\omega} \left[\lim_{\Delta S\to 0}\left(\frac{\Delta F_s}{\Delta S}\right)\right] dS$$

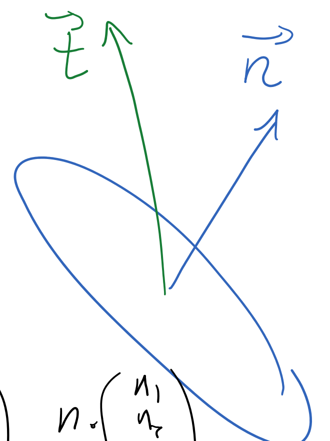


intensity of force

"force per area" $\lim_{\Delta S\to 0}\frac{\Delta F_s}{\Delta S} = \vec t$

Pa
MPa
psi

$$\boxed{F_s = \int_{\partial\omega} \vec t \, dS}$$



$$\vec t = \sigma \cdot \vec n$$

σ → stress tensor

$t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$ $\quad n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & — & \\ & & \sigma_{33} \end{pmatrix}$$

stress tensor

2D

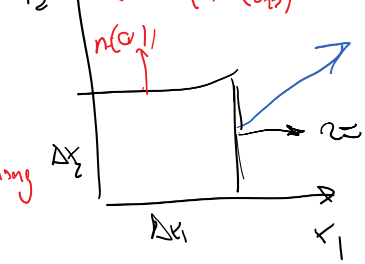
$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

traction for a surface with normal along x_1 axis

traction, normal along x_2

$t_{(2)} = \sigma\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix}$

$n(0,1)$



$$t_{(1)} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \end{bmatrix}$$

n

$t_i = \begin{bmatrix} \sigma_{11} & \sigma_{12} \end{bmatrix}\begin{bmatrix} n_1 \end{bmatrix}$

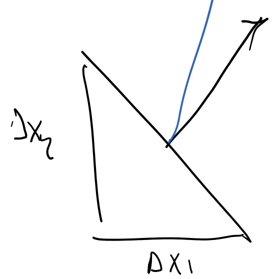
traction

for a surface with normal along x_1 axis

$\sigma = \sigma^T$, $\sigma_{12} = \sigma_{21}$ (Balance of angular momentum

$t = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$

$n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$



for arbitrary direction

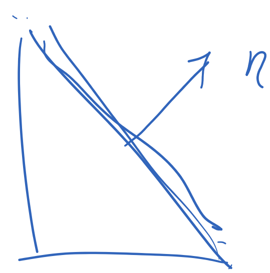
force per area (traction)

$t = \sigma \cdot n$

σ: stress tensor

n: normal vector

heat conduction



$q_n = q \cdot n$

q_n: energy flux per surface area for normal n

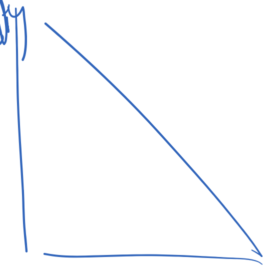
$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = q_1 n_1 + q_2 n_2$

In general

$(f_x)_n = f_x \cdot n$

f_x: always one tensor order higher

spatial flux density



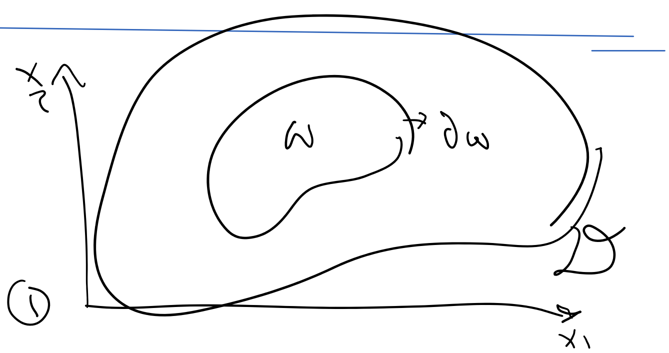
Let's continue with balance of forces

$\Sigma F = F_V + F_S$

$\forall \omega = \int_{\omega} \rho b \, dV + \int_{\partial \omega} \sigma \cdot n \, dS = 0$ (1)

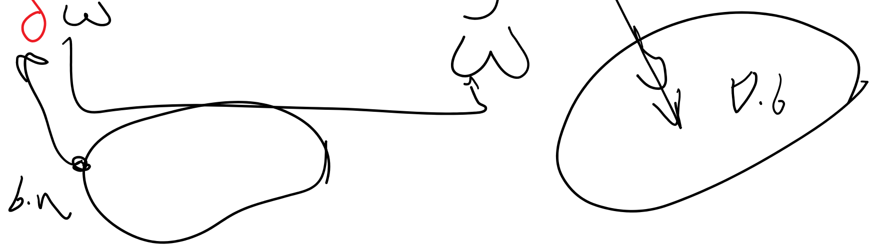
ω: integral over interior of ω

$\partial\omega$: boundary of ω



Divergence / Gauss theorem

$$\int_{\partial\omega} \sigma \cdot n \, dS = \int_{\omega} \nabla \cdot \sigma \, dV \quad (2)$$



$$\Sigma F = \int_{\omega} \rho b \, dV + \int_{\omega} \nabla \cdot \sigma \, dV = 0$$

$$\forall \omega \subseteq D \int_{\omega} \underbrace{(\nabla \cdot \sigma + \rho b)}_{\text{integrand}} dV = 0$$

Integral $= 0$

$$\forall \omega \int (\text{integrand}) \, dV = 0$$

integrand $= 0$

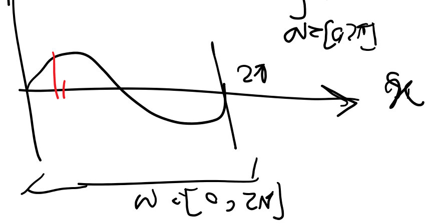
PDE / Strong form $\boxed{\nabla \cdot \sigma + \rho b = 0}$

Example 1D

$f(x) = \sin x$

$\int_{\omega} f(x) \, dx = 0$ But $f(x) \neq 0$

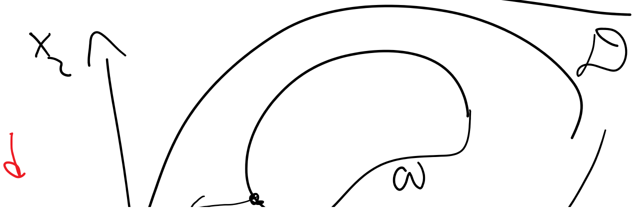
$\omega = [0, 2\pi]$



General Balance law

$\forall \omega \subseteq D$

f : quantity to be balanced



$\forall\, \omega \subset \Omega$

f_t : quantity to be balanced
density

$$F_t = \int_\omega f_t \, dV$$

F_t: energy, f_t: volumetric energy density

$$\underbrace{\frac{d}{dt}\underbrace{\int f_t \, dV}_{\text{total } F_t}}_{\text{rate of total } F_t} = \int_\omega \overset{\text{source term}}{S} \, dV - \int_{\partial\omega} \overset{\text{spatial flux density}}{f_x} \cdot n \, dS$$

example heat conduction: $C_v T$

$$\frac{d}{dt}\int e \, dV = \int_\omega Q \, dV - \int_{\partial\omega} q \cdot n \, dS$$

$$\frac{d}{dt}\int_\omega f_t \, dV = \int_\omega S \, dV - \int_{\partial\omega} f_x \cdot n \, dS$$

$$\boxed{\int_\omega \frac{d}{dt} f_t \, dV = \int_\omega S \, dV - \int_\omega \nabla \cdot f_x \, dV}$$

(3a) Dynamic Balance law for $F_t = \int_\omega f_t \, dV$

First dynamic: $\forall \omega \quad \int_\omega \left(\frac{d}{dt} f_t + \nabla \cdot f_x - S \right) dV = 0$

$$\left| \frac{d}{dt} f_t + \nabla \cdot f_x - S = 0 \right| \quad \text{(3b)}$$

Dynamic PDE

$$\boxed{\frac{d}{dt} f_t + \nabla \cdot f_x - S = 0} \quad (3b)$$

Dynamic PDE

heat eqn

$$e = C_v T \qquad q \qquad Q$$

$$\boxed{\frac{d}{dt}(C_v T) + \nabla \cdot q - Q = 0} \quad (3c)$$

- Example 1 heat conduction

$$f_t = p = \rho \vec{v} \qquad f_x = -\sigma \qquad S = \rho b$$

Example 2 solid mechanics

$$\frac{d}{dt}\rho\vec{v} + \nabla \cdot (-\sigma) - \rho b = 0$$

Equation of motion

Static / steady state problems

$$\cancel{\frac{d}{dt}\int_{\omega} f_t \, dV} = -\int_{\partial\omega} f_x \cdot n \, dS + \int_{\omega} S \, dV$$

$= 0$

no change in time

$$\boxed{\int_{\partial\omega} f_x \cdot n \, dS = \int_{\omega} S \, dV} \quad (4)$$

balance law for statics

$$\int_{\omega} \nabla \cdot f_x = \int_{\omega} S \, dV \rightarrow \boxed{\nabla \cdot f_x - S = 0}$$

Differential equation

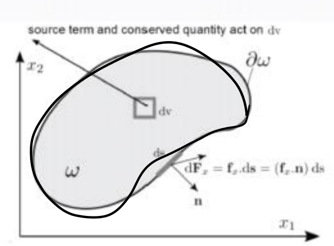
Summary slide

General form of balance laws

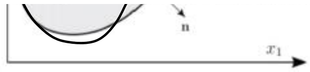
For a general conservation law let:

- $\mathbf{f}_t$: conserved quantity = temporal flux
- $\mathbf{f}_x$: total outward spatial flux
- $\mathbf{r}$: source term

then the balance law for dynamics reads:



then the balance law for dynamics reads:



$$\boxed{\forall \omega \subset \mathcal{D} \wedge \forall t : \int_\omega \mathbf{r}\, \mathrm{dv} - \int_{\partial\omega} \mathbf{f}_x . \mathrm{d}\mathbf{s} = \int_\omega \mathbf{r}\, \mathrm{dv} - \int_{\partial\omega} (\mathbf{f}_x . \mathbf{n})\, \mathrm{ds} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \mathbf{f}_t\, \mathrm{dv}} \quad (13)$$

For static case the RHS is zero (*i.e.*, the quantity $\int_\omega \mathbf{f}_t\, \mathrm{dv}$ remains constant). The static balance law reads:

$$\boxed{\forall \omega \subset \mathcal{D} : \int_\omega \mathbf{r}\, \mathrm{dv} - \int_{\partial\omega} \mathbf{f}_x . \mathrm{d}\mathbf{s} = \int_\omega \mathbf{r}\, \mathrm{dv} - \int_{\partial\omega} (\mathbf{f}_x . \mathbf{n})\, \mathrm{ds} = \mathbf{0}} \quad (14)$$

These can be directly compared to $\mathbf{F} = \mathrm{d}\mathbf{P}/\mathrm{d}t$ and $\mathbf{F} = 0$ in previous discrete examples.

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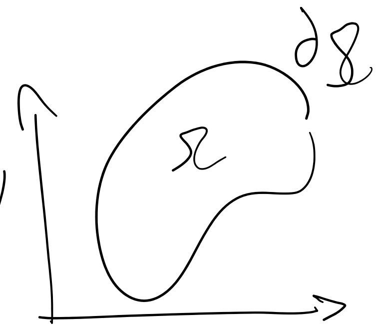
For any balance law we have

$$\int_{\partial\Omega} \underbrace{\bar{F}}_{\text{spatial (spacetime) flux}} \cdot \underbrace{n}_{\text{space (spacetime normal)}} \, ds = \int_\Omega \underbrace{r}_{\text{source term}} \, dV$$



FYI

$$\underbrace{\nabla}_{\text{space (space time divergence)}} \cdot \bar{F} = r$$

You can skip slides 6-14, except slide 13, the static part of it

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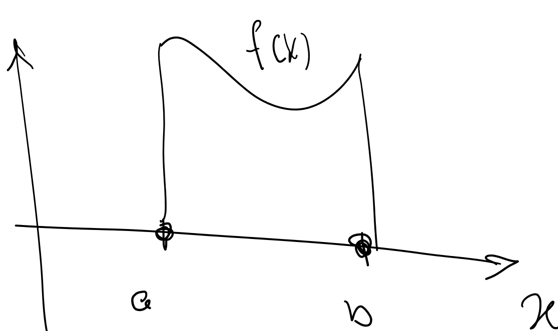
We'll discuss divergence theorem and localization theorem

1D

Divergence theorem

$$\int_a^b F(x)\, dx = f(b) - f(a)$$

$$(f = \int F \Rightarrow F = f')$$



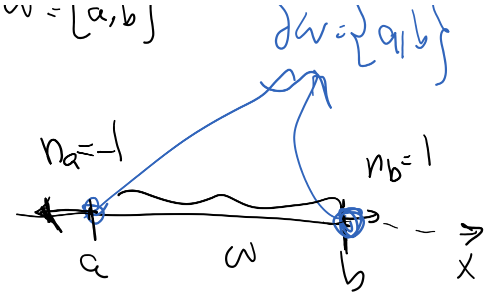
$\omega = [a, b]$

$\partial\omega = \{a, b\}$

$$\left(f = \int F \Leftarrow F = f' \right) \qquad \omega = [a,b]$$

that is

$$\int_{\omega=[a,b]} f'(x)\,dx = f(b)\cdot 1 + f(a)\cdot(-1)$$

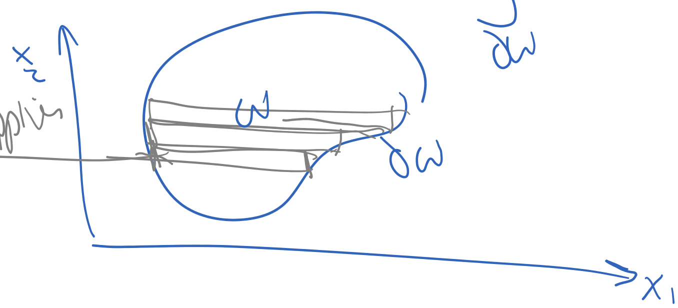


(★) $$\int_\omega \underbrace{f'(x)}_{\nabla\cdot f}\,dx = f(a)\,n_a + f(b)\,n_b = \int_{\partial\omega} f(x)\cdot n(x)\,ds$$

3 like

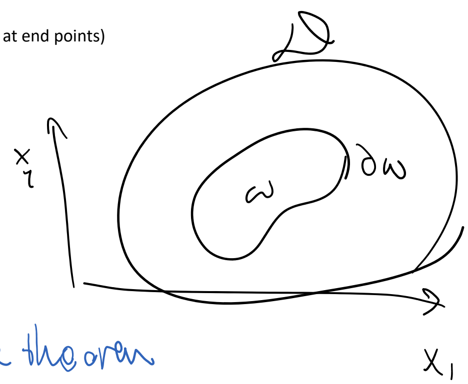
Compare this with divergence theorem

$$\int_\omega \nabla\cdot f(x)\,dV = \int_{\partial\omega} f(x)\cdot n\,ds$$



Divergence theorem is multi-dimension version of fundamental theorem of calculus (turning line integral to values at end points)

Which one is more general



Motivation:

$$\Sigma F = \int_\omega \rho b\,dV + \int_{\partial\omega} \sigma\cdot n\,dS$$

using divergence theorem

$$\int_\omega \nabla\cdot\sigma\,dV$$

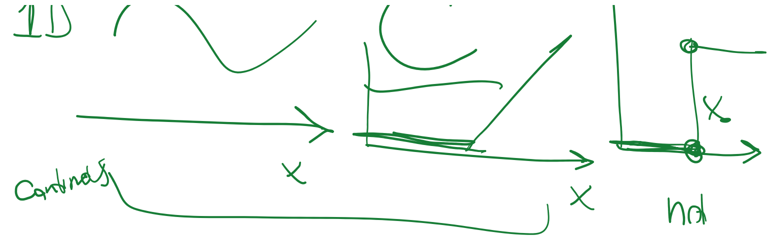
we have derivative :(

For the balance law, all we need is that the spatial flux (sigma here) to be integrable over the boundary of omega.

However, for getting to DE, ie. the blue term divergence of spatial flux should exist and in fact should be continuous.

C^0 function is continuous 1D

C^0 function is continuous 1D f:



C^1 if f & f' are continuous

C^2 if f, f', f'' are continuous

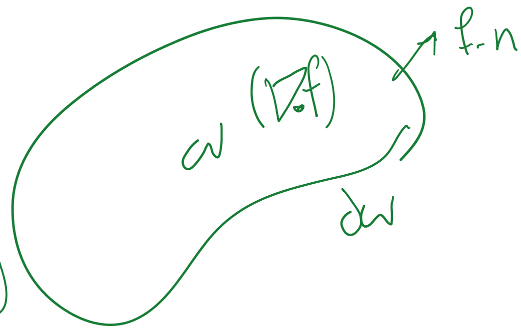
C^n means $f, f', \ldots, f^{(n)}$ derivates exist and all are continuous

1D

2D $\quad f, \dfrac{\partial f}{\partial x_i}, \dfrac{\partial^2 f}{\partial x_i \partial x_j} \ldots$ n partial derivatives

Divergence theorem

$$\int_{\partial\omega} f \cdot n \, dS = \int_{\omega} (\nabla \cdot f) \, dV$$



Divergence theorem requires $\nabla \cdot f$ to exist & to be continuous $\boxed{f \in C^1(D)}$

this is restrictive. For balance law, f should be integrable this is restrictive. For balance law, f should be integrable