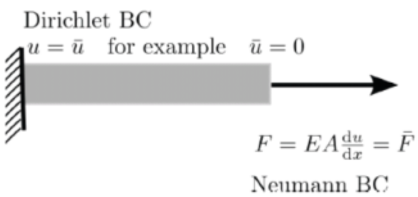
2022/09/13

Tuesday, September 13, 2022 11:09 AM

Bar problem



$$\underbrace{(EAu')'}_{\to\ M=2 \text{ order of DE}} - q = 0$$

$$\boxed{m = \frac{M}{2} = 1}$$

Operator	Sample	1D bar	operator order
$L_{2m}(\mathbf{u}) = \mathbf{r}$	$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = -q$	$L_{2m} = \frac{d}{dx}\left(EA\frac{d(.)}{dx}\right)$	$m = 1 (M = 2)$
$L_u(\mathbf{u}) = \bar{\mathbf{u}}$	$u = \bar{u}$	$L_u = (.)$	$M_u = 0$
$L_f(\mathbf{u}) = \bar{\mathbf{f}}$	$EA\frac{du}{dx} = \bar{f}$	$L_f = EA\frac{d(.)}{dx}$	$M_f = 1$

BC's are order 0 to $M-1$

$M=2 \rightarrow$ BC's 0th order $u = \bar{u}$ Essential $u = \bar{u}$

1st order $AEu' = \bar{F}$ Natural $F = AEu' = AEu' = \bar{F}$

BC's: 0 — lower half — Essential

$M-1$ — upper half — Natural

Slides 32 to 34 provide the formulation of the beam problem.
I'll cover it later, but it's good to read it and see how the essential and natural BCs are divided

WRS and Weak statement:

Weighted Residual Statement (WRS)

$R_i = (EAu')' + q$ in $\mathcal{D}$ $\partial\mathcal{D}_u$ $u = \bar{u}$ $\mathcal{D}$ $F = EAu' = \bar{F}$

$R_u = \bar{u} - u$ essential BC residual on $\partial\mathcal{D}_u$ $\partial\mathcal{D}_f$

DE: $(EAu')' + q = 0$

$R_f = \bar{F} - F = \bar{F} - EAu'$ on $\partial\mathcal{D}_f$

Multiply by weights

some function of w e.g. w'

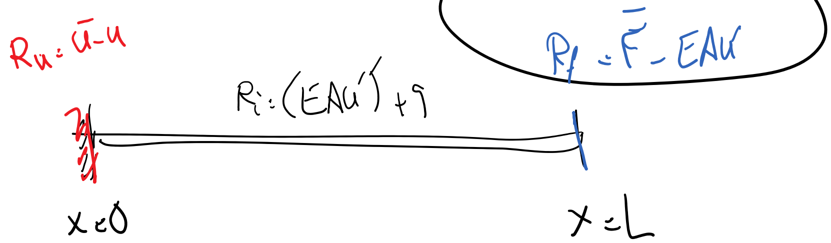
$\int wR_i \, dV$

$$\int_{\Omega} \underbrace{w}_{\text{weight function}} R_i \, dV + \int_{\partial\Omega_u} f(w) R_u \, dS + \int_{\partial\Omega_f} w R_f \, dS = 0$$

1D case, this problem: $f(w) R_u \big|_{x=0}$

1D, this problem: $w R_f \big|_{x=L} = 0$

In continuous FEM (this course), I'll show that we don't have the middle term (integral of residual on essential BC)



$$\int_0^L w R_i \, dx + w R_f \big|_{x=L}$$

$$= \int_0^L \underbrace{w}_{0^{th}\text{ order}} \big(\underbrace{(EAu')'}_{2nd\text{ order}} + q \big) dx + w(\bar{F} - EAu') \big|_{x=L} = 0$$

$$\int_0^L UV' \, dx = \int_0^L \big((UV)' - U'V \big) dx$$

$$= UV \big|_0^L - \int U'V \, dx$$

IBP

$$\int_0^L \underbrace{w}_{U} \underbrace{(EAu')'}_{V'} dx \qquad V = EAu'$$

$$= w EAu' \big|_0^L - \int \big(\underbrace{w'}_{U'} \underbrace{EAu'}_{V} \big) dx$$

$$\int_0^L w (EAu')' \, dx = w EAu' \big|_{x=L} - w EAu' \big|_{x=0} - \int_0^L w' EAu' \, dx$$

$$|_{x=L} \qquad |_{x=0} \qquad \int_0 w EA u\,dx$$

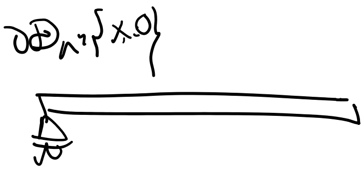
1st order

$$\underbrace{\int_0^L w(EAu')'\,dx + \int_0^L wq\,dx}_{\int_0 wR_i\,dx} + \underbrace{w(\bar{F} - EAu')\Big|_{x=L}}_{wR_f} = 0$$

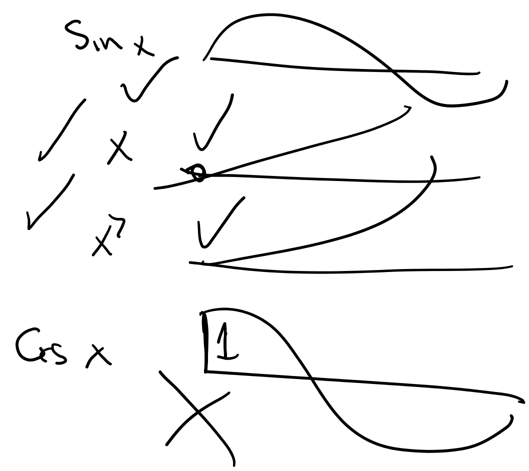
$$\underbrace{wEAu'\Big|_{x=L} - \overbrace{wEAu'\Big|_{x=0}}^{F(x=0)} - \int_0^L w'EAu'\,dx}_{\int_0^L w(EAu')'dx} + \int_0^L wq\,dx + w\bar{F}\Big|_{x=L} - wEAu'\Big|_{x=L}$$

$=0$

When we go from WRS to weak statement we only limit ourselves to weight functions that are zero on essential BC.



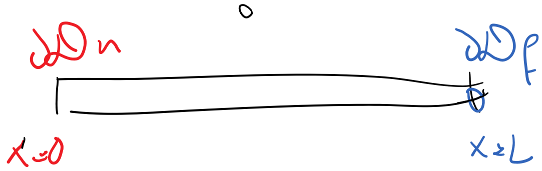
w :



good for WRS

bad for weak statement

$$\underbrace{\cancel{-w(x=0)\,EAu'(x=0)}}_{0} - \int_0^L w'EAu'\,dx + \int_0^L wq\,dx + w(L)\,\bar{F}\Big|_{x=L} = 0$$



1 derivative

$$\int_0^L w'EAu'\,dx = \int_0^L wq\,dx + w\bar{F}\Big|_{x=L}$$ Weak statement

$$\int_0^L w\left(EAu'' \right) dx + w(\bar{F} - EAu')\big|_{x=L} = 0 \quad \text{WRS}$$

0 der — 2 derivatives

WRS

Find $u(x) \in \mathcal{V} = \{ f \in C^2(\Omega) \mid \forall x \in \partial\Omega_u \;\; u(x) = \bar{u} \}$

space of sol'n

$\Omega = [0, L]$

here for this 1D problem $u(0) = \bar{u}$

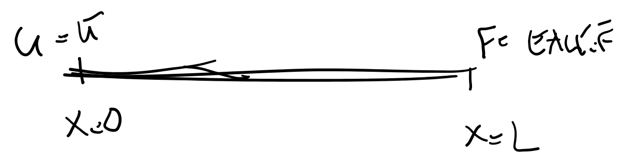
such that for all $w(x) \in \mathcal{W} = \{ f \in C^0(\Omega) \mid \text{———} \}$

we have

$$\int_0^L w\left((EAu')' + q \right) dx + w(\bar{F} - EAu')\big|_{x=L} = 0$$

2 derivatives

0th order



Because we did not add weight time residual essential boundary to the WRS

$$\cancel{\int_{\partial\Omega_u} w R_e \, ds}$$

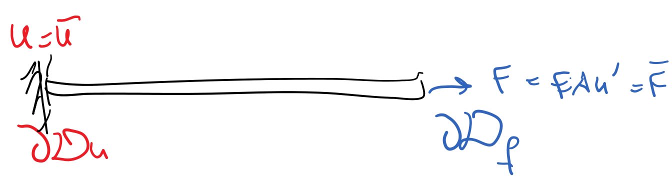
WRS

Weak statement

Find $u(x) \in \mathcal{V} = \{ f \in C^1(\Omega) \mid f(0) = \bar{u} \}$

such that for all weight functions $w \in \mathcal{W} = \{ f \in C^1(\Omega) \mid f(0) = 0 \}$

$$\int_0^L w' EA u' \, dx = \int_0^L w q \, dx + w\bar{F}\big|_{x=L}$$

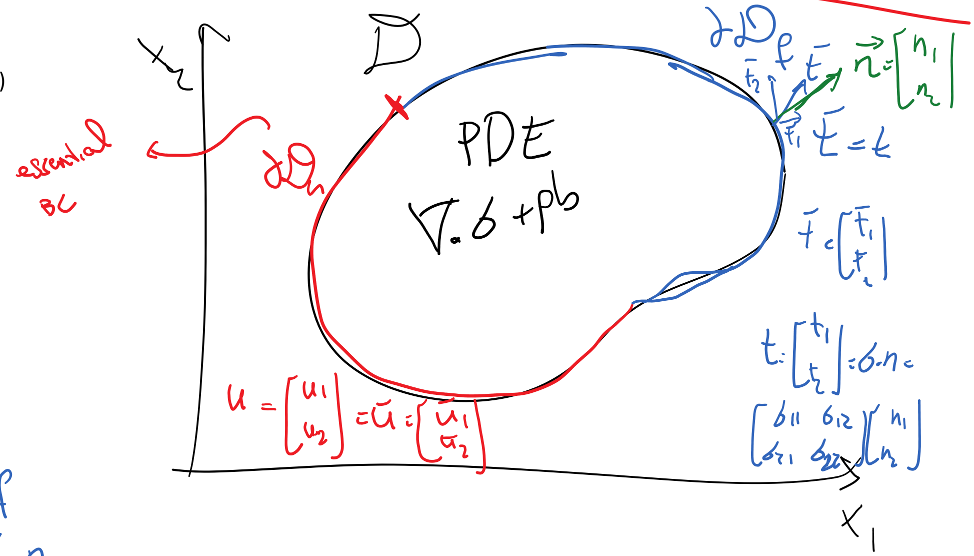


Key points for the weak statement:

- Derivative orders are balanced for weight and solution
- Both w and u satisfy the essential BC.
  - For solution the actual essential BC
  - For the weight, the homogenous (e.g. 0) version of that.

2D examples:

1. Elastostatics (in class)
2. Heat conduction (HW2)



$R_i: \forall x \in D$

$R_i = \nabla \cdot \sigma + \rho b$

$R_f: \forall x \in \partial D_f$

$R_f = \bar{t} - \sigma \cdot n$

$R_u: \forall x \in \partial D_u \quad R_u = \bar{u} - u$

Weighted residual statement

we strongly satisfy essential BC

Find $u \in V = \{ f \in C^2(D) \mid \forall x \in \partial D_u \; u(x) = \bar{u} \}$

such that $\forall \; w \in W = \{ f \in C^0(D) \mid \quad \}$

$$\int_D w \, R_i \, dV + \int_{\partial D_f} w \, R_f \, ds + \cancel{\int_{\partial D_u} h(w) \, R_u \, ds} = 0$$

$\nabla \cdot \sigma + \rho b$

2 der of u

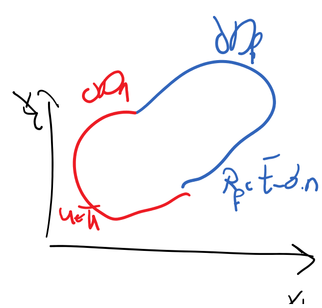
some function of weight

we cross this term as we satisfy it strongly

The process of deriving the weak statement

$$\int_D w(\nabla \cdot \sigma + \rho b) \, dV + \int_{\partial D_f} w(\bar{t} - \sigma \cdot n) \, dS = 0$$

$(EAu')'$



$\mathcal{D}$ $(EAu')'$ $\partial\mathcal{D}_f$

similar to 1D $\int w((A\sigma)' + f) + w(\bar{F} - EAu')$ $u=\bar{u}$ $\bar{F} - F = \bar{F} - EAu'$

$$\int_{\mathcal{D}} w(\nabla\cdot\sigma)\,dV$$

2 derivative for u

2D version of IBP

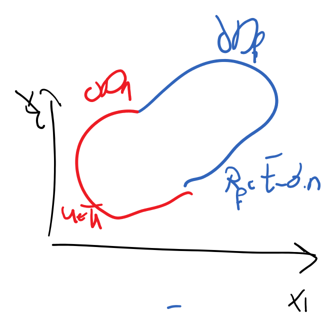
$$\int_{\mathcal{D}} w(\nabla\cdot\sigma)\,dV = \int_{\partial\mathcal{D}} w\,\sigma\cdot n\,dS - \int_{\mathcal{D}} \nabla w\cdot\sigma\,dV$$

WRS $\int_{\mathcal{D}} w(\nabla\cdot\sigma)\,dV + \int_{\mathcal{D}} w(\rho b)\,dV + \int_{\partial\mathcal{D}_f} w(\bar{t} - \sigma\cdot n)\,dS$

$$\left(\int_{\partial\mathcal{D}} w(\sigma\cdot n)\,dS - \int \nabla w\cdot\sigma\,dV\right) + \int_{\mathcal{D}} w\,\rho b\,dV$$

$$+ \int_{\partial\mathcal{D}_f} w\,\bar{t}\,dS - \int_{\partial\mathcal{D}_f} w\,\sigma\cdot n\,dS = 0$$

$$\int_{\partial\mathcal{D}_u \cup \partial\mathcal{D}_f} w(\sigma\cdot n)\,dS - \int_{\partial\mathcal{D}_f} w\,\sigma\cdot n\,dS$$



$$= \int_{\partial\mathcal{D}_u} w\,\sigma\cdot n\,dS$$

$w = 0$ on $\partial\mathcal{D}_u$