2022/09/08

Thursday, September 8, 2022 11:23 AM

Refer to slides 19-20 for divergence theorem

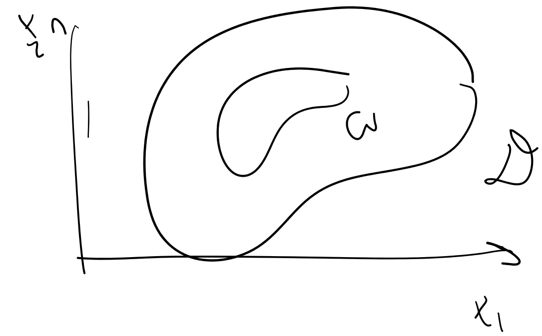
Second point from last time

# Localization theorem:

$\forall \omega \subseteq D \quad \int_\omega g(\vec{x})\, dV = 0$

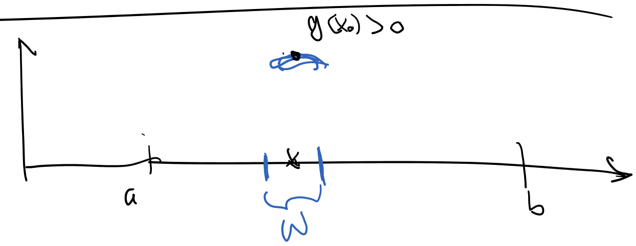
& $g$ is continuous

$\Longrightarrow \forall \vec{x} \in D \quad g(\vec{x}) = 0$




Proof: 1D

$D = [a, b]$




$\exists x_0 \in D$ such that $g(x_0) > 0$

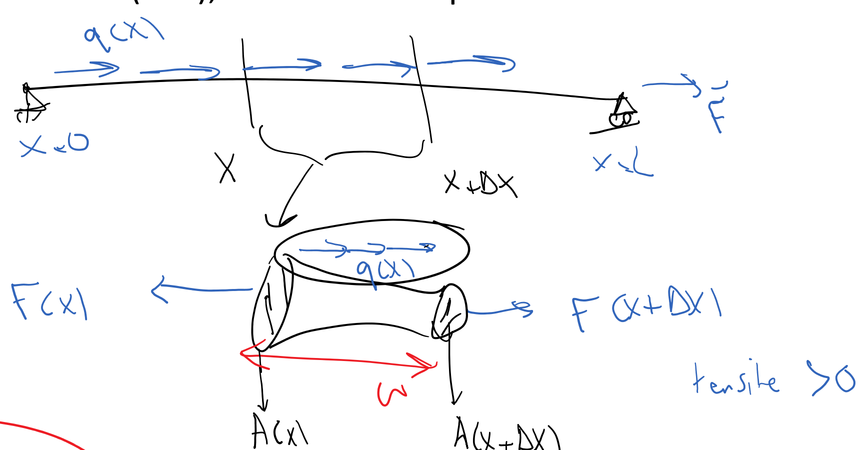
there is a neighborhood $\omega$ of $x_0 \Rightarrow g(x) > 0$ from continuity

$\Longrightarrow \int_\omega \underbrace{g(x)}_{>0} dV > 0$ negates $\forall \omega \subseteq D \int_\omega g(x)\, dV = 0$

$\Longrightarrow$ false

$\forall x \in D \quad g(x) = 0$

## 1D bar problem: Balance law (balance of forces), Differential Equation, Boundary Conditions (BCs), constitutive equation




Balance of forces

"Balance law"

$F(x+\Delta x) - F(x) + q(x)\Delta x \ldots$

$A(x)$ $A(x+\Delta x)$

$F(x+\Delta x) - F(x) + q(x)\Delta x = 0$ $\Delta x$ small

divide by $\Delta x$

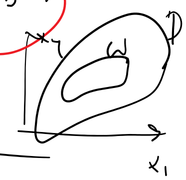
let $\Delta x \to 0$ $\frac{F(x+\Delta x) - F(x)}{\Delta x} = q(x)$

DE $F'(x) = \frac{dF(x)}{dx} + q(x) = 0$

comparison with 2D

$\Sigma F = 0$

Balance law $\int_{\partial\omega} \sigma n \, ds + \int_{\omega} \rho b \, dv = 0$

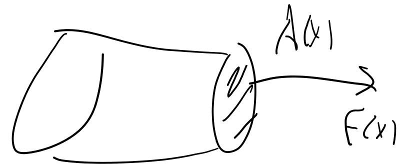


PDE $\nabla \cdot \sigma + \rho b = 0$

$F(x)$ $q(x)$

(1D) DE $\frac{dF(x)}{dx} + q(x) = 0$

Closing the system



stress $\sigma(x) = \frac{F(x)}{A(x)}$

① $\boxed{\frac{d\, A(x)\sigma(x)}{dx} + q(x) = 0}$ Strong form 1D bar problem

Some BC's are in terms of $u$ & displacement $u$ is the primary field

we need to express $\sigma$ in terms of $u$:

② $\boxed{\sigma(x) = E(x)\,\varepsilon(x)}$

elastic modulus, strain

Constitutive eqns are empirical eqns

③ $\boxed{\varepsilon(x) = \frac{du(x)}{dx}}$ first class Compatibility eqn

①,②,③ $\longrightarrow$

DE

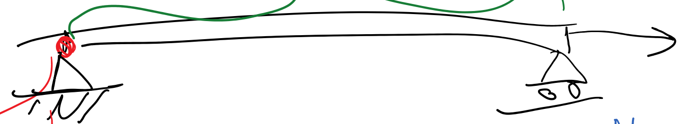
④ $\frac{d\, A(x)\sigma(x)}{dx} + q(x) = \frac{d\left(E(x)A(x)\frac{du}{dx}\right)}{dx} + q(x) = 0$

$(A(x)\sigma(x))' + q(x) = (EA(x)u'(x))' + q(x) = 0$

$AE(x) = A(x)E(x)$

$u(x=0)=\bar{u}$ $\quad D \quad$ $u(x=L)=\bar{F}$

$\partial D_u$ = essential boundary of $D$



$x=0$

Essential BC

" specify the primary field $u$

Here $\partial D_u = \{0\}$

Natural boundary

"specify the spatial flux

$\partial D = \{0, L\}$

$\partial D_f$ = natural boundary

$\partial D_f = \{L\}$

$\partial D_u \cup \partial D_f = \partial D$

$\partial D_u \cap \partial D_f = \emptyset \longrightarrow$ at one point we can specify only 1 BC

## Boundary value Problem (BVP)

$\partial D_u$ $\quad D \quad$ $\bar{F}$

$\partial D_f$

DE $\forall x \in D$ $\quad (EAu')' + q = 0$

BCs:
$$\begin{cases} u(x=0) = \bar{u} & @\ \partial D_u = \{0\} \\ F(x=L) = AEu'(x=L) = \bar{F} & @\ \partial D_f = \{L\} \end{cases}$$

(2)

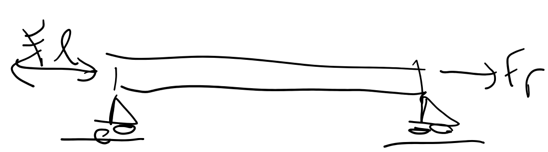
Other BCs in 1D

$\partial D_f$ $\quad$ $\partial D_u$ $\quad$ $\partial D_u = \{0, L\}$

$\partial D_f = \emptyset$

not valid for statics

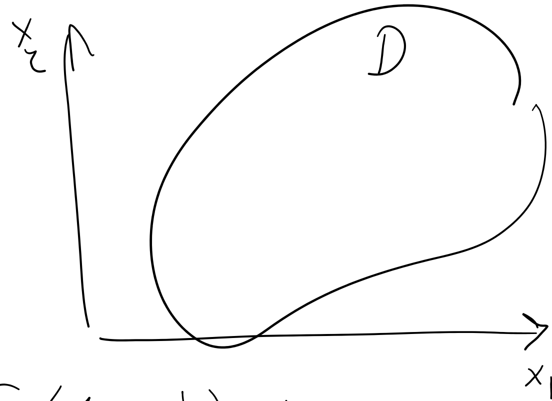
$F_l$ $\quad$ $F_r$



$\partial D_u = \emptyset$

$\partial D_f : \{c, L\}$

## (2D) Balance law



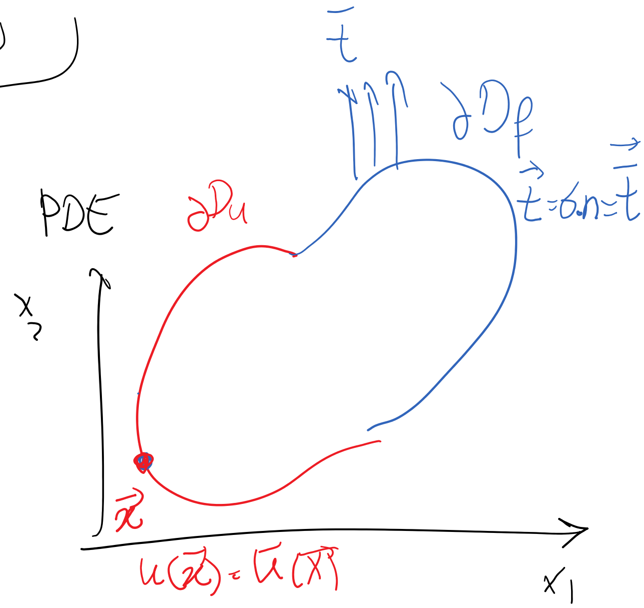

$$\int_{\partial \omega} \sigma \cdot n \, dS + \int_{\omega} \rho b \, dV = 0$$

divergence theorem

$$\int_{\omega} \nabla \cdot \sigma \, dV + \int \rho b \, dV = 0 \rightarrow \int_{\omega} (\nabla \cdot \sigma + \rho b) \, dV = 0$$

localization problem

(3) $\boxed{\nabla \cdot \sigma + \rho b = 0}$ Strong form PDE




Can I solve this? No

we need to close the system

I'll do this in 2D

(3) $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$

$\sigma_{21} = \sigma_{12}$ sym

$\nabla \cdot \sigma = \begin{bmatrix} \sigma_{11,1} + \sigma_{12,2} \\ \sigma_{12,1} + \sigma_{22,2} \end{bmatrix}$ $\rho b = \begin{bmatrix} \rho b_1 \\ \rho b_2 \end{bmatrix}$

$\nabla \cdot \sigma + \rho b = 0 \longrightarrow$
$$\boxed{\begin{aligned} &\text{Eq1)} \quad \sigma_{11,1} + \sigma_{12,2} + \rho b_1 = 0 \\ &\text{Eq2)} \quad \sigma_{12,1} + \sigma_{22,2} + \rho b_2 = 0 \end{aligned}}$$

2 eqn's 3 unknowns $\sigma_{11}, \sigma_{12}, \sigma_{22}$ :(

$\sigma$ is a linear function of strain

2D & 3D $\quad \sigma$ is a linear function of strain

2D

$$
\begin{array}{l} \text{Eq 3} \\ \text{Eq 4} \\ \text{Eq 5} \end{array}
\underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}}_{\text{Voigt stress notation}} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}}_{\text{Voigt stiffness}} \underbrace{\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}}_{\text{Voigt strain notation}}
$$

$2\epsilon_{12} \rightarrow \gamma_{12}$ ↗ Engineering shear strain

2D version of $\sigma = E\varepsilon$ Constitutive eqn

I added 3 more equations
I add 3 unknowns $(\epsilon_{11}, \epsilon_{22}, \epsilon_{12})$

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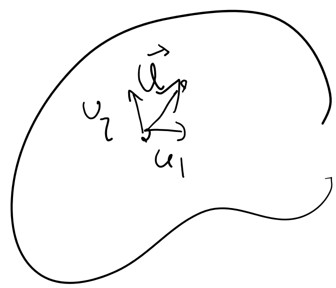
$\varepsilon \xrightarrow{\text{expressed}}$ in terms of $U$ $\quad$ Compatibility equation

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \qquad \bar{\bar{\varepsilon}} = \frac{\nabla u + (\nabla u)^T}{2}$$




$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{bmatrix} + \begin{bmatrix} u_{1,1} & u_{2,1} \\ u_{1,2} & u_{2,2} \end{bmatrix}\right)$$

independent strain components

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} = \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) \\ \text{same} & u_{2,2} \end{bmatrix}$$

$\epsilon_{12} \leftarrow \epsilon_{21}$

$$
\boxed{
\begin{array}{ll}
\text{Eq 6} & \epsilon_{11} = u_{1,1} \\
\text{Eq 7} & \epsilon_{22} = u_{2,2} \\
\text{Eq 8} & \epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1})
\end{array}
}
$$

Added 3 more eqns

All unknowns
$\sigma = [\sigma_{11} \ \sigma_{12}]$

Added 3 more eqns

(2) unknowns $(u_1, u_2)$

All unknowns

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

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Summary

$$\nabla \cdot \sigma + \rho b = 0 \qquad \text{PDE} \qquad i$$

Constitutive equ

$$\sigma = \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}}_{\text{2 indices}} = \mathbb{C} \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix} \qquad ii$$

4th Elasticity tensor

2 indices

2+2 = 4 indices

Compatibility

$$\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T) \qquad iii$$

i, ii $\quad \nabla \cdot \sigma + \rho b = \nabla \cdot (\mathbb{C}\,\varepsilon) + \rho b = 0$

iii $\quad \varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$

$$\boxed{\nabla \cdot \left(\mathbb{C}\left(\frac{1}{2}\left[\nabla u + \nabla u^T\right]\right)\right) + \rho b = 0}$$

2 equs

2 unknowns $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

2nd order differential equation

2D version of

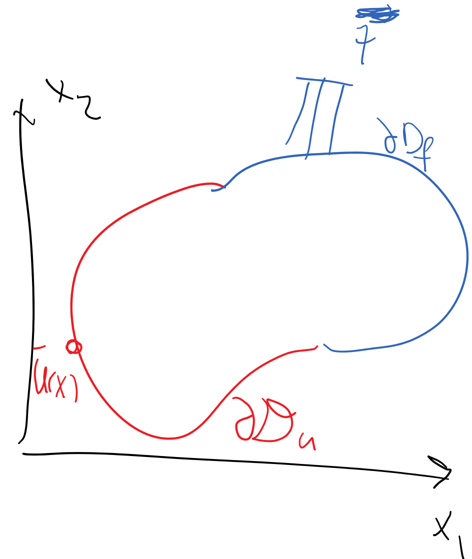
$$(EAu')' + q = 0$$

# BVP for 2D elasticity

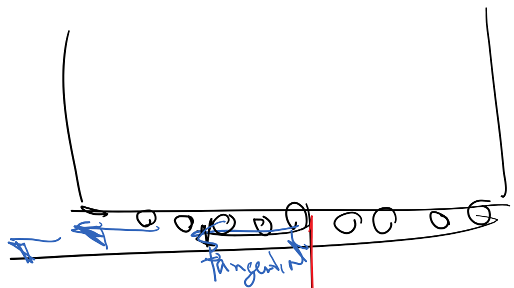
1) PDE $\forall x \in D$: $\nabla\cdot\sigma + \rho b = \nabla\cdot C(\nabla u) + \rho b = 0$

2) BCs

- Essential BCs $\forall x \in \partial D_u$: $u(x) = \bar{u}(x)$
- Natural BC $\forall x \in \partial D_f$: $t = \sigma\cdot n = \bar{t}$




Note:




$t_{\text{tangential}} = \bar{t}_{tan}$

$n$ $\quad u_n = 0$ or sth else

## Closing the system of equations (Statics)

Strong form (23) of balance of linear momentum for statics is:

$$\nabla.(-\sigma) - \rho\mathbf{b} = \mathbf{0}, \quad \Rightarrow \nabla.\sigma + \rho\mathbf{b} = \mathbf{0} \quad \Rightarrow \quad \sigma_{ij,j} + \rho b_i = 0 \qquad (24)$$

where $\mathbf{f} = -\sigma$, $\mathbf{r} = \rho\mathbf{b}$, and $\nabla(.) = \frac{\partial(.)}{\partial x_1} + \frac{\partial(.)}{\partial x_2} + \frac{\partial(.)}{\partial x_3}$.

| Type | Equation | $n_e$ | new unknowns | $n_u$ | $N_e - N_u$ |
|---|---|---|---|---|---|
| Balance law | $\sigma_{ij,j} + \rho b_i = 0$ | 3 | $\sigma_{ij} = \sigma_{ji}$, $i,j \in \{1,2,3\}$ | 6 | 3 |
| Constitutive equation | $\sigma_{ij} = C_{ijkl}E_{kl}$ | 6 | $E_{kl} = E_{lk}$ | 6 | 3 |
| kinematic compatibility | $E_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$ | 6 | $u_k$ | 3 | 0 |

$n_e$ = number of new equations $\quad n_u$ = number of new unknowns
$N_e$ = total number of equations $\quad N_u$ = total number of unknowns

- We need other equations (constitutive equations and kinematic compatibility equations) to balance the number of unknowns and equations.

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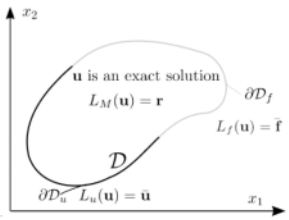
Slide 26, FYI

## Different types of spatial boundary conditions (BC)

- $L_M(\mathbf{u}) = \mathbf{r}$ is the strong form after incorporating the "constitutive" and "compatibility" conditions.
- $L_u(\mathbf{u}) = \bar{\mathbf{u}}$: Dirichlet BC, order $M_u$.
- $L_f(\mathbf{u}) = \bar{\mathbf{f}}$: Neumann BC, order $M_f$.
- $\mathbf{u}$ is a primary field, (*e.g.*, displacement for solid mechanics; temperature for heat conduction)
- $M$ is typically even (*e.g.*, $M = 2m$)




| $\partial\mathcal{D}_u$ | $\partial\mathcal{D}_f$ |
|---|---|
| Dirichlet BC | Neumann BC |
| Essential BC<br>(typically strongly enforced) | Natural BC<br>("naturally" derived from balance law fluxes) |
| "primary" or "kinematic" BC | "flux" or "force" BC |

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Next step

1D

DE: $\forall x \in \mathcal{D}$ $\quad R_i = \sigma' + q = (EAu')' + q = 0$

BC: $\partial\mathcal{D}_u$ $\quad R_u = \bar{u} - u = 0$

$\partial\mathcal{D}_f$ $\quad R_f = \bar{F} - F = \bar{F} - AEu' = 0$

$\partial\mathcal{D}_u$ $u = \bar{u}$

$\partial\mathcal{D}_f$