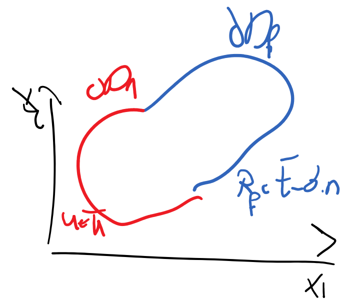
2022/09/15

Thursday, September 15, 2022 11:10 AM

The process of deriving the weak statement

$$\int_{\mathcal{D}} w(\nabla\cdot\sigma + \rho b)\,dV + \int_{\partial\mathcal{D}_f} w(\bar{t} - \sigma\cdot n)\,dS = 0$$



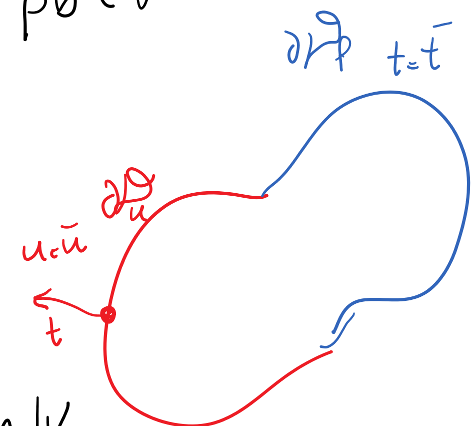
Note

$$\int_{\mathcal{D}} w(\nabla\cdot\sigma)\,dV = \int_{\partial\mathcal{D}} w\,\sigma\cdot n\,dS - \int_{\mathcal{D}} \nabla w\cdot\sigma\,dV$$

By plugging the RHS we obtained:

$$\left(\int_{\partial\mathcal{D}} w(\sigma\cdot n)\,dS - \int_{\mathcal{D}} \nabla w\cdot\sigma\,dV\right) + \int_{\mathcal{D}} w\,\rho b\,dV$$

$$+ \int_{\partial\mathcal{D}} w\,\bar{t}\,dS - \int_{\partial\mathcal{D}_f} w\,\sigma\cdot n\,dS = 0$$



$$\int_{\partial\mathcal{D}_f} w(\sigma\cdot n)\,dS + \int_{\partial\mathcal{D}_u} w(\sigma\cdot n)\,dS - \int_{\mathcal{D}} \nabla w\cdot\sigma\,dV + \int_{\mathcal{D}} w\,\rho b\,dV$$

$$+ \int_{\partial\mathcal{D}} w\,\bar{t}\,dS - \int_{\partial\mathcal{D}_f} w\,\sigma\cdot n\,dS = 0$$

now we want to get rid of this

We will choose weight functions that are zero on Essential BC

$\forall x \in \partial\mathcal{D}_u \; w(x) = 0$

$\sigma(u) = C\,\varepsilon(u)$

1D

$\sigma = E\varepsilon$

weak statement $\rightarrow \varepsilon(w)$

$$\int (\nabla w)\, C\,\varepsilon(u)\,dV = \int w\,\rho b\,dV + \int w\,\bar{t}\,dS$$

$$\int_D (\nabla w) \cdot C\, \varepsilon(u)\, dV = \int_D w \rho b\, dV + \int_{\partial D_f} w \bar{t}\, ds$$

$\varepsilon(u) = \frac{\nabla u + \nabla u^T}{2}$ 1D $\varepsilon(u) = u'$

$\varepsilon(w) = \frac{\nabla w + \nabla w^T}{2}$

we can show $\nabla w \cdot C\, \varepsilon(u) = \underbrace{\left(\frac{\nabla w + \nabla w^T}{2}\right)}_{\varepsilon(w)} \cdot C\, \varepsilon(u)$ from symmetries of C

Final form of the weak statement:

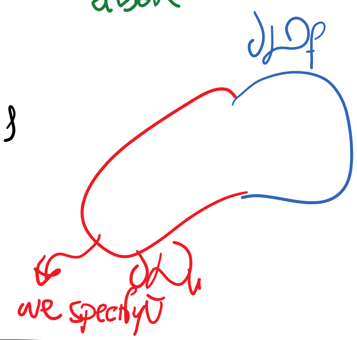
because we strongly satisfy essential BC

Find $u \in \mathcal{V} = \{ f \in C^1(D) \,|\, \forall x \in \partial D_u \;\; f(x) = \bar{u}(x) \}$

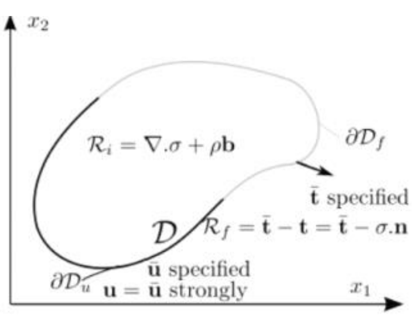
such that $\forall w \in \mathcal{W} = \{ \underbrace{f}_{\text{funct}} \in C^1(D) \,|\, \forall x \in \partial D_u \;\; f(x) = 0 \}$ because we got rid of ∂D term above

$$\int_D \underbrace{\varepsilon(w)}_{\frac{\nabla w + \nabla w^T}{2}} : C\, \underbrace{\varepsilon(u)}_{\frac{\nabla u + \nabla u^T}{2}}\, dV = \int_D w \rho b + \int_{\partial D_f} w \bar{t}\, ds$$

weak statement



Compare this with the WRS:



The Weighted Residual Statement reads as,

Find $\mathbf{u} \in \mathcal{V}^{\mathrm{WRS}} = \{\mathbf{v} \in C^2(\mathcal{D}) \,|\forall \mathbf{x} \in \partial\mathcal{D}_u \;\; \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}\}$, such that, (66a)

$\forall w \in \mathcal{W}^{\mathrm{WRS}} = C^0(\mathcal{D})$ no need to enforce the homogeneous essential BCs for WRS (66b)

$0 = \int_{\mathcal{D}} \mathbf{w}.(\underbrace{\nabla.\sigma}_{C_{ijkl}\, u_{k,lj}} + \rho \mathbf{b})\, dv + \int_{\partial \mathcal{D}_f} \mathbf{w}.(\bar{\mathbf{t}} - \mathbf{t})\, ds$ (66c)

$$0 = \int_{\mathcal{D}} \mathbf{w}.(\underbrace{\nabla.\sigma}_{C_{ijkl}\, u_{k,lj}} + \rho \mathbf{b})\, dv + \int_{\partial \mathcal{D}_f} \mathbf{w}.(\bar{\mathbf{t}} - \mathbf{t})\, ds \qquad (66c)$$

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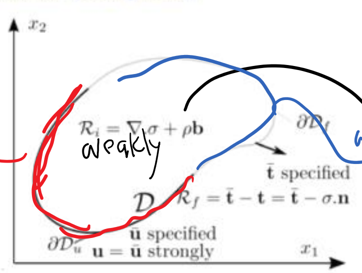
2 derivatives for u

Q: distinction between weak and strong

FEM

∂D_u strongly



weakly

WRS:

$R_i = \nabla.\sigma + \rho b$ in D

$R_f = \bar{t} - t = \bar{t} - \sigma.n$ on ∂D_f

$R_u = \bar{u} - u$ on $\partial \Omega_u$

The Weighted Residual Statement reads as,

we need to satisfy this a priori

Find $\mathbf{u} \in \mathcal{V}^{\mathrm{WRS}} = \{\mathbf{v} \in C^2(\mathcal{D}) \mid \forall \mathbf{x} \in \partial\mathcal{D}_u\ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}\}$, such that, (66a)

$\forall w \in \mathcal{W}^{\mathrm{WRS}} = C^0(\mathcal{D})$ no need to enforce the homogeneous essential BCs for WRS (66b)

$$0 = \int_{\mathcal{D}} \mathbf{w}.(\underbrace{\nabla.\sigma}_{C_{ijkl}\, u_{k,lj}} + \rho \mathbf{b})\, dv + \int_{\partial \mathcal{D}_f} \mathbf{w}.(\bar{\mathbf{t}} - \mathbf{t})\, ds \qquad (66c)$$

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WRS

$$\int_{\Omega} w R_i + \int_{\partial\Omega_f} w R_f \, ds + \cancel{\int_{\partial\Omega_u} w R_u \, ds = 0}$$

"weak"

Strong -> the equation is satisfied at all points
Weak -> the equation is satisfied in "integral" form, where a weight function multiplies the equation

FEM
we decide to satisfy the last term strongly

Specific meaning of weak statement -> The great looking :) equation we get after "integtration by part" of the WRS

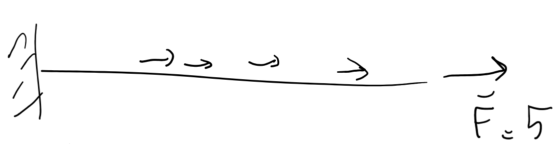
Weak statement is much better than WRS because the solution and the weight have the same regularity requirement and this enables continuous FE formulation.

A brief note on how to satisfy the essential boundary condition for the solution and the homogeneous version of that for the weight when dealing with the Weak Statement.

1D Example

the exact sol is
$u(x)$ → is discretized to

$u = \bar{u} = 1$



$u(x) \longrightarrow$ is discretized to

n unknowns

discretized

$$u^h(X) = \sum_{i=1}^{n} \underbrace{\phi_i(x)}_{\text{the functions we choose}} \underbrace{a_i}_{\text{unknowns}} + \phi_p(X)$$

Motivation from DE

$\dot{x} + 5x = 10$ $\quad x_p(t) = \frac{10}{5} = 2$ $\quad$ source term

$x(0) = 0$ $\quad$ satisfies non-homogeneous $\neq 0$

$x = \phi_1(t) a_1 + \phi_p(t)$

$\underline{\phi_1(t) = e^{-5t}} \rightarrow$ satisfies homogeneous DE

$\dot{\phi}_p + 5\phi_p = \underline{10}$ $\quad \dot{\phi}_1 \; 5\phi_1 = \underline{\underline{0}}$

$$\dot{x} + 5x = \overbrace{(\dot{\phi}_p + 5\phi_p)}^{10} + \overbrace{(\dot{\phi}_1 + 5\phi_1)}^{0} a_1 = 10$$

We do the same trick to satisfy the essential BCs

discretized

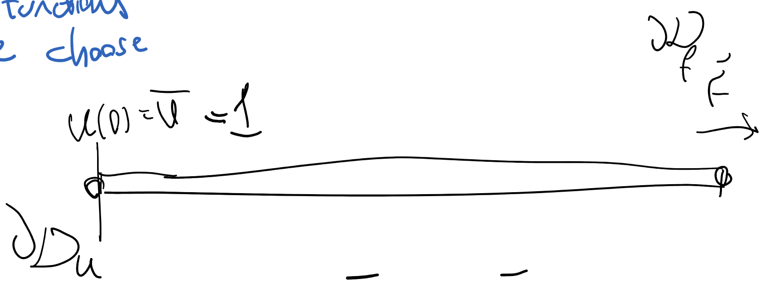
$$u^h(X) = \sum_{i=1}^{n} \underbrace{\phi_i(x)}_{\text{the functions we choose}} \underbrace{a_i}_{\text{unknowns}} + \phi_p(X)$$

<u>where</u>

$\partial\Omega_u : \; x = 0$

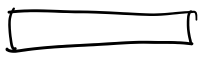
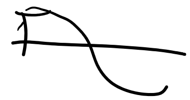
$\phi_p(x=0) = \bar{u}$

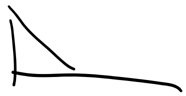
$\forall_{i=1..n} \; \phi_i(x=0) = 0$



$$u^h(x=0) = \sum_{i=1}^{n} \phi_i(x=0) a_i + \phi_p(0) = \sum_{i=1}^{n} 0 \cdot a_i + \bar{u} = \bar{u}$$

Examples of ϕ_p : $\quad \phi_p = 1$

Examples of ϕ_p : $\phi_p < 1$  $\phi_p = \cos x$ 

$1 - x$ 

ϕ_i's $\quad 1, \quad x, \quad x^2, \quad x^3, \quad \sin x, \quad \cos x$

$\phi_i(0) = 0 \quad$ ✗ $\quad$ ✓ $\quad$ ✓ $\quad$ ✓ $\quad$ ✓ $\quad$ ✗

I can choose $\phi_1 = x \qquad \phi_2 = \sin x$

$\phi_p = 1$

$u^h = \phi_1 a_1 + \phi_2 a_2 + \phi_p = a_1 x + a_2 \sin x + 1$ this satisfies essential BC a priori

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Since ϕ_i's are already zero on Essential BC they can readily be used as weight functions in weak statement.

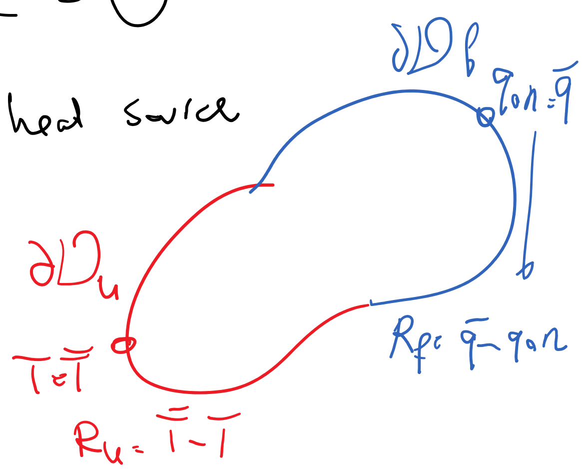
HW hint

PDE $\quad +\nabla \cdot q - Q = 0$

$$+\underbrace{\nabla \cdot q}_{\text{heat flux}} - \underbrace{Q}_{\text{heat source}} = 0$$

WRS

$$\int_\Omega w \underbrace{(\nabla \cdot q - Q)}_{R_i} dV + \int_{\partial\Omega_f} w \underbrace{(\bar{q} - q \cdot n)}_{R_f} ds = 0$$



$$\int w \nabla \cdot q \, dV$$

bad

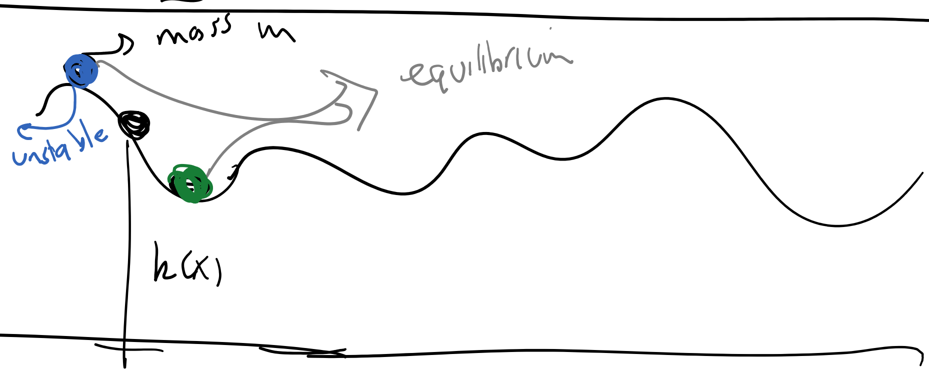
$$\int \underbrace{w(\cdots)}_{\text{bad}} dV$$

Use

$$\int_D w \underbrace{\nabla \cdot q}_{\text{bad}}\, dV = \int_D \nabla\cdot(wq)\, dV - \int_D \underbrace{\nabla w \cdot q\, dV}_{\text{good}} =$$

$$\int_{\partial D} w q\cdot n\, ds - \int \nabla w \cdot q\, dV$$

Energy method



$$\Pi(x) = m g h(x)$$

mass

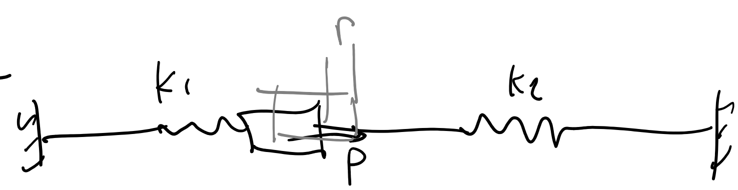
Equilibrium $\dfrac{d\Pi}{dx} = 0$ Π is an extremum (min or max)

unstable local max $\dfrac{d\Pi}{dx} > 0$, ...

stable: local min $\dfrac{\partial \pi}{\partial x} < 0$

Stable solution → we want to find a local minimum

Discre system → Ex1, Ex2

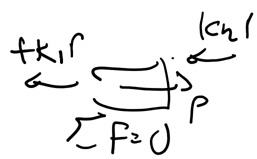


internal energy

$$\Pi = V - W$$

external work

$$V = \frac{k_1 r^2}{2} + \frac{k_2 r^2}{2} \qquad W = P r$$



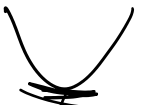
$$\Pi(r) = \frac{k_1 r^2 + k_2 r^2}{2} - P r$$

$d\Pi$ ... equilibrium ... P

$$\frac{d\Pi}{dr} = (k_1+k_2)r - P = 0 \quad \text{equilibrium} \qquad r = \frac{P}{k_1+k_2}$$

matches our force approach

$$\frac{d^2\Pi}{dr^2} = k_1+k_2 > 0$$



→ local minimum

stable equilibrium

Continuum version

Energy Method for Solid Mechanics

The total energy in solid mechanics is,

$$\Pi = (V - W) - T = \text{Total energy} \qquad (85a)$$

V → internal, W → external

$$T = \int_{\mathcal{D}} \frac{1}{2}\rho \mathbf{v}.\mathbf{v}\, dv = \text{Kinetic energy} \qquad (85b)$$

$=0$ for static problems

$$\boxed{V = \int_{\mathcal{D}} e(\epsilon)\, dv = \text{Internal energy}} \qquad (85c)$$

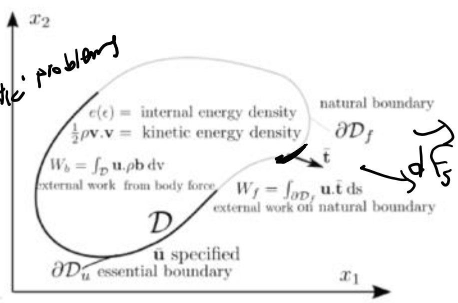
$$W = W_b + W_f = \text{External work} \qquad (85d)$$

$$W_b = \int_{\mathcal{D}} \mathbf{u}\, \rho \mathbf{b}\, dv \qquad (85e)$$

$\mathbf{u}$: displacement, $\rho\mathbf{b}\,dv$: dF_r

$$W_f = \int_{\partial\mathcal{D}_f} \mathbf{u}\, \bar{\mathbf{t}}\, ds \qquad (85f)$$

$\bar{\mathbf{t}}\,ds$: dF_s



- For static problems $T = 0$.
- Internal energy density, $e(\epsilon) = \frac{1}{2}\epsilon : \sigma(\epsilon) = \frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl}$ for linear solid.
- Natural boundary forces are naturally incorporated into the energy (W_f).
- Essential boundary conditions are incorporated into function space:

$$\mathbf{u} \in \mathcal{V} = \{\mathbf{v}\ |\mathbf{v} \in C^1(\mathcal{D}) : \ \forall \mathbf{x} \in \partial\mathcal{D}_u \ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x})\}, \text{ is a solution if}$$
$$\forall \tilde{\mathbf{u}} \in \mathcal{V}, \quad \Pi(\mathbf{u}) \leq \Pi(\tilde{\mathbf{u}}). \qquad (86)$$

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1D

$$e = \frac{\sigma.\varepsilon}{2} = \frac{1}{2}E\varepsilon^2$$

internal energy density

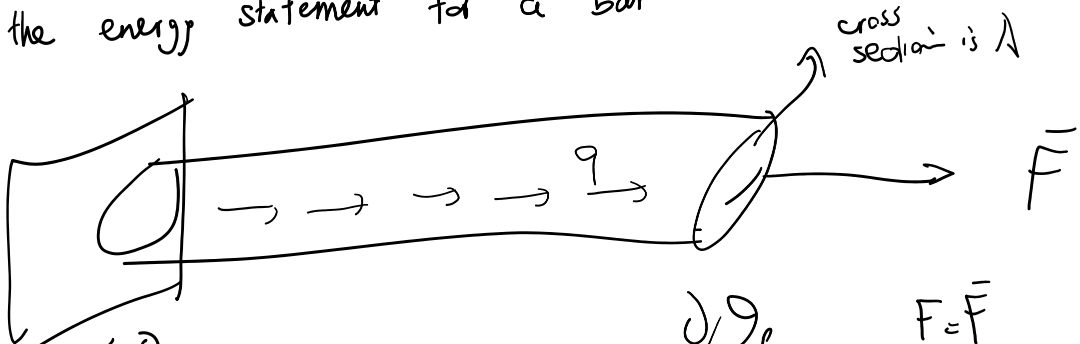
2D, 3D linear elasticity

$$\frac{1}{2}\varepsilon : \sigma = \frac{1}{2}\varepsilon : C : \varepsilon$$

C → stiffness matrix

For the problems we'll do (static) $T = 0$

Derive the energy statement for a bar



$\Pi = V - W$ ... Work $\quad \Pi = \int e(\varepsilon)\, dV = \int_0^L (e(\varepsilon)\, dA\, dx$

$u = \bar{u}$

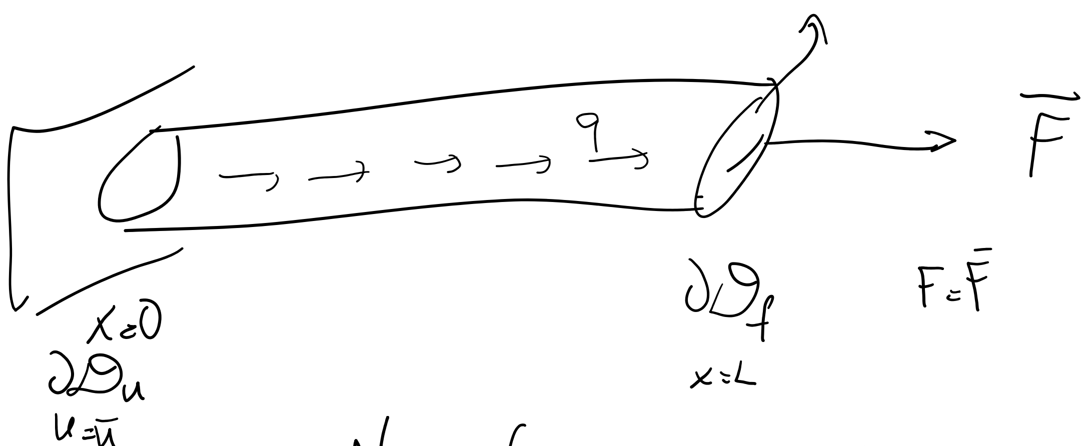
$\Pi = V - W$ (internal energy) (external work) $\Big|$ $V = \int_V e(\varepsilon)\, dV = \int_{x=0}^{L} \int_A e(\varepsilon)\, \underbrace{dA\,dx}_{dV}$, $e(\varepsilon) = \frac{E\varepsilon^2}{2}$ constant across the section

$= \int_0^L \int_A \frac{E}{2}\varepsilon^2\, dA\,dx$ constant across area for each x

$\sigma(x)$ constant so is $\varepsilon(x) = \frac{\sigma(x)}{A}$

$= \frac{1}{2} \int_0^L E(x) A(x)\, \varepsilon^2\, dx \qquad \varepsilon = \frac{du}{dx} = u'$

$$\boxed{V = \frac{1}{2} \int EA(x)\, u'^2(x)\, dx} \quad (i)$$

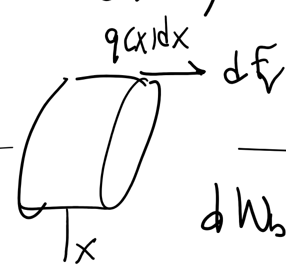


$W_f = \bar{F}u(L)$ (iii)

$F = \bar{F}$

$W = \underbrace{W_b}_{\text{body / bulk}} + \underbrace{W_f}_{\text{natural boundary}} \quad (ii)$

$W_b = \int_0^L u(x)\, q(x)\, dx \quad (iv)$

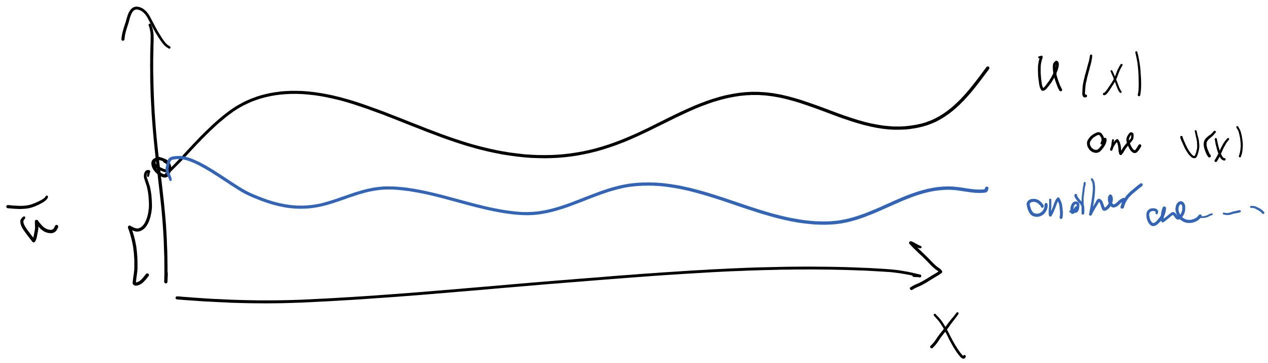


$dW_b = u(x)\, df_v = u(x)\, q(x)\, dx$

$\Pi(u(x)) = V(u(x)) - W_b - W_f$

$$\rightarrow \boxed{\Pi(u(x)) = \frac{1}{2} \int_0^L EA(x)\, u'^2(x)\, dx - \int_0^L u(x)\, q(x)\, dx - u(L)\bar{F}}$$

$$\rightarrow \Pi(u(x)) = \frac{1}{2}\int_0 EA(x)\,u'(x)\,dx - \int_0 \dots$$



A function of a function is called a functional.

1. Useful links for energy method (not necessary to apply energy approach in the derivation of weak statement) – link Functional optimization: How an equation for first variation of a functional (e.g. equations 93, 95 on slide 78) can be derived. You clearly do not need to read this document for this course and this is only provided as a related material for students that want to understand the logic behind the derivation of equations 93, 95. – link Exact calculation of total, first, and second variations for a simple example: In this document the total variation of the energy functional for the bar problem is directly calculated. The first and second variations are directly obtained and higher variations are zero for this simple functional. It is observed that the first variation is exactly the same as what we would have obtained by equation 96 on slide 78.