2022/08/25

Thursday, August 25, 2022 10:13 AM

Course webpage:
http://rezaabedi.com/teaching/me-517-finite-elements/

Selected Bibliography

- Jacob, Fish, and Belytschko Ted. *A first course in finite elements*. Wiley, 2007. link → easy to read, good reference for bars, beams, trusses,
- K. J. Bathe; Finite Element Procedures. Cambridge, MA: Klaus-Jurgen Bathe, 2007. ISBN: 9780979004902 (B). → link quadrature (numerical integration), 2D elements
- T. J. R. Hughes; The Finite Element Method: Linear Static and Dynamic Finite Element Analysis, Dover Publications, 2000. ISBN : 978-0486411811 (H). link
- R.D. Cook, D.S. Malkus, M.E. Plesha, R.J. Witt, Concepts and Applications of Finite Element Analysis, Wiley, 4th Edition, 200 1.ISBN: 0471356050 (C). link
- o O.C. Zienkiewicz, R.L. Taylor, J.Z. Zhu; The Finite Element Method: Its Basis and Fundamentals, Butterworth -Heinemann; 7th edition, 2013. ISBN: 1856176339 (Z). link

From <http://rezaabedi.com/teaching/me-517-finite-elements/>

About 40% of the course is about balance laws, strong form, weak form, and finite element formulation (beginning of the course).
This part is more mathematical and the course notes are the best reference for this part.

You can download course notes from:
http://rezaabedi.com/wp-content/uploads/Courses/FEM/c_FEM.pdf

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Course outline:

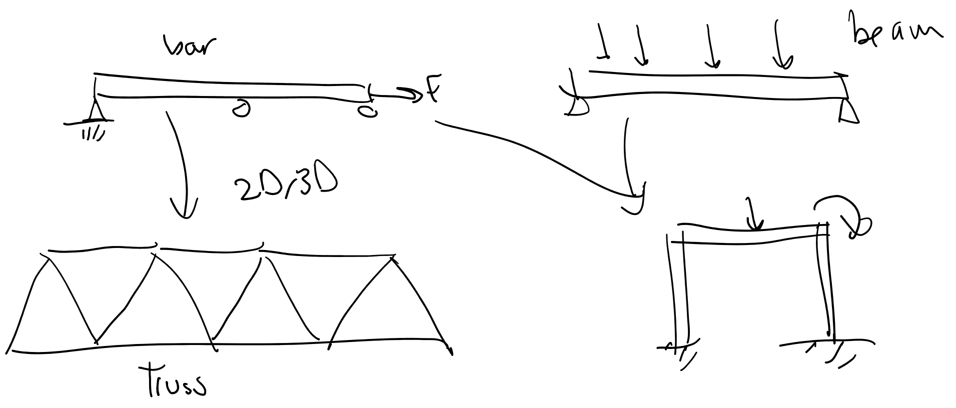
1. Finite element (&spectral methods, a bit finite difference) formulation
   a. Balance laws
   b. Strong Form
   c. Weighted Residual Statement (WRS)
   d. Weak Form
   e. Discretization
   f. Energy method

   long process, but very general

   energy method can directly provide the final weak statement

   40%, a bit more mathematical

2. 1D elements (bar, beam, truss, and frame problems)



   Each element type provides a new concept

3. 2D/3D problems:
   a. Elastostatics (with some notes of elastodynamics)
   b. Heat conduction

 - Numerical integration (quadrature)
 - Isoparametric 2D/3D elements

4. Finite element implementation (how to code an FEM using

C++, Matlab): lectures on objected-oriented programming
of FEM

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often takehome

- Exam(s) (subject to change): 8%
- Assignments: Homework assignments take up 50% of the grade. Assignments typically involve a computational part that requires writing/modifying small computer codes (Matlab, C++) or using commercial packages such as COMSOL. The assignments include challenge problems" that can add up to 5-10% to the final grade. Percentage can be subject to change. 60%
- Term project(s): Computer FEM code (18%)& commercial FEM software (1[illegible]%) 32%

  Ansys (me) , ... Abaqus
- Absences and excused grades: Excuses will be given only under the following circumstances: for fluid / thermal you can chose another project
  - illness
  - personal crisis (e.g. automobile accident, death of a close relative)

  otherwise there is a 15% penalty per day for late assignments.

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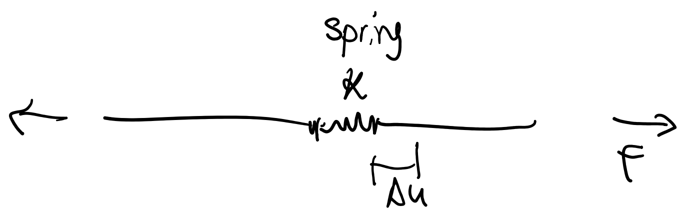
1. Finite element formulation

If you understand this part well:
- (Continuous) Finite element method
- Spectral method
- Discontinuous Galerkin (DG)
- Finite Volume
- Finite Difference

tangently related

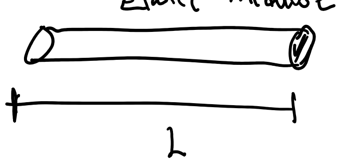
Bar problem



$F = k \Delta u$

① what if we have a finite bar

Elastic modulus E



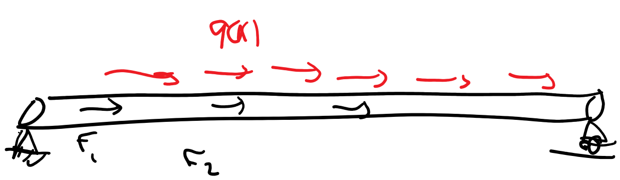
A area section A

$F = k \Delta u$

$\downarrow$

?

②



unknown is displacement u

what is the differential equation

---

①

ρA

①

change of length L

total displacement change for the bar

① $\varepsilon = \frac{\Delta u}{L}$

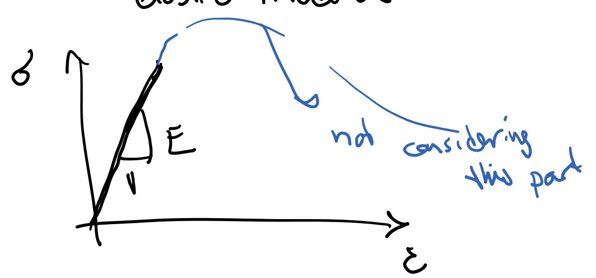
original length

free differential $d\vec{F}$

dA surface area differential dA

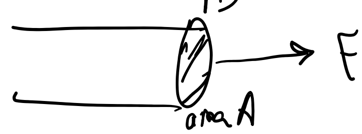
② $\sigma = E\,\varepsilon$ linear elastic

elastic modulus



traction $\vec{t} = \frac{d\vec{F}}{d\vec{A}}$

1D



③ $\sigma = \frac{F}{A}$

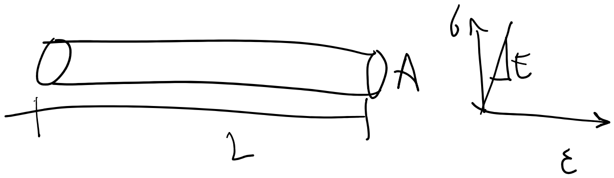
$\sigma = E\varepsilon = E\frac{\Delta u}{L}$

③ $F = \sigma A$

(1a)

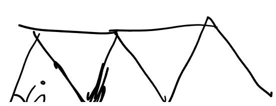
$$F = \underbrace{\left(\frac{EA}{L}\right)}_{k} \Delta u$$

spring stiffness



$F = k\,\Delta u$

part 2 of the course



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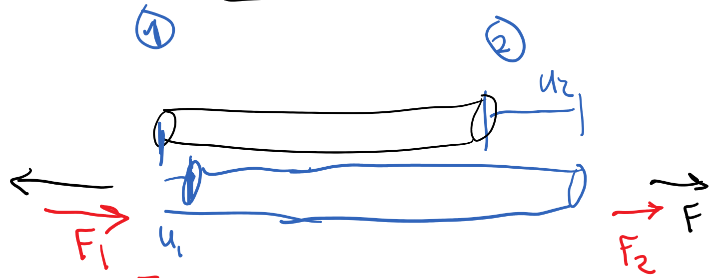
Ia → Ib writing $\boxed{F = \frac{AE}{L}\Delta u}$ in FEM-friendly form



$\Delta u = u_2 - u_1$

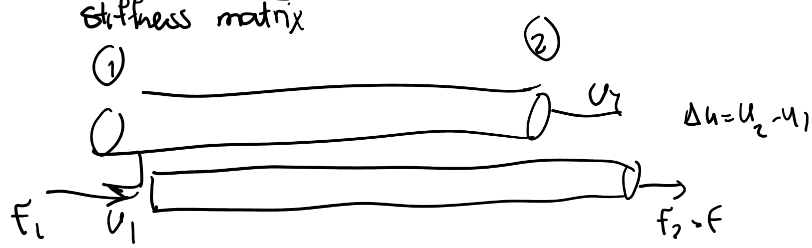
$F = \frac{AE}{L}(u_2 - u_1)$ $\quad F_1 = -F \quad F_2 = F$

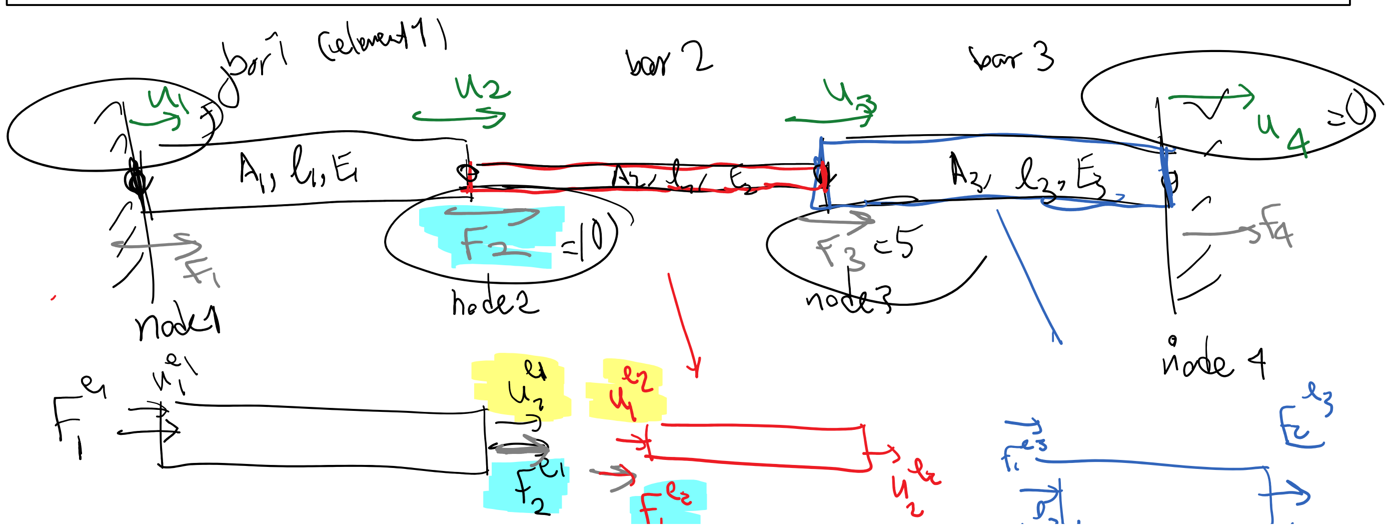
$$F_1 = -F = \frac{AE}{L}(u_1 - u_2)$$

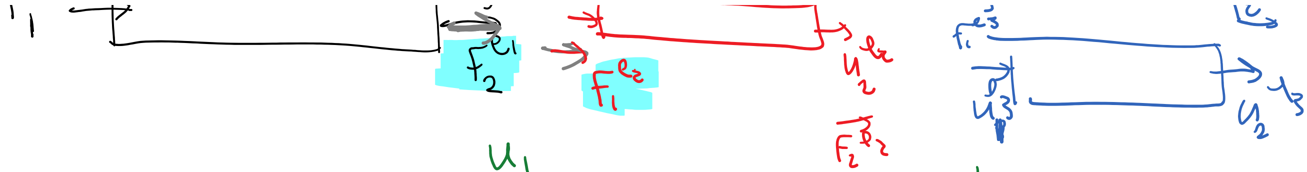
$$F_2 = F = \frac{AE}{L}(u_2 - u_1)$$

Finite Element version of Ic

Ib $\quad \overbrace{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{L} \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{stiffness matrix}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}} \equiv$ Ia $F = \frac{AE}{L}\Delta u$







$$\begin{bmatrix} F_1^{e_1} \\ F_2^{e_1} \end{bmatrix} = \left(\frac{AE}{L}\right)_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{e_1} \\ u_2^{e_1} \end{bmatrix} \quad u_1,\ u_2$$

$$\begin{bmatrix} f_1^{e_2} \\ F_2^{e_2} \end{bmatrix} = \left(\frac{AE}{L}\right)_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{e_2} \\ u_2^{e_2} \end{bmatrix} \quad u_2,\ u_3$$

$$\begin{bmatrix} f_1^{e_3} \\ f_2^{e_3} \end{bmatrix} = \left(\frac{AE}{L}\right)_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{e_3} \\ u_2^{e_3} \end{bmatrix} \quad u_3,\ u_4$$

Q1) What is the relationship between $u_2^{e_1} \to u_1^{e_2}$

Q2) What is the relation between $F_2^{e_1}$, $F_1^{e_2}$ & F_2

$$F_2 = F_2^{e_1} + F_1^{e_2}$$

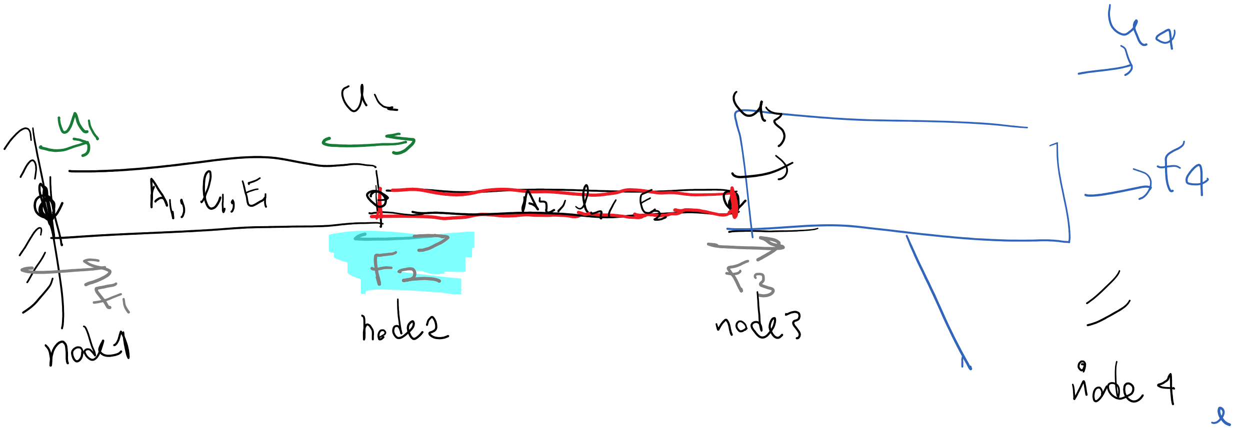
$$F_2 = F_2^{e_1} + F_1^{e_2} = \left(\frac{AE}{L}\right)_1 (u_2 - u_1) + \left(\frac{AE}{L}\right)_2 (u_2 - u_3)$$

$$F_2 = \left(\frac{AE}{L}\right)_1 (-u_1) + \left(\left(\frac{AE}{L}\right)_1 + \left(\frac{AE}{L}\right)_2\right) u_2 - \left(\frac{AE}{L}\right)_2 (u_3)$$

This process:

- Noting that neighboring element displacements (unknowns) are the same
- Their "forces" ADD with each other
- Add contribution of all elements

is called Assembly



$$\begin{bmatrix} F_1 \neq ? \\ F_2 = 5 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ -\left(\frac{AE}{L}\right)_1, & \left(\frac{AE}{L}\right)_1 + \left(\frac{AE}{L}\right)_2, & -\left(\frac{AE}{L}\right)_2, & 0 \end{bmatrix} \begin{bmatrix} u_1 \to 0 \\ u_2\ ? \end{bmatrix}$$

$$\begin{Bmatrix} \vdots \\ F_2 \to 5 \\ F_3 \to 0 \\ F_4 \to ? \end{Bmatrix} = \underbrace{\begin{bmatrix} \vdots & & & \\ -\left(\frac{AE}{L}\right)_1, & \left(\frac{AE}{L}\right)_1 + \left(\frac{AE}{L}\right)_2, & -\left(\frac{AE}{L}\right)_2, & 0 \\ - & - & - & \\ & & & K_{44} \end{bmatrix}}_{\text{Global system stiffness}} \begin{Bmatrix} \vdots \\ u_2 \; ? \\ u_3 \; ? \\ u_4 \to 0 \end{Bmatrix}$$

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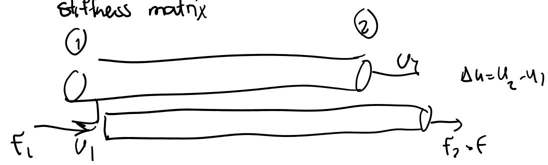
Tuesday, August 30, 2022 11:16 AM

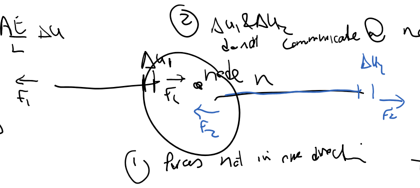
From last time:

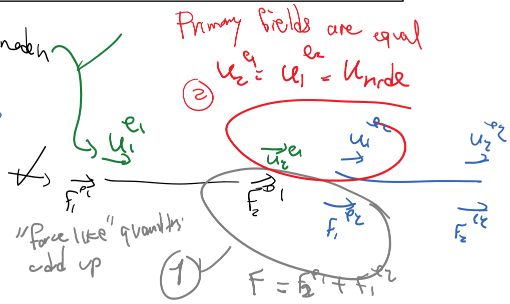
Ib
$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{L} \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{stiffness matrix}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv \text{Ia } F = \frac{AE}{L}\Delta u$$



$F = \frac{AE}{L}\Delta u$

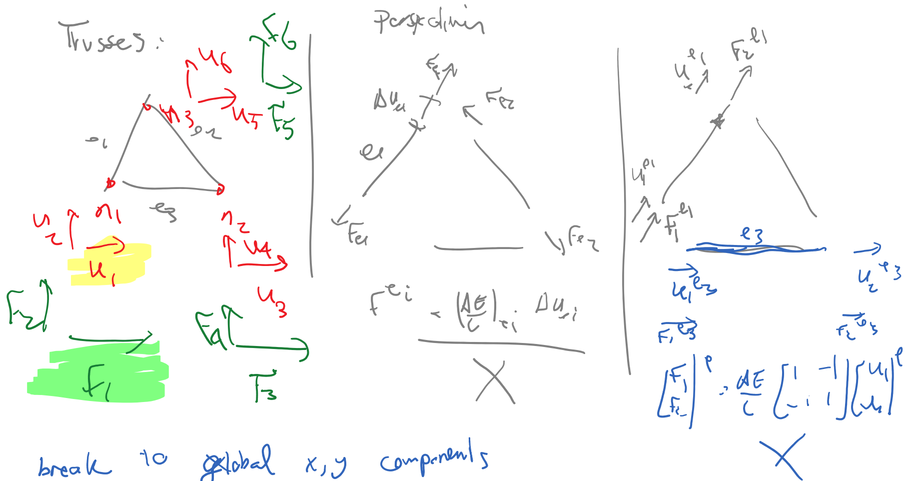
Problems



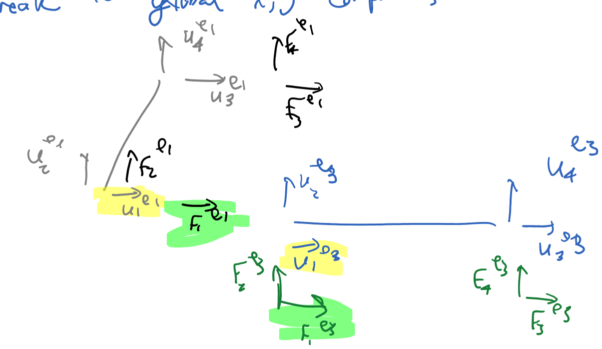


$$\begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}$$

Trusses:

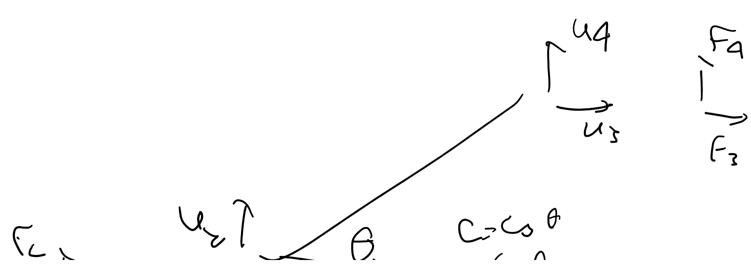


break to global x, y components

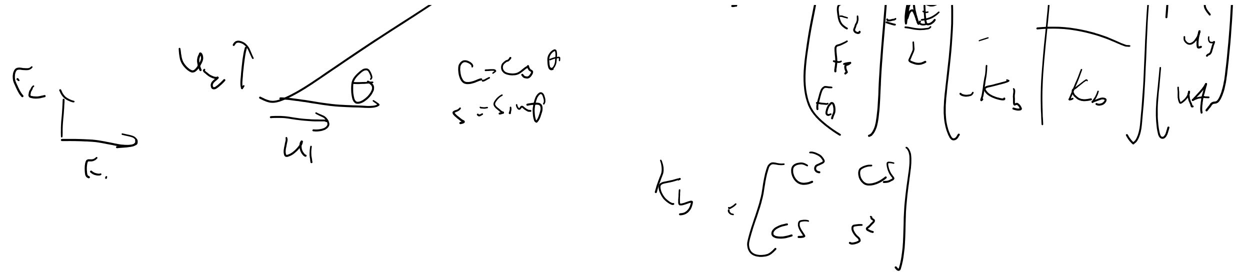


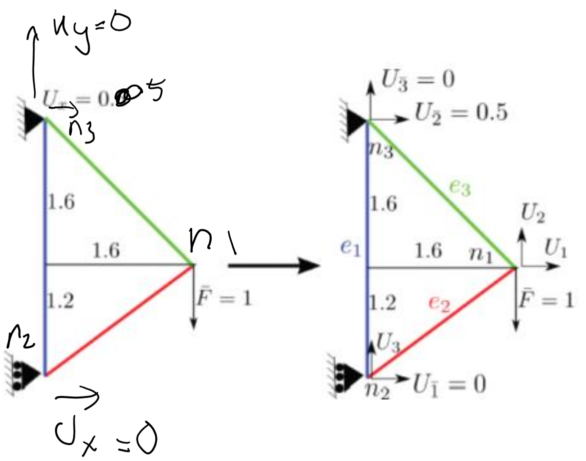
$u_1^{e_1} = u_1^{e_3} = u_1$ displacements are equal

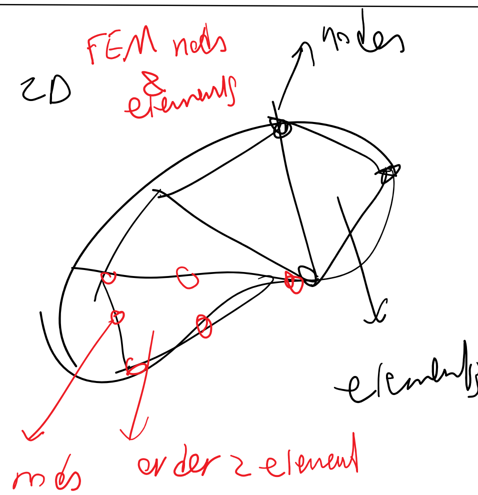
$F_1^{e_1} + F_1^{e_3} = F_1$ Force add

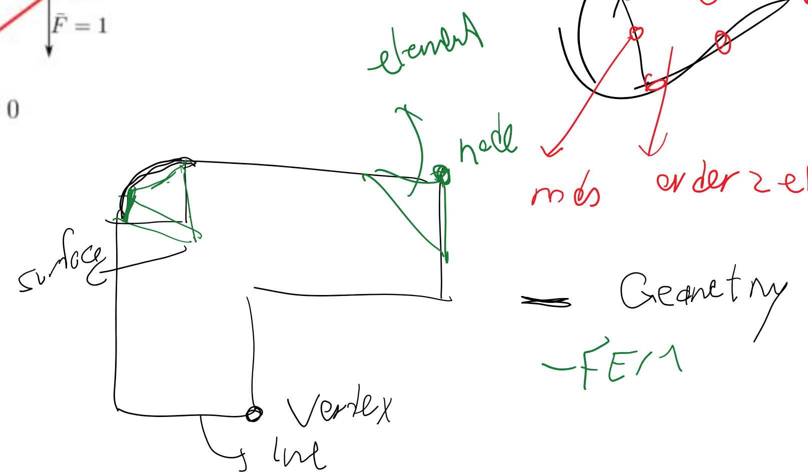


$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} K_b & -K_b \\ -K_b & K_b \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$









FEM packages:

- We always work with geometry objects (vertices, lines, surfaces, volumes). For example, we even apply the loads and other BCs on geometry. Finally, we mesh the geometry.
- The only exception is for domains with 1D elements. In this case, we directly specify FEM objects (nodes and elements)

materials

$E_1 = 200$

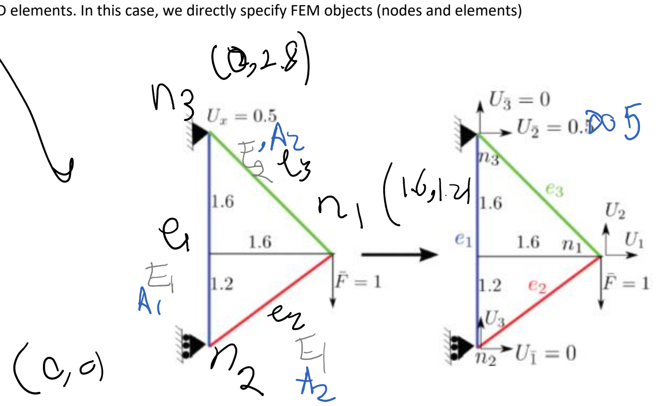
$E_2 = 10$

section

2 section

$A_1 = 10$

$A_2 = 100$



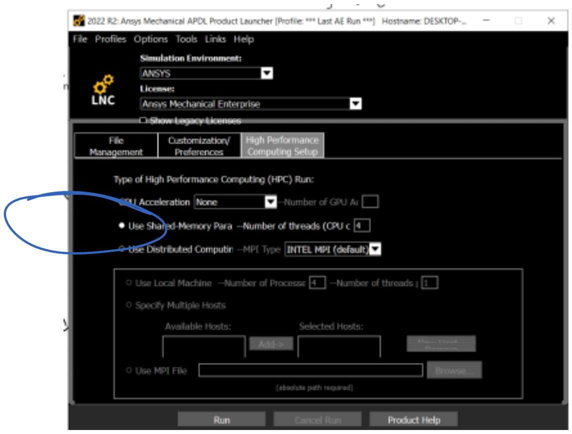
$K = \frac{AE}{L} \begin{bmatrix} & \\ & \end{bmatrix}$

material property

geometry Ansys calculates it

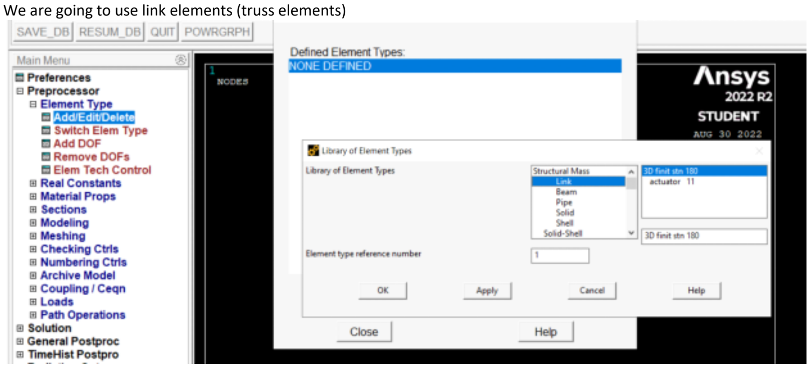
section property

Launch Ansys Product Launcher



1. Define elements to be used

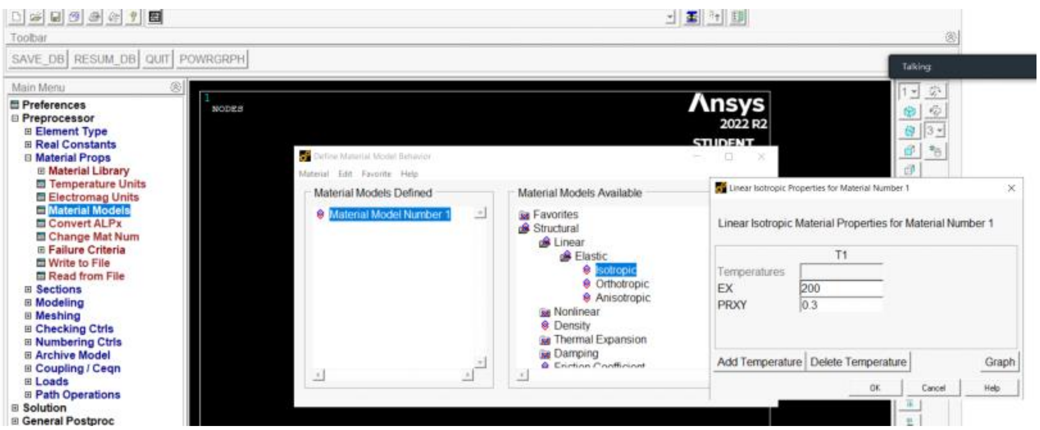
We are going to use link elements (truss elements)

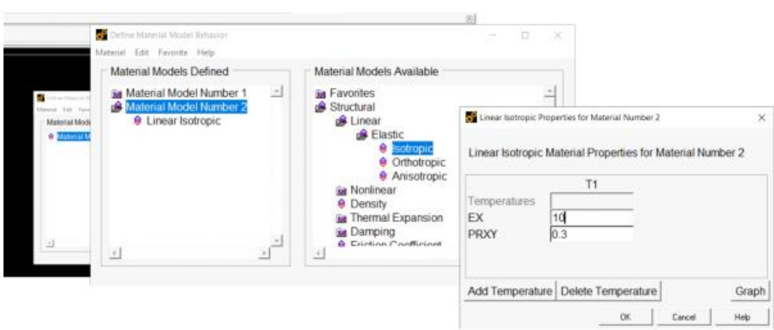


2. Add material properties

E1 = 200

E2 = 10

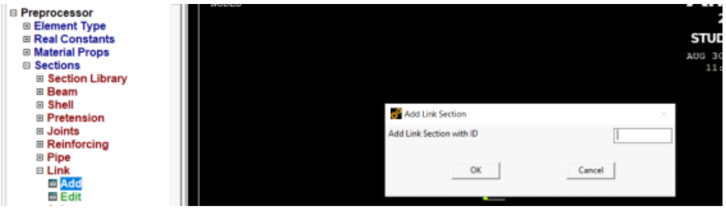


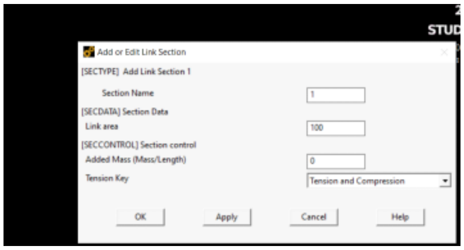


3. Define section properties

$A_1 = 10$

$A_2 = 100$

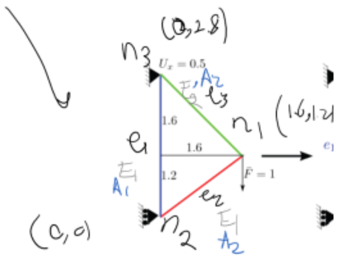


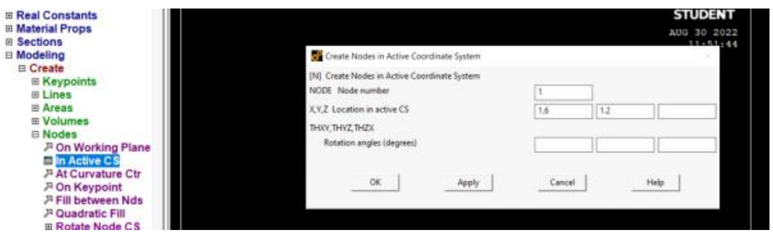


Add section 2

You can list materials and sections

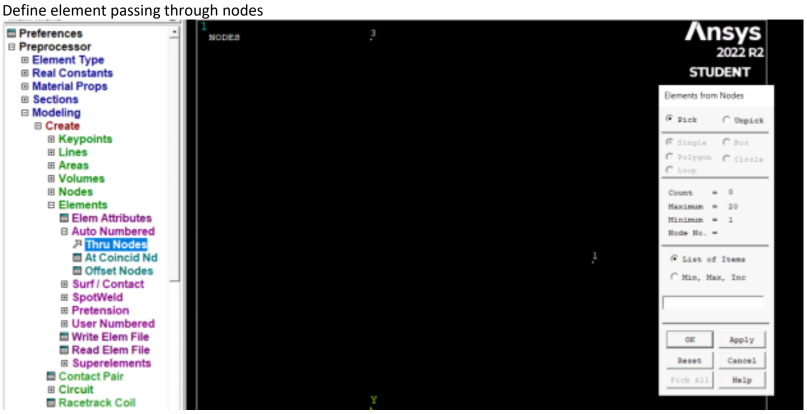
4. Define nodes:

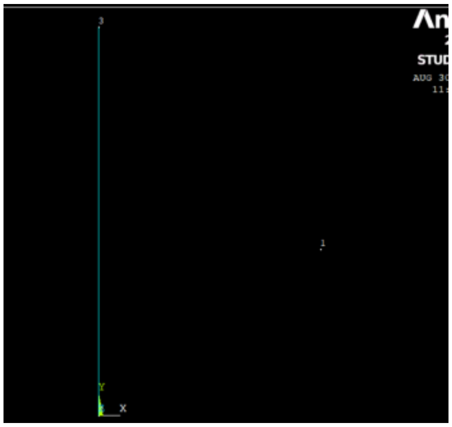




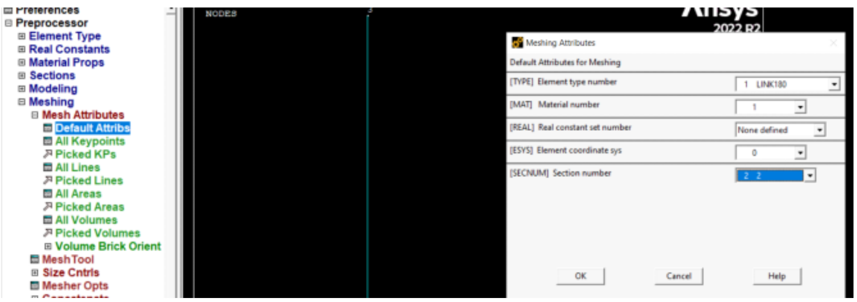
And nodes 2, 3

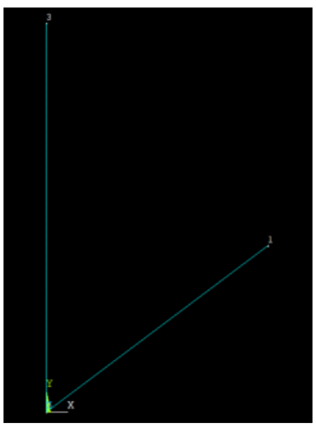
5. Step 4: define elements
- Choose default material number
- Choose default section number
- Define element passing through nodes



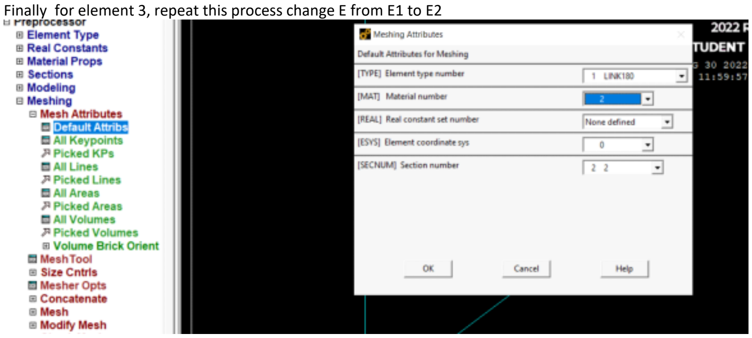


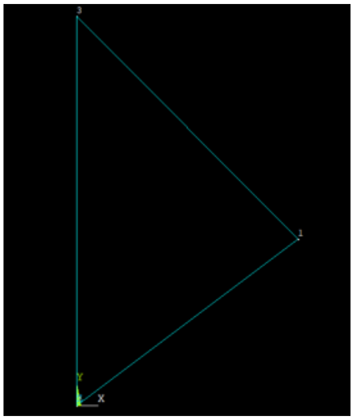
For element 2: change A from A1 to A2



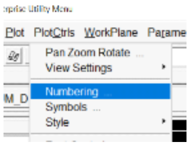


Finally for element 3, repeat this process change E from E1 to E2

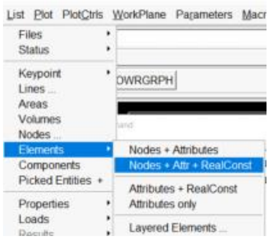




Show the element numbering



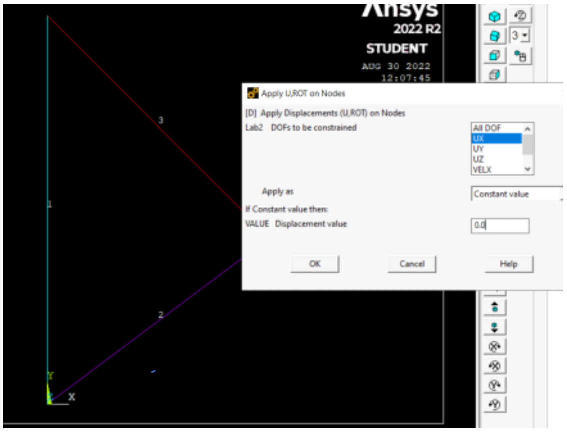
If want to check elements are formed correctly

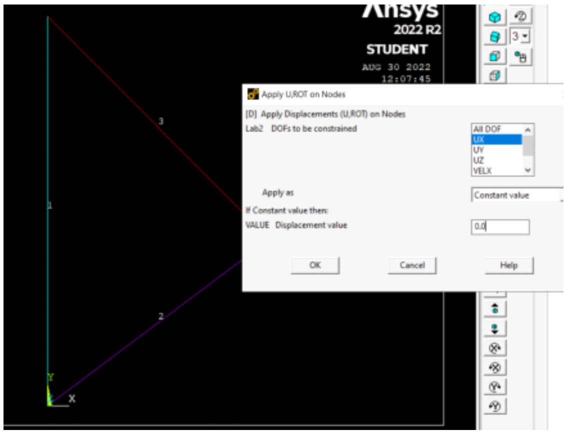


LIST ALL SELECTED ELEMENTS. (LIST NODES)

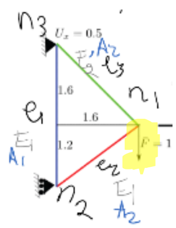
ELEM	MAT	TYP	REL	ESY	SEC	NODES	
1	1	1	1	0	1	2	3
2	1	1	1	0	2	2	1
3	2	1	1	0	2	3	1

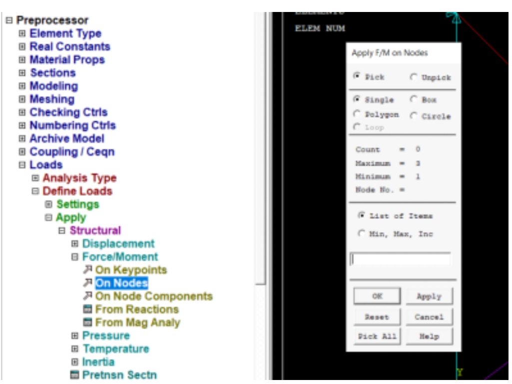
6. BCs:
   a. Displacements:

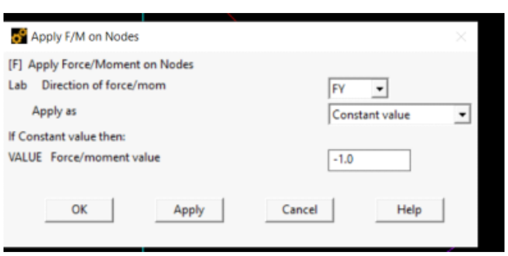




b. Forces



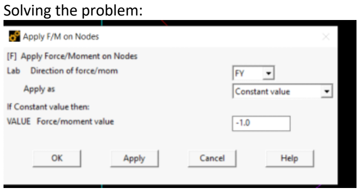


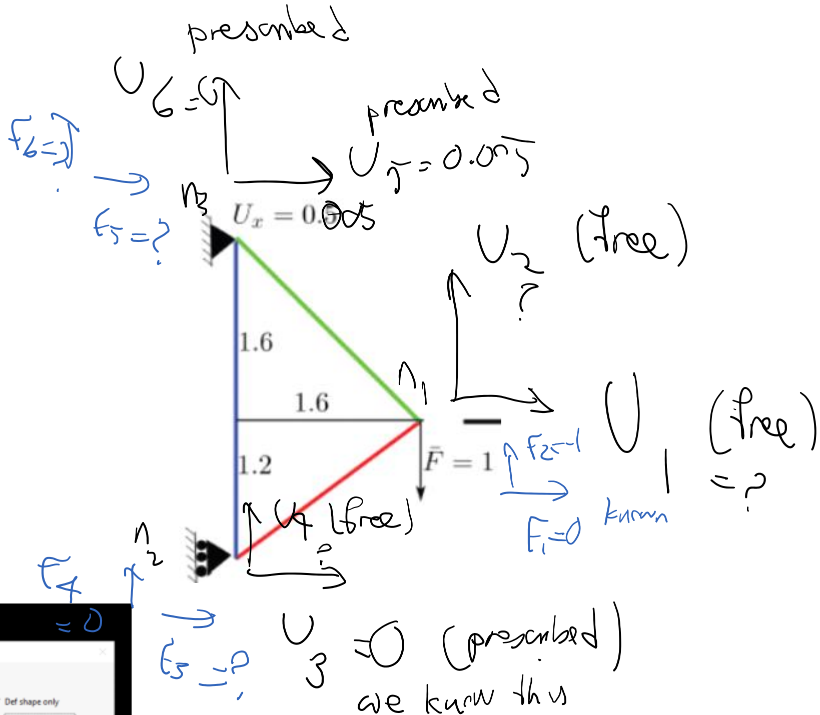


prescribed

Preprocessor stage is finished.
Solving the problem:

Preprocessor stage is finished.
Solving the problem:

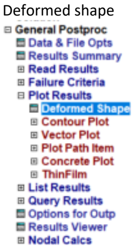


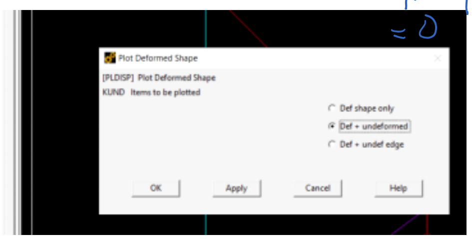


Postprocess:
View and list the results

1. Deformed shape
2. Elements solutions (axial force and stress)
3. Displacements for free degrees of freedom (dof)
4. Forces for prescribed dofs (called Reaction forces)

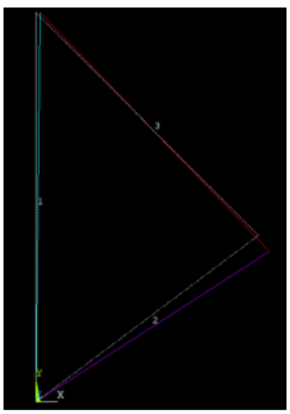
1. Deformed shape

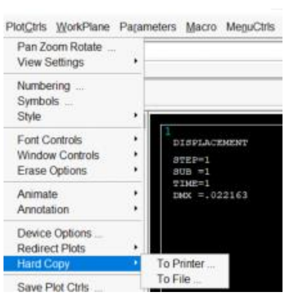




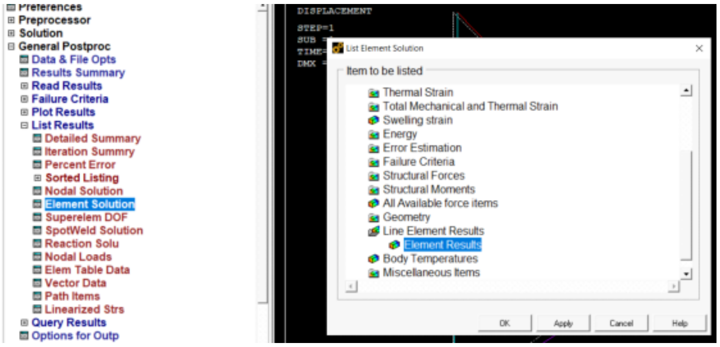
Prescribed, Dirichlet dof:
- we know the displacement (primary field)
- We don't know the force

Free (Neumann) dof
- We don't know the displacement (primary field)
- We know the force



2. 

2. Elements solutions (axial force and stress)



PRINT ELEM ELEMENT SOLUTION PER ELEMENT

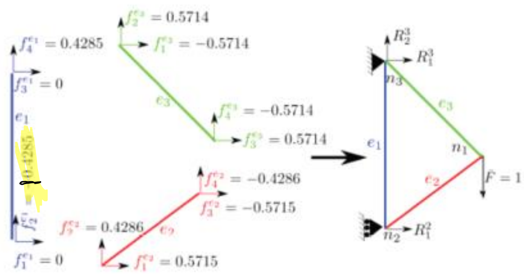
***** POST1 ELEMENT SOLUTION LISTING *****

LOAD STEP 1 SUBSTEP= 1
TIME= 1.0000 LOAD CASE= 0

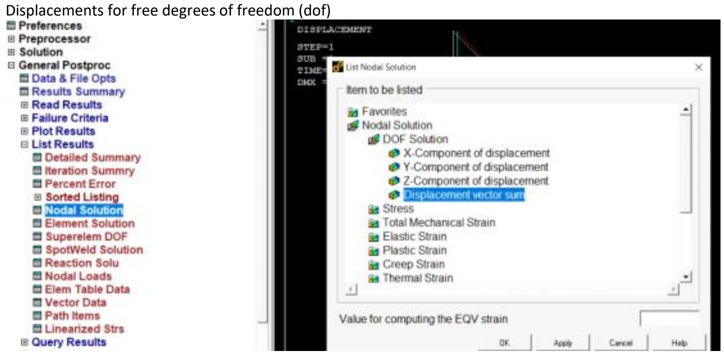
EL= 1 NODES= 2 3 MAT= 1 XC,YC,ZC= 0.000 1.400 0.000 AREA= 100.00 LINK180
FORCE= 0.42857 STRESS= 0.42857E-02 EPEL= 0.21429E-04
TEMP= 0.00 0.00 EPTH= 0.0000

EL= 2 NODES= 2 1 MAT= 1 XC,YC,ZC= 0.8000 0.6000 0.000 AREA= 10.000 LINK180
FORCE=-0.71429 STRESS=-0.71429E-01 EPEL=-0.35714E-03
TEMP= 0.00 0.00 EPTH= 0.0000

EL= 3 NODES= 3 1 MAT= 2 XC,YC,ZC= 0.8000 2.000 0.000 AREA= 10.000 LINK180
FORCE= 0.80812 STRESS= 0.80812E-01 EPEL= 0.80812E-02
TEMP= 0.00 0.00 EPTH= 0.0000



3. Displacements for free degrees of freedom (dof)



PRINT U NODAL SOLUTION PER NODE

***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

LOAD STEP= 1 SUBSTEP= 1
TIME= 1.0000 LOAD CASE= 0

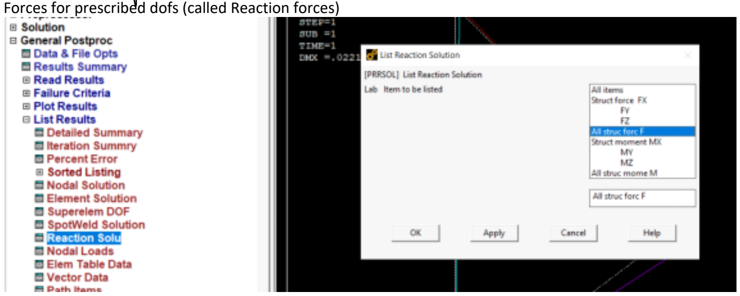
THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

free dofs

NODE	UX	UY	UZ	USUM
1	0.12690E-001	-0.18170E-001	0.0000	0.22163E-001
2	0.0000	-0.60000E-004	0.0000	0.60000E-004
3	0.50000E-002	0.0000	0.0000	0.50000E-002

prescribed

4. Forces for prescribed dofs (called Reaction forces)



PRINT F REACTION SOLUTIONS PER NODE

***** POST1 TOTAL REACTION SOLUTION LISTING *****

LOAD STEP= 1 SUBSTEP= 1
TIME= 1.0000 LOAD CASE= 0

THE FOLLOWING X,Y,Z SOLUTIONS ARE IN THE GLOBAL COORDINATE SYSTEM

NODE FX FY FZ

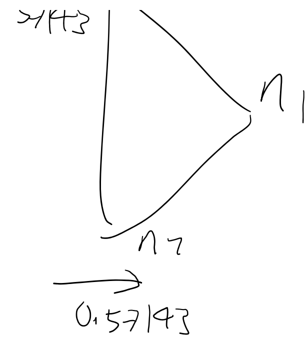
TIME= 1.0000 LOAD CASE= 0

THE FOLLOWING X,Y,Z SOLUTIONS ARE IN THE GLOBAL COORDINATE SYSTEM

NODE	FX	FY	FZ
2	0.57143		
3	-0.57143	1.0000	

TOTAL VALUES

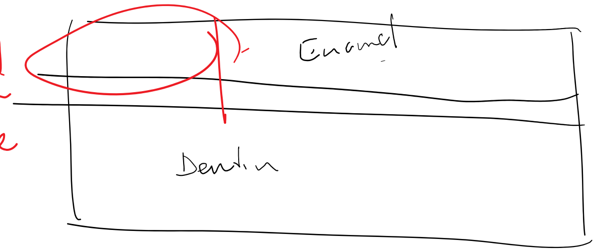
VALUE	-0.77716E-015	1.0000	0.0000

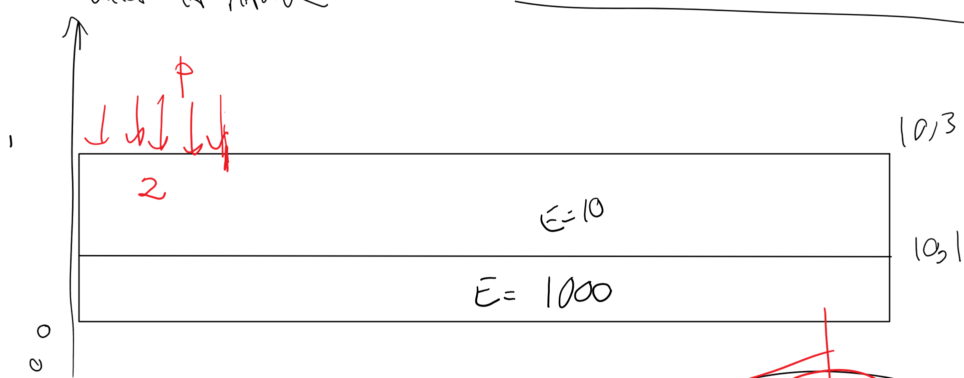


HW1 and final project is a truss problem

2D example
Project 1, a dental crown problem

what's the best gradient for the filling

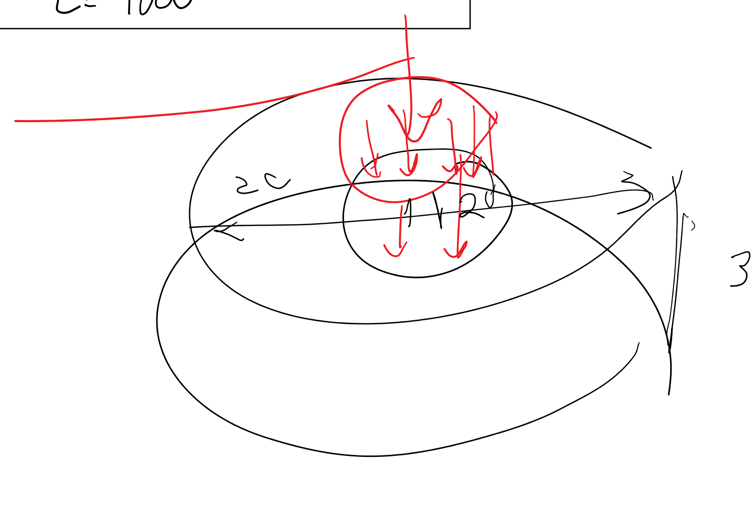


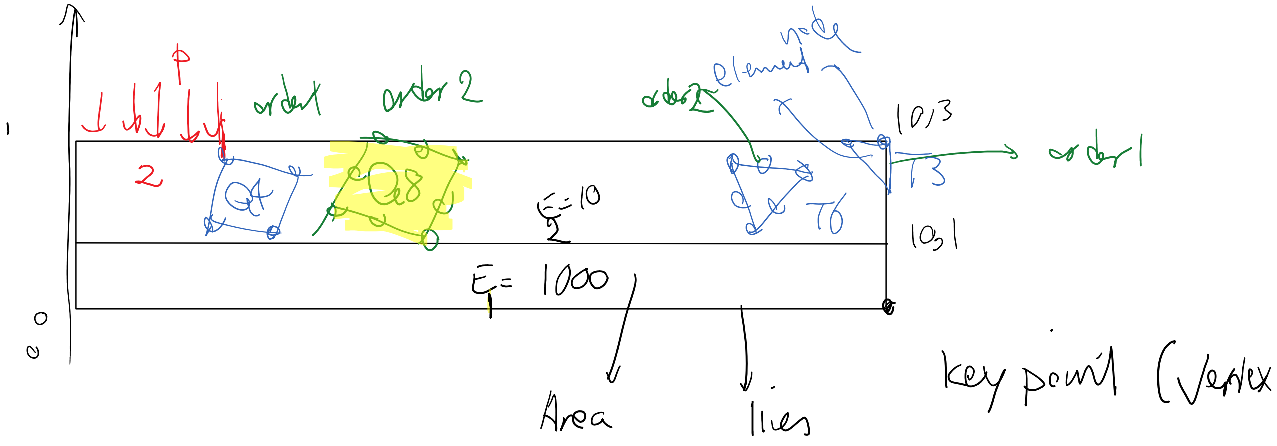


resultant force

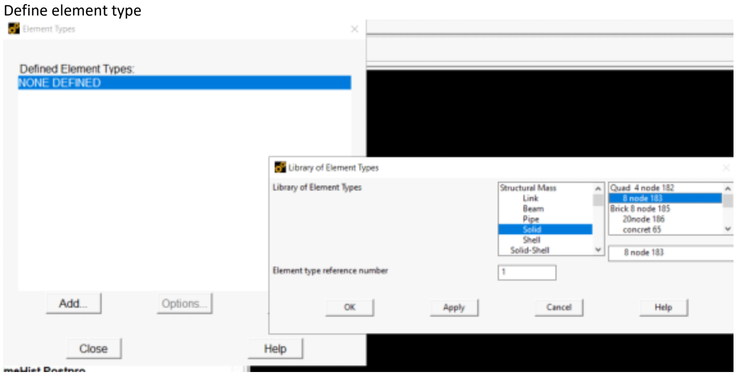
$$F = \pi r^2 p$$

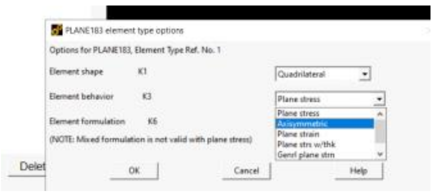
$$P = \frac{F}{\pi r^2}$$



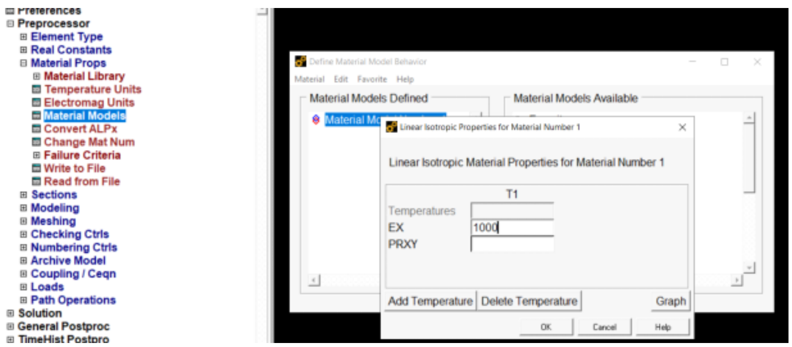


1. Define element type





2. Materials

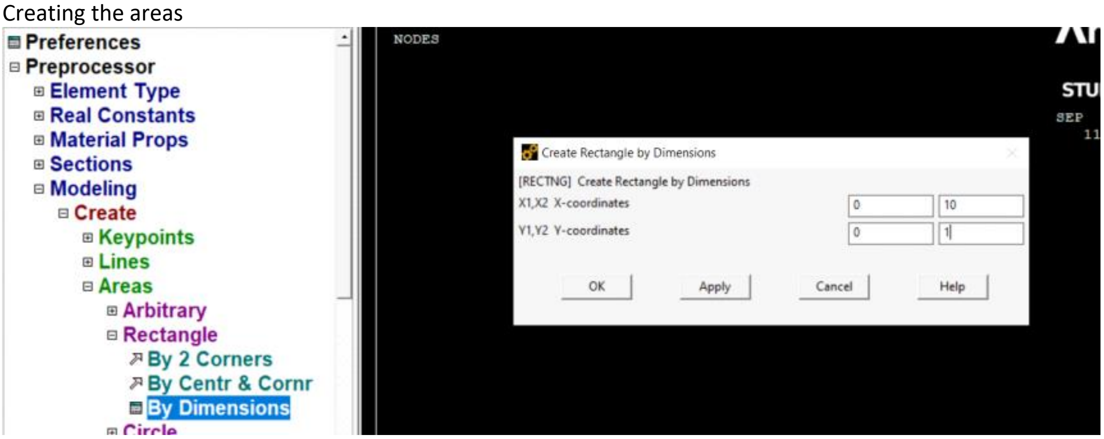


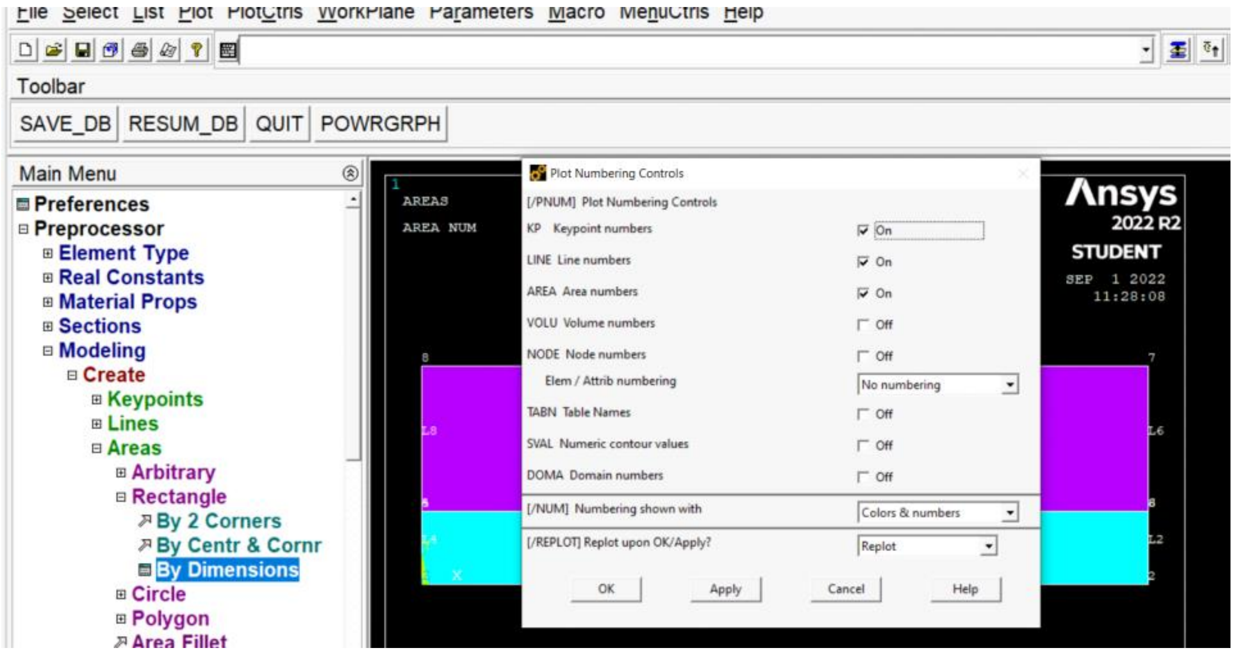
And then 2

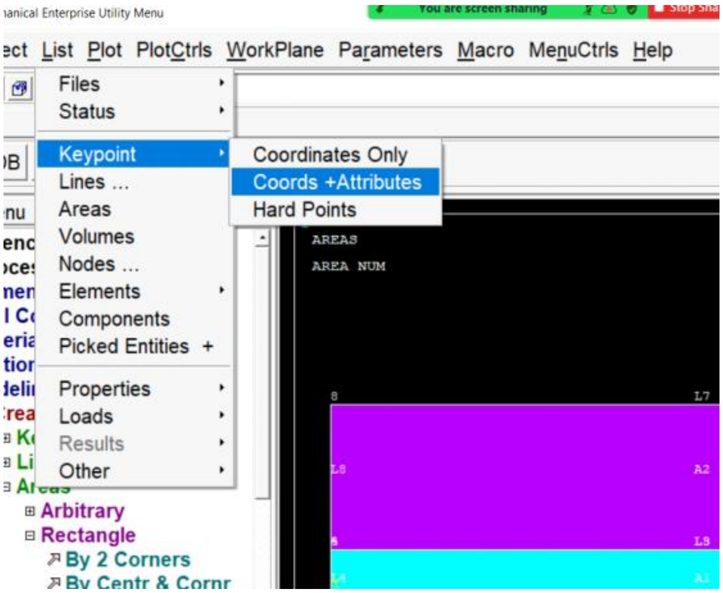
2022/09/01

Thursday, September 1, 2022 11:25 AM

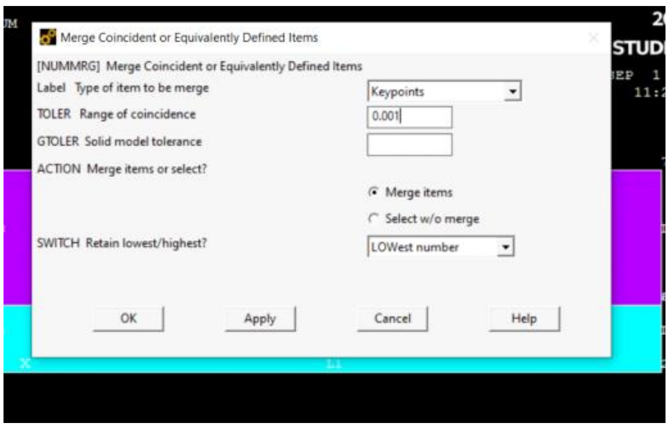
Creating the areas

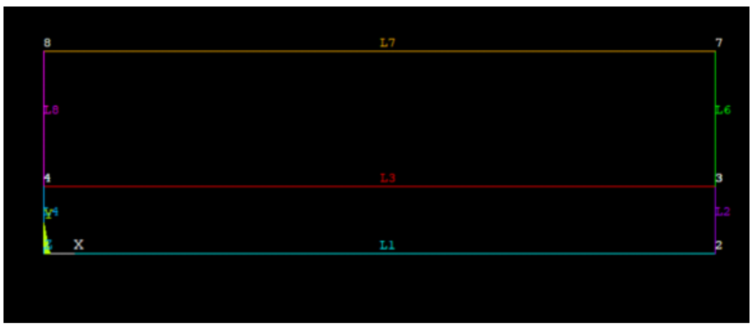






We need to merge the keypoionts and after that the connecting lines will also merge

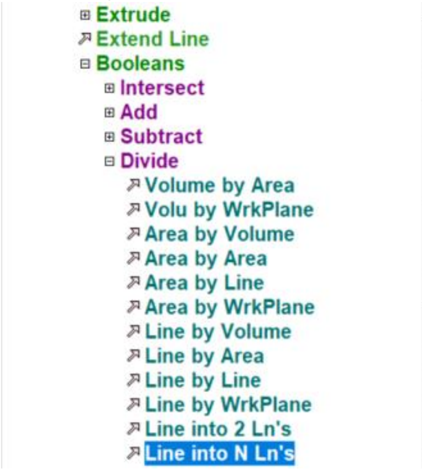


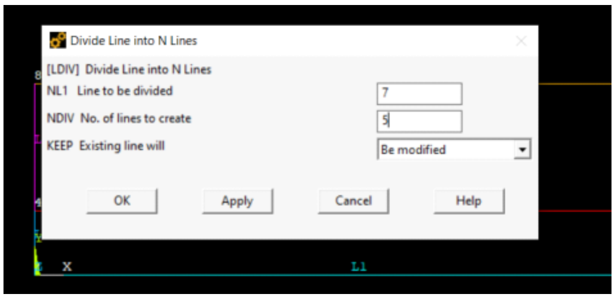


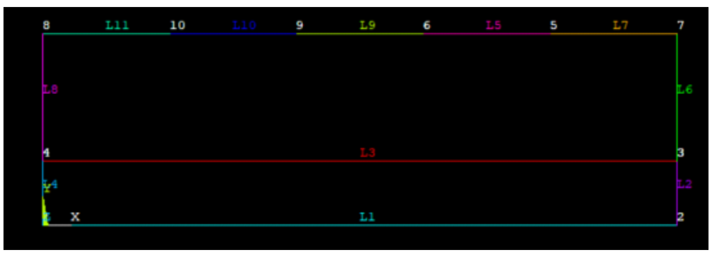
LIST ALL SELECTED KEYPOINTS. DSYS= 0

NO.	X,Y,Z LOCATION			KESIZE	NODE	ELEM	MAT	REAL	TYP	ESYS
1	0.00	0.00	0.00	0.00	0	0	0	0	0	0
2	10.0	0.00	0.00	0.00	0	0	0	0	0	0
3	10.0	1.00	0.00	0.00	0	0	0	0	0	0
4	0.00	1.00	0.00	0.00	0	0	0	0	0	0
7	10.0	3.00	0.00	0.00	0	0	0	0	0	0
8	0.00	3.00	0.00	0.00	0	0	0	0	0	0

Dividing the top line to 5 segments so we can apply the load on the first created segment from the left





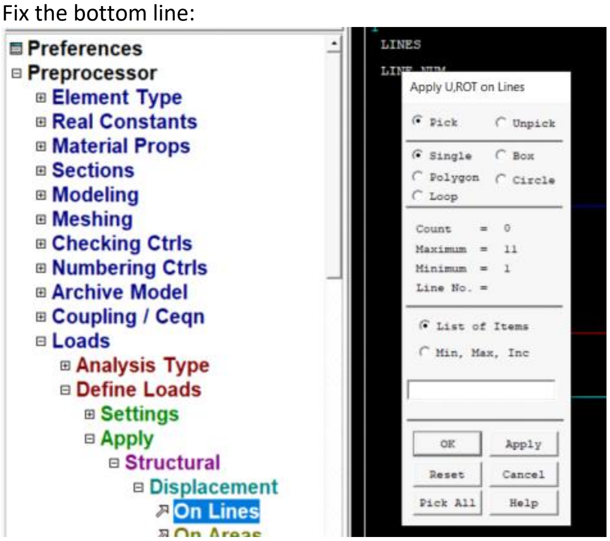


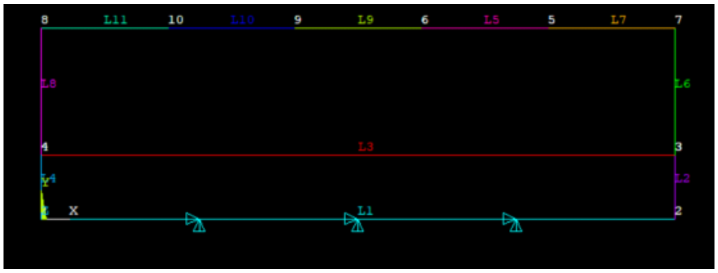
Boundary conditions

- We don't need to specify that the left edge is the axis of symmetry

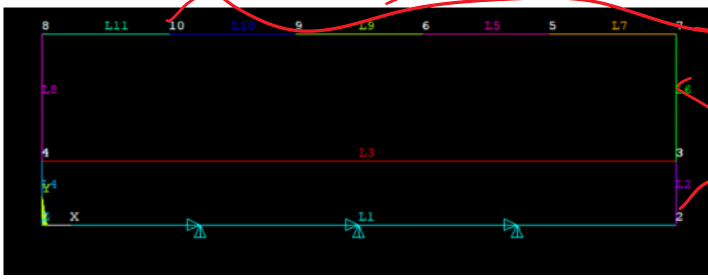
- Fix the bottom line:

-





After fixing all dofs on the bottom surface

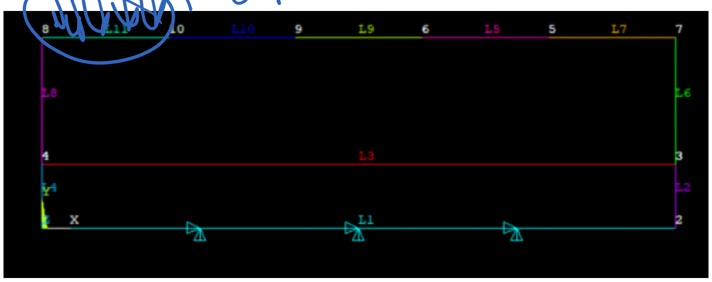


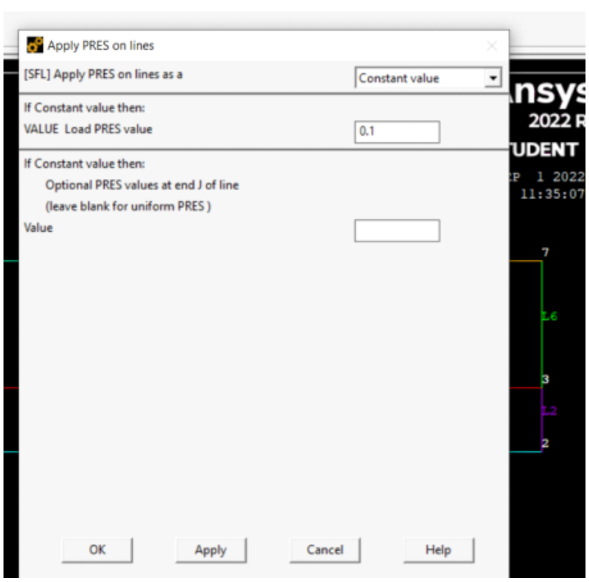
traction free

no need to define BC here as this is FEM's default

Apply the pressure on the top surface - first segment

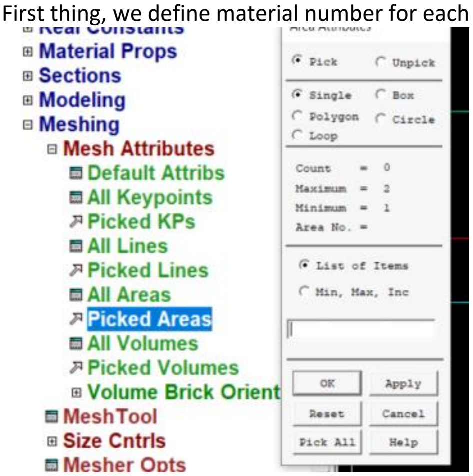
P=0.1

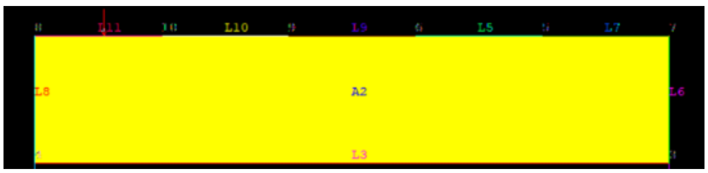


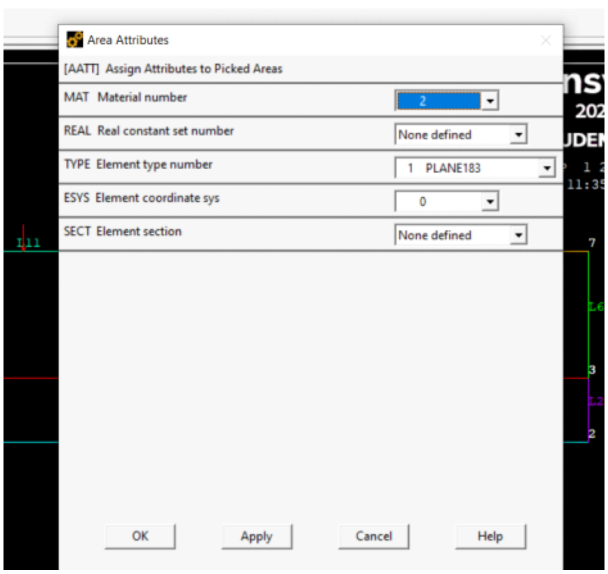


Meshing

First thing, we define material number for each area

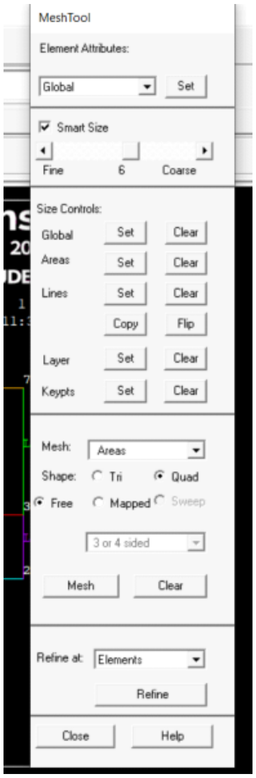




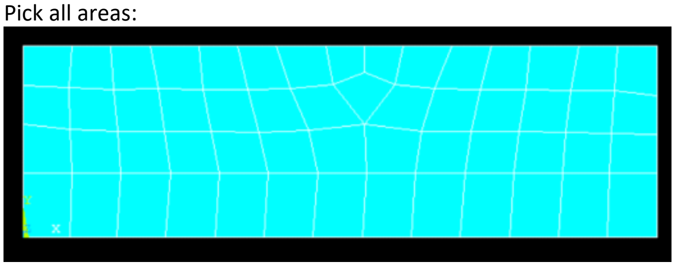


List the areas

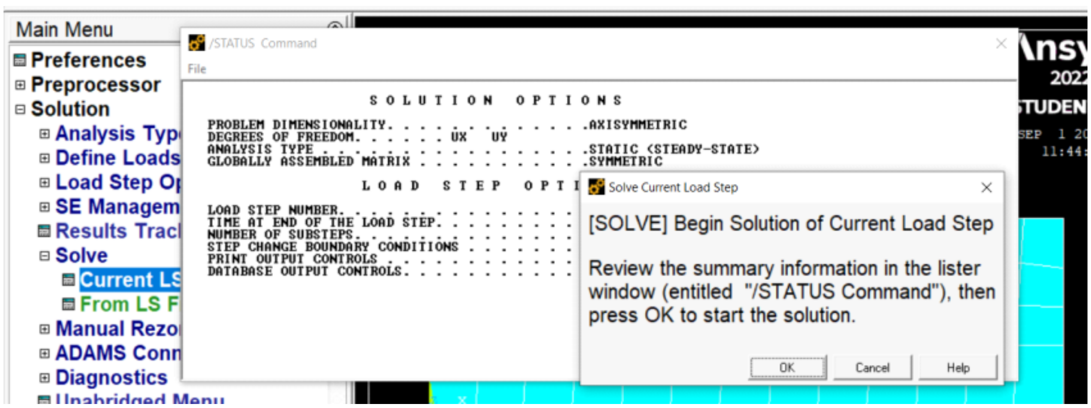
```
LIST ALL SELECTED AREAS.
NUMBER LOOP  LINES              AREA    ELEM SIZE  #NODES #ELEM   MAT  REAL  TYP  ESYS  SECN
   1 1    1    2    3    4  N/A    0.000      0    0    1    0    1    0    0
   2 1    3    6    7    5  N/A    0.000      0    0    2    0    1    0    0
          9   10   11    8
```



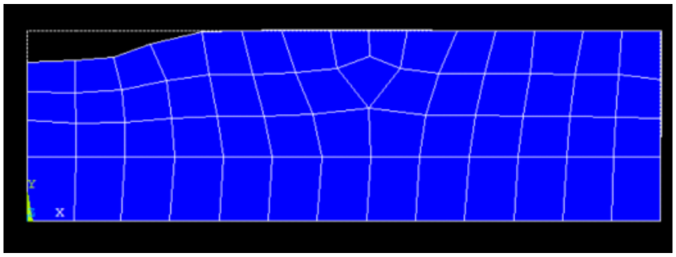
Pick all areas:



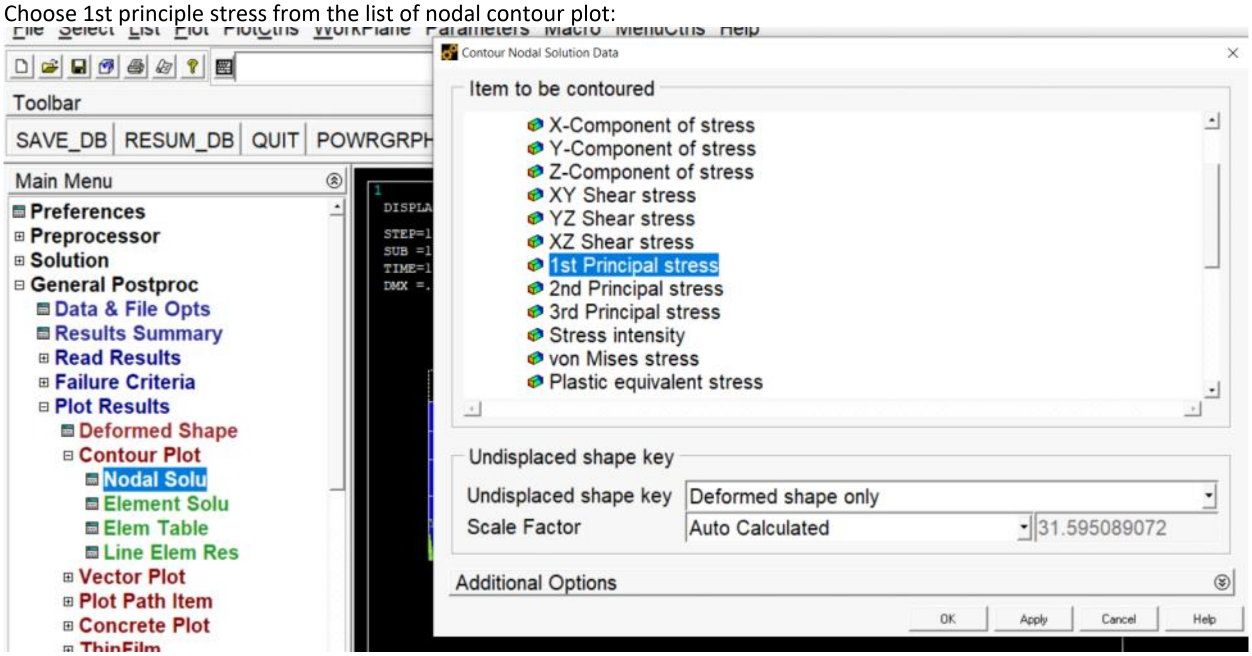
Solution:

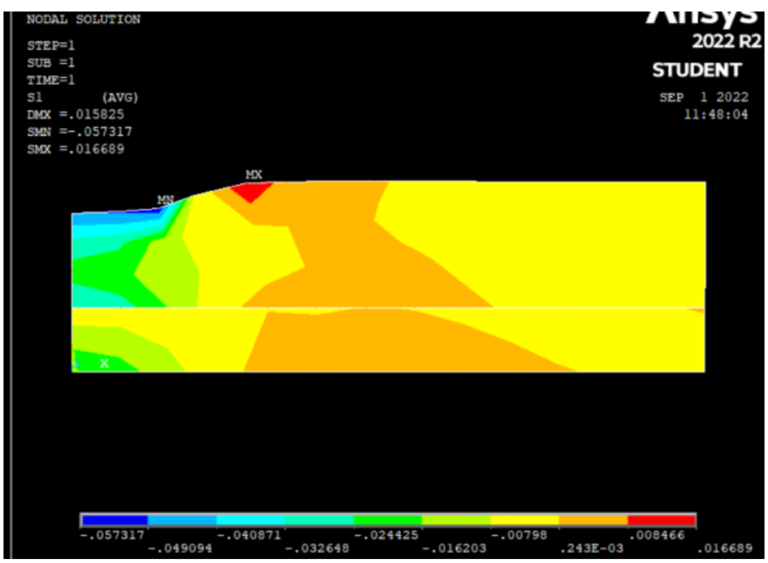


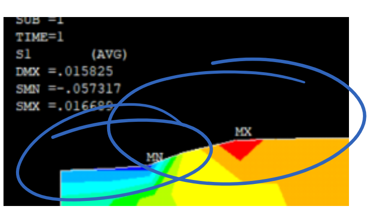
Postprocess:



Contour plots
Choose 1st principle stress from the list of nodal contour plot:

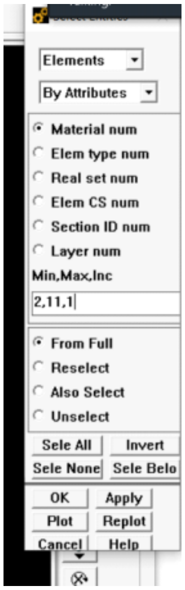




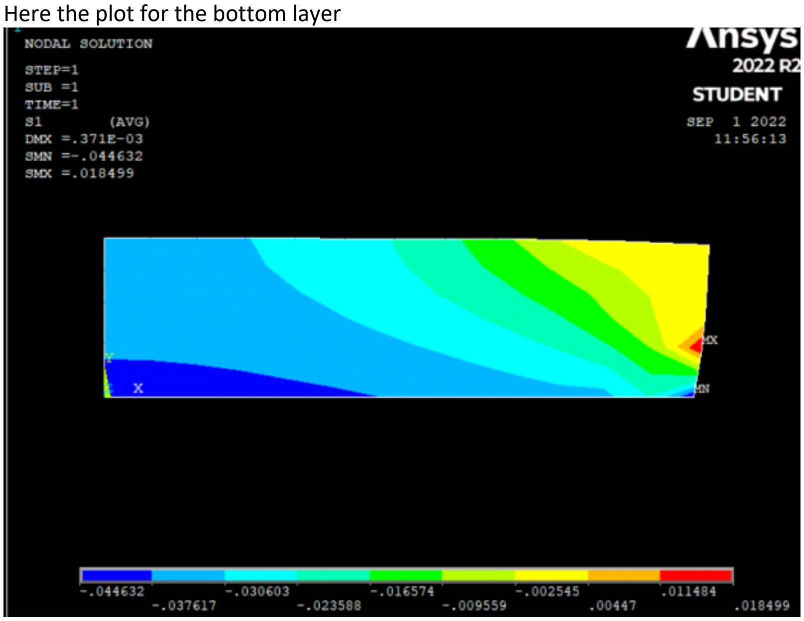


Min and max sigma_1 for the whole domain

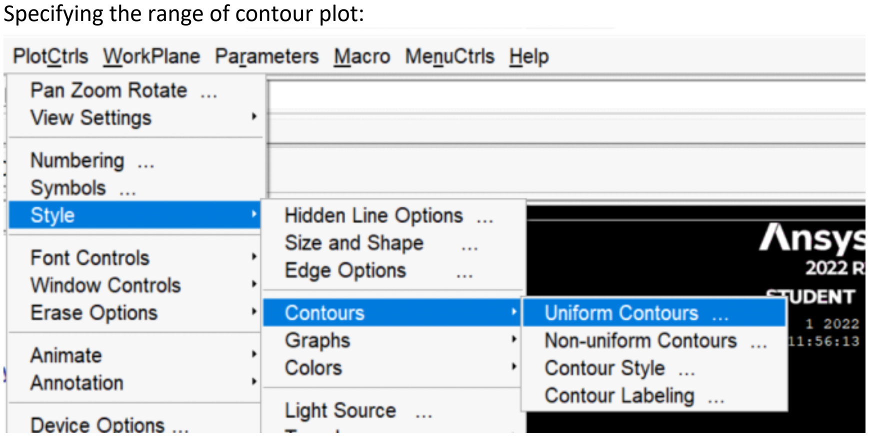
Plotting the results for certain number of layers
Select -> entities ->

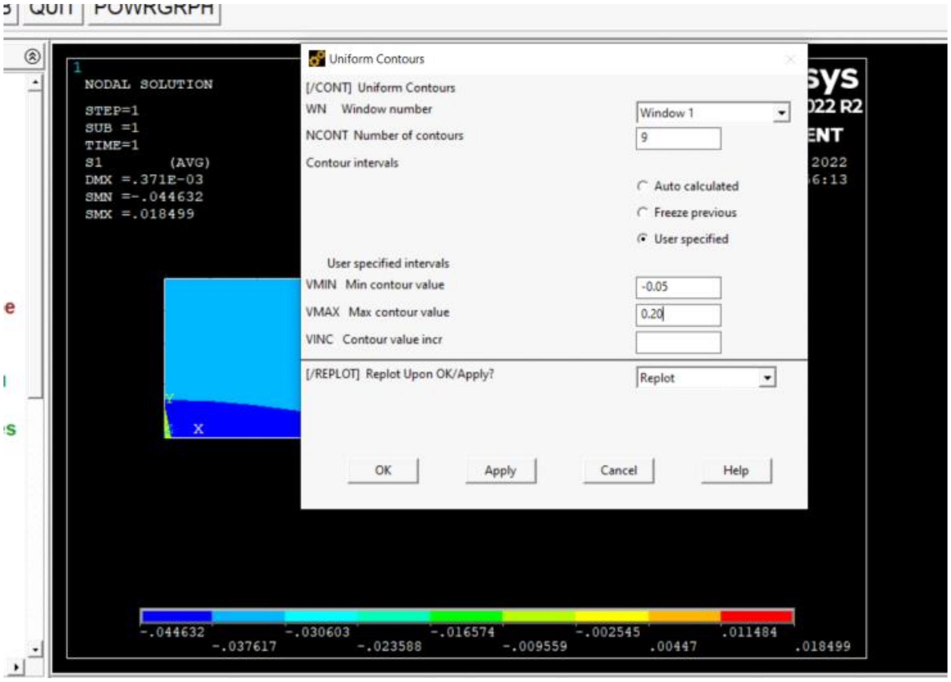


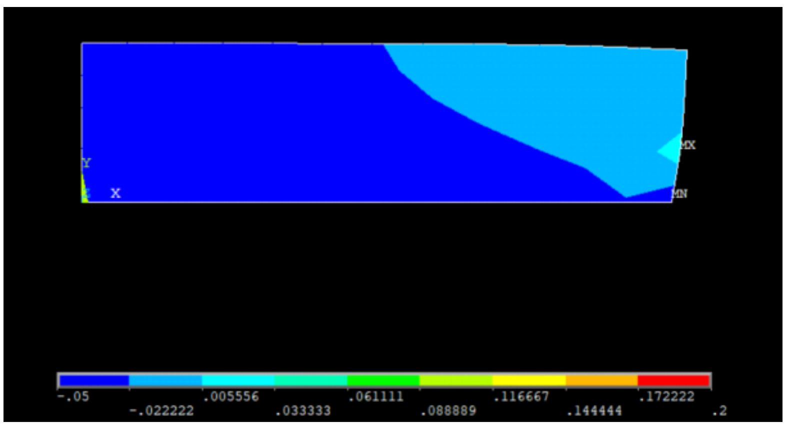
Here the plot for the bottom layer

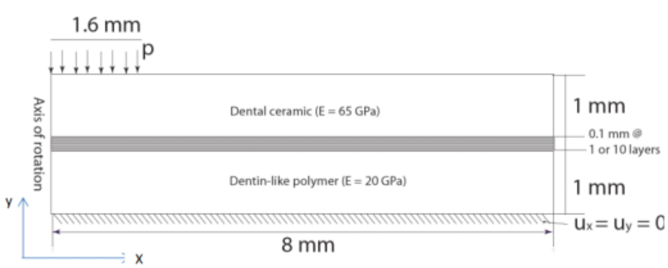


Specifying the range of contour plot:







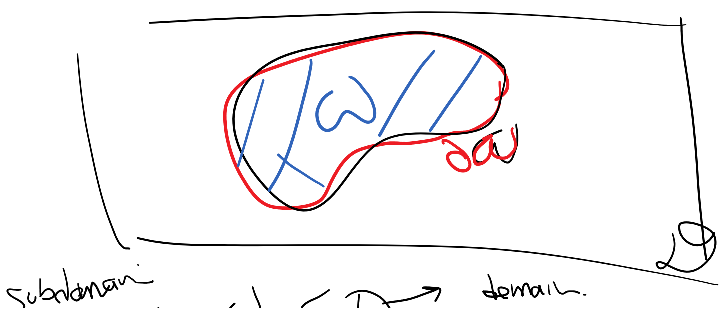


make sure total length is 0.1 mm

FEM Formulation

1. Balance laws

$$\sum F_{on\ \omega} = 0$$



$\sum \vec{F}_{\text{on } \omega} = 0$

subdomain $\leftarrow \omega \subseteq D \rightarrow$ domain

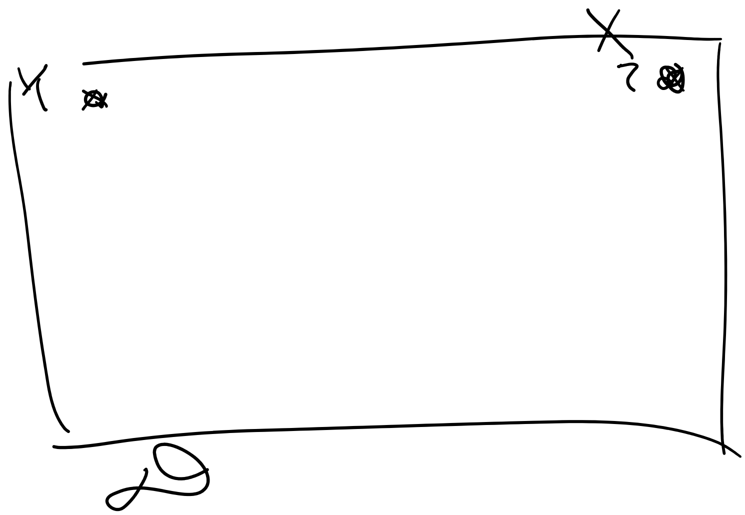
$$\equiv \underbrace{\int_{\partial \omega}}_{\text{boundary of } \omega} \underbrace{\sigma}_{\text{stress tensor}} \cdot n \, dS + \underbrace{\int_{\omega}}_{\text{arbitrary}} \rho \underbrace{b}_{\text{body force}} = 0$$

(2) $\longrightarrow$ we will derive

strong form = (Partial) Differential Equation (PDE)

$$\underbrace{\nabla \cdot}_{\text{divergence theorem}} \sigma + \rho b = 0 \quad \forall x \in \Omega$$

(3)



$$\underbrace{R}_{\substack{\text{residual} \\ \text{"error"}}} = \nabla \cdot \sigma + \rho b$$

Weighted Residual Statement (WRS)

weight function $\longrightarrow$ arbitrary

$$\forall w \quad \int_{\Omega} w \underbrace{(\nabla \cdot \sigma + \rho b)}_{R} = 0 \quad (\star)$$

fixed

$\forall w$

weight functions

domain

R = Residual

4

$$u = \sum_{i=1}^{N} a_i \, \phi_i(x)$$

a_i: unknowns

ϕ_i: shape functions, known → we choose this

$\phi_1 = x$
$\phi_2 = x^2$
$\phi_3 = x^3$

Discretization

∞ unknowns → N unknowns $(a_1, \dots, a_n)$

→ N equations

we'll satisfy (A) for $w_1, w_2, \dots, w_N$

we'll choose these functions

$$\forall i \in \{1, N\} \quad \int_\Omega w_i (\nabla \cdot \sigma + \rho b) = 0$$

1D

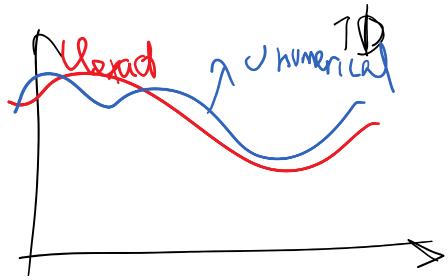
$w_1 = x$
$w_2 = x^2$ example
$w_3 = x^3$
⋮

Find a_i's

$$u = \sum a_i \, \phi_i(x)$$

u: fully known

a_i: is obtained



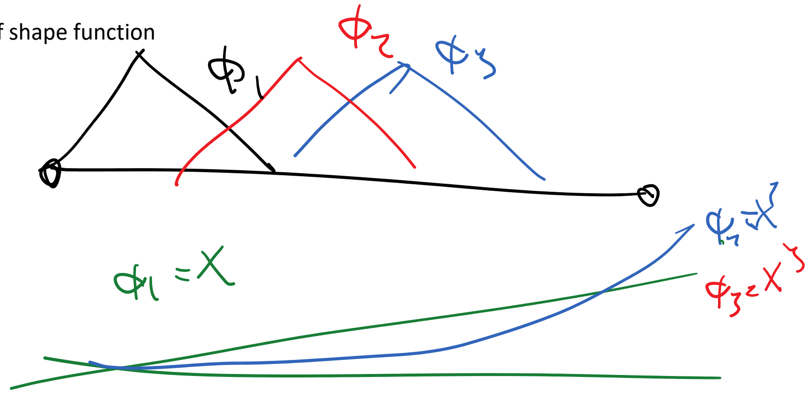
$u \rightarrow \varepsilon \rightarrow \sigma, \dots$

Different methods have different forms of shape function

ϕ_2 ϕ_3

Different methods have different forms of shape function

FEM



Spectral method

or
$\sin x$
$\sin 2x$
$\sin 3x$

Discontinuous Galerkin -> Different basis functions

FEM formulation in detail

1. Balance law

- Why start with a balance law?
  - They are the actual physics laws.
  - They contain more *information* than their corresponding PDEs.
  - Larger solution space than the PDEs.
- Can we directly start the FE formulation from a PDE?
  - Yes, FE formulation starts from a differential equation.
  - A PDE may not be derived from a balance law.

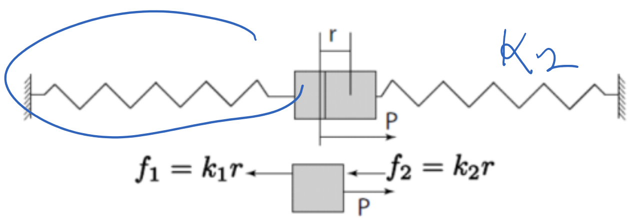
Balance law

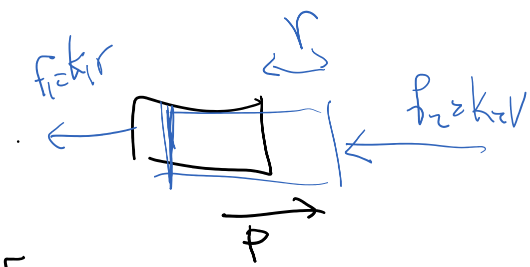
Diff. Eqn

Jump condition

Balance of mass, force (linear momentum), energy, …

Example of balance of force in discrete setting:





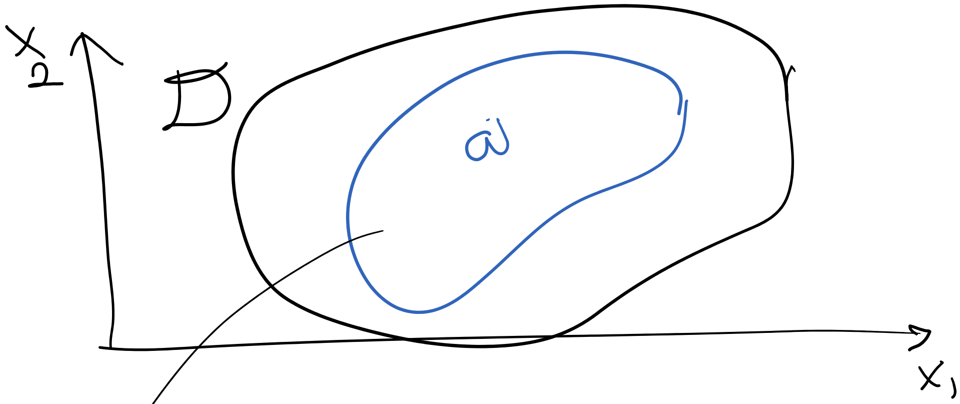
$-f_1 - f_2 + P = 0$

$\rightarrow (k_1 + k_2) r = P \rightarrow r = \dfrac{P}{k_1 + k_2}$

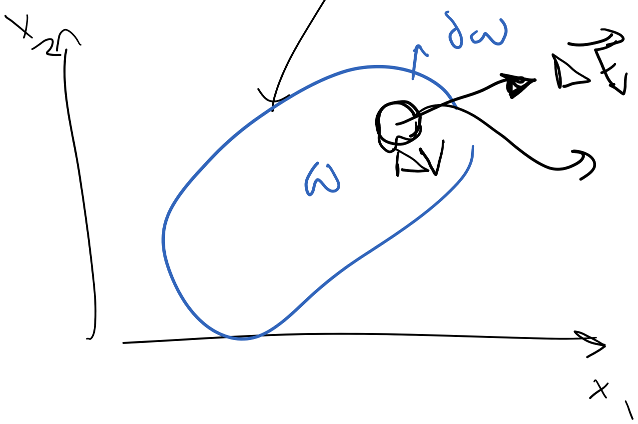
Continuum:

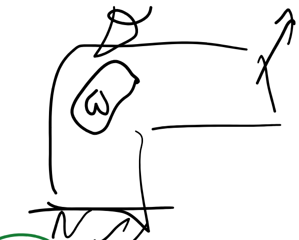
Balance of forces

$$\sum F = 0$$



subdomain $\omega \subseteq D$ domain





$$\frac{\Delta \vec{F}_v}{\Delta V} = \frac{\Delta \vec{F}_v}{\Delta m} \cdot \frac{\Delta m}{\Delta V} \rightarrow \rho \text{ - density}$$

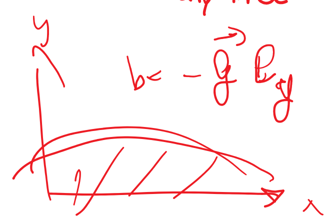
force per volume

incremental mass

$$\lim_{\Delta m \to 0} \frac{\Delta \vec{F}_v}{\Delta m} = \frac{d\vec{F}_v}{dm} = \vec{b}$$

body force

Example $b = -g\,\vec{e}_y$



Types of forces: 1. Volumetric force

$$\vec{F}_v = \sum \Delta \vec{F}_v = \sum \frac{\Delta \vec{F}_v}{\Delta V} \Delta V$$

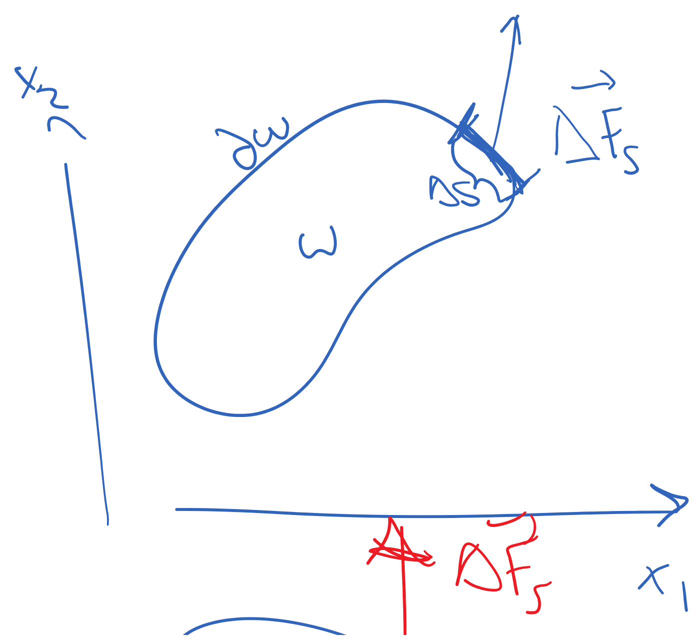
$$= \int_\omega (\rho b)\, dV$$

Surface force

$$\vec{F}_s = \sum \Delta \vec{F}_s$$

$$= \sum \frac{\Delta \vec{F}_s}{\Delta S} \Delta S$$

let $\Delta S \to 0$



let $\Delta S \to 0$

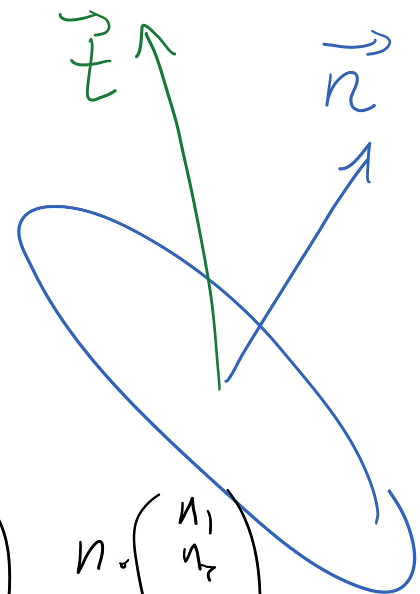
$$= \int_{\partial\omega} \left[\lim_{\Delta S \to 0} \left(\frac{\Delta F_s}{\Delta S} \right) \right] dS$$

intensity of force

"force per area" $\lim_{\Delta S \to 0} \frac{\Delta F_s}{\Delta S} = \vec{t}$

Pa
MPa
psi

$$F_s = \int_{\partial\omega} \vec{t} \, dS$$



$$\vec{t} = \sigma \cdot \vec{n}$$

stress tensor

$$t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \quad n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & - & \\ & & \sigma_{33} \end{pmatrix}$$

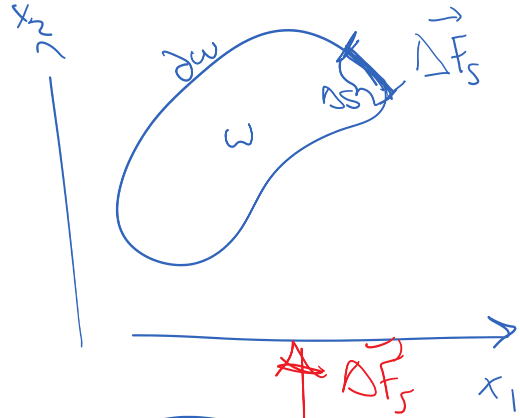
stress tensor

2022/09/06

Tuesday, September 6, 2022 11:16 AM

From last time

Surface force

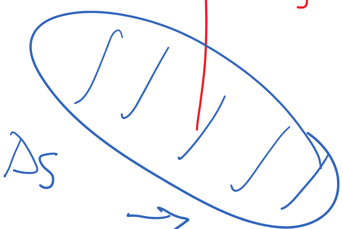


$$\vec F_s = \sum \Delta \vec F_s$$

$$= \sum \frac{\Delta \vec F_s}{\Delta S}\,\Delta S$$

let $\Delta S \to 0$

$$= \int_{\partial\omega}\left[\lim_{\Delta S\to 0}\left(\frac{\Delta F_s}{\Delta S}\right)\right] dS$$



intensity of force

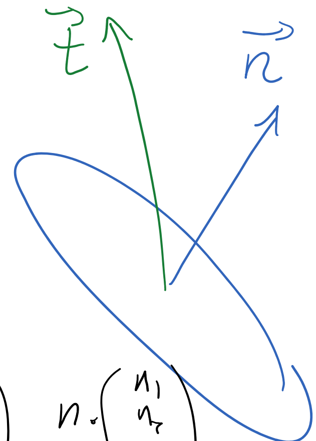
"force per area" $\lim_{\Delta S\to 0}\frac{\Delta F_s}{\Delta S} = \vec t$

Pa

MPa

psi

$$\boxed{F_s = \int_{\partial\omega} \vec t \; dS}$$



$$\vec t = \sigma \cdot \vec n$$

stress tensor

$t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$ $\quad n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & — & \\ & & \sigma_{33} \end{pmatrix}$$

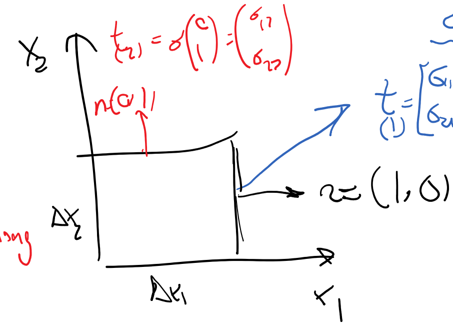
stress tensor

2D

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

traction for a surface with normal along x_1 axis

traction, normal along x_2



$$t_{(1)} = \overbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}}^{\sigma} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{n} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \end{bmatrix}$$

$$t_1 = \begin{bmatrix} \sigma_{11} & \sigma_{12} \end{bmatrix}\begin{bmatrix} n_1 \end{bmatrix}$$

traction

for a surface with normal along x_1 axis

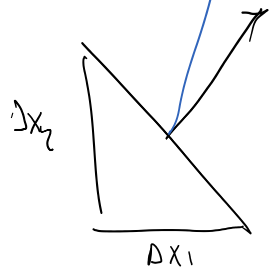
Δx_1 $\quad x_1$

$$t = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

$\sigma = \sigma^T$, $\sigma_{12} = \sigma_{21}$ (Balance of angular momentum

$n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$

for arbitrary direction



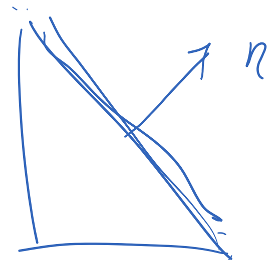
force per area (traction)

$t = \sigma \cdot n$

σ: stress tensor

n: normal vector

heat conduction



$$q_n = q \; n$$

q_n: energy flux per surface area for normal n

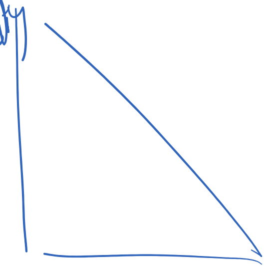
$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = q_1 n_1 + q_2 n_2$$

In general

$$(f_x)_n = f_x \cdot n$$

spatial flux density

f_x: always one tensor order higher



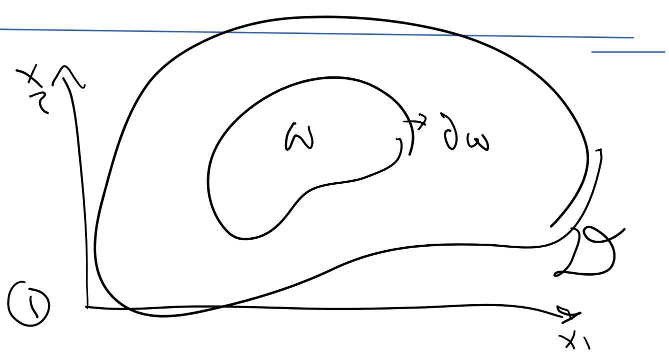
Let's continue with balance of forces

$$\Sigma F = F_v + F_s$$

$$\forall \omega \quad = \int_\omega \rho b \, dV + \int_{\partial\omega} \sigma \cdot n \, dS = 0 \quad (1)$$

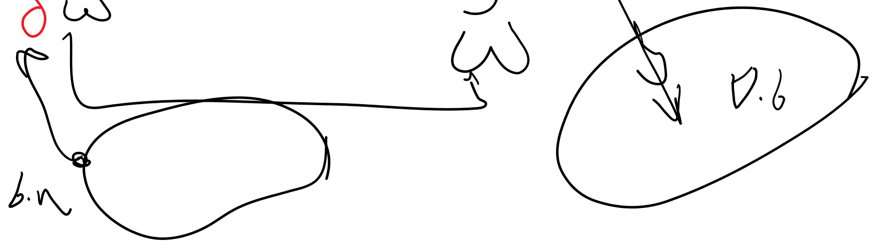
$\int_\omega$: integral over interior of ω

$\partial\omega$: boundary of ω



Divergence / Gauss theorem

$$\int_{\partial\omega} \sigma\cdot n \, dS = \int_{\omega} \nabla\cdot\sigma \, dV \qquad (2)$$



$$\Sigma F = \int_{\omega} \rho b \, dV + \int_{\omega} \nabla\cdot\sigma \, dV = 0$$

$$\forall \omega \subseteq D \int_{\omega} (\underbrace{\nabla\cdot\sigma + \rho b}_{\text{integrand}}) \, dV = 0$$

Integral $= 0$

$$\forall \omega \int (\text{integrand}) \, dV = 0$$

integrand $= 0$

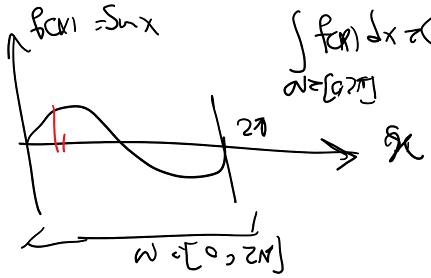
PDE / strong form

$$\boxed{\nabla\cdot\sigma + \rho b = 0}$$

Example 1D

$f(x) = \sin x$

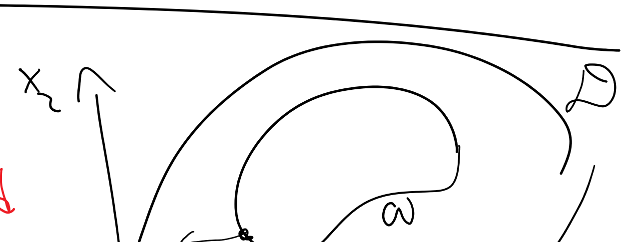
$\int_{\omega = [0, 2\pi]} f(x) \, dx = 0$ But $f(x) \neq 0$



General Balance law

$\forall \omega \subseteq D$

f : quantity to be balanced



$\forall\ \omega \supset \mathcal{D}$

f_t : quantity to be balanced
density

$$F_t = \int_\omega f_t \, dV$$

$F_t \to$ energy, $f_t \to$ volumetric energy density

source term, spatial flux density

$$\underbrace{\frac{d}{dt} \underbrace{\int f_t \, dV}_{\text{total } F_t}}_{\text{rate of total } F_t} = \int_\omega S \, dV - \int_{\partial\omega} f_x \cdot n \, dS$$

example heat conduction: $C_v T$

$$\frac{d}{dt} \int e \, dV = \int_\omega Q \, dV - \int_{\partial\omega} q \cdot n \, dS$$

$$\frac{d}{dt} \int_\omega f_t \, dV = \int_\omega S \, dV - \int_{\partial\omega} f_x \cdot n \, dS$$

$$\boxed{\therefore \int_\omega \frac{d}{dt} f_t \, dV = \int_\omega S \, dV - \int_\omega \nabla \cdot f_x \, dV}$$

(3a) Dynamic Balance law for $F_t = \int_\omega f_t \, dV$

First dynamic: $\forall \omega \quad \int_\omega \left(\frac{d}{dt} f_t + \nabla \cdot f_x - S \right) dV = 0$

$$\left| \frac{d}{dt} f_t + \nabla \cdot f_x - S = 0 \right| \quad (3b)$$

Dynamic PDE

$$\boxed{\frac{d}{dt} f_t + \nabla\cdot f_x - S = 0} \quad (3b)$$

Dynamic PDE

heat eqn

$$e = c_v T \qquad q \qquad Q$$

$$\boxed{\frac{d}{dt}(c_v T) + \nabla\cdot q - Q = 0} \quad (3c)$$

- Example 1 heat conduction

$$f_t = p = \rho\vec{v} \qquad f_x = -\sigma \qquad S = \rho b$$

Example 2 solid mechanics

$$\frac{d}{dt}\rho\vec{v} + \nabla\cdot(-\sigma) - \rho b = 0$$

Equation of motion

Static / steady state problems

$$\underbrace{\cancel{\frac{d}{dt}\int_\omega f_t \, dV}}_{=0} = -\int_{\partial\omega} f_x\cdot n \, dS + \int_\omega S \, dV$$

no change in time

$$\boxed{\int_{\partial\omega} f_x\cdot n \, dS = \int_\omega S \, dV} \quad (4)$$

balance law for Statics

$$\int_\omega \nabla\cdot f_x = \int_\omega S \, dV \rightarrow \boxed{\nabla\cdot f_x - S = 0}$$

Differential equation

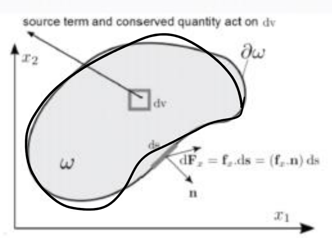
Summary slide

General form of balance laws

For a general conservation law let:

- $\mathbf{f}_t$: conserved quantity = temporal flux
- $\mathbf{f}_x$: total outward spatial flux
- $\mathbf{r}$: source term

then the balance law for dynamics reads:



then the balance law for dynamics reads:

$$\forall\omega \subset \mathcal{D} \wedge \forall t : \int_\omega \mathbf{r}\,\mathrm{dv} - \int_{\partial\omega} \mathbf{f}_x.\mathrm{d}\mathbf{s} = \int_\omega \mathbf{r}\,\mathrm{dv} - \int_{\partial\omega} (\mathbf{f}_x.\mathbf{n})\,\mathrm{ds} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\omega \mathbf{f}_t\,\mathrm{dv} \qquad (13)$$

For static case the RHS is zero (*i.e.*, the quantity $\int_\omega \mathbf{f}_t\,\mathrm{dv}$ remains constant). The static balance law reads:

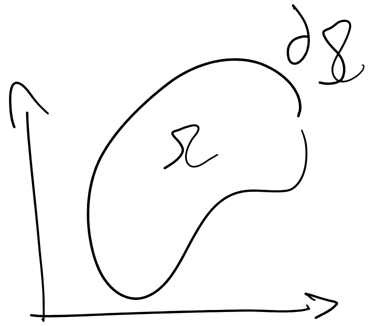
$$\forall\omega \subset \mathcal{D} : \int_\omega \mathbf{r}\,\mathrm{dv} - \int_{\partial\omega} \mathbf{f}_x.\mathrm{d}\mathbf{s} = \int_\omega \mathbf{r}\,\mathrm{dv} - \int_{\partial\omega} (\mathbf{f}_x.\mathbf{n})\,\mathrm{ds} = \mathbf{0} \qquad (14)$$

These can be directly compared to $\mathbf{F} = \mathrm{d}\mathbf{P}/\mathrm{d}t$ and $\mathbf{F} = 0$ in previous discrete examples.

13 / 456

For any balance law we have

$$\int_{\partial\Omega} \underbrace{\bar{F}}_{\text{spatial (spacetime) flux}} \cdot \underbrace{n}_{\text{space (spacetime normal)}} \, dS = \int_\Omega \underbrace{r}_{\text{source term}} \, dV$$



FYI

$$\underbrace{\nabla}_{\text{space (space time divergence)}} \cdot \bar{F} = r$$

You can skip slides 6-14, except slide 13, the static part of it

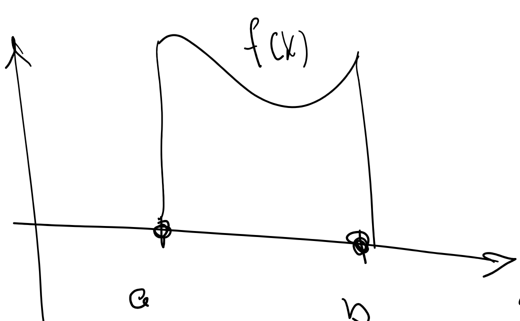
We'll discuss divergence theorem and localization theorem

1D

Divergence theorem

$$\int_a^b F(x)\,dx = f(b) - f(a)$$

$$\left(f = \int F \quad F = f' \right)$$



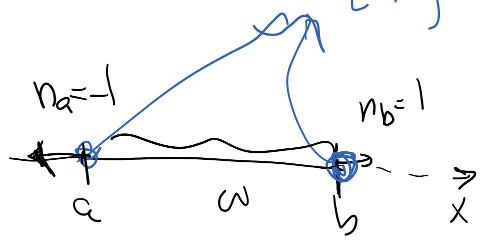
$\omega = [a,b]$

$\partial\omega = \{a,b\}$

$$\left(f = \int F \Rightarrow F = f' \right) \qquad \omega = [a,b] \qquad \partial\omega = \{a,b\}$$

that is

$$\int_{\omega=[a,b]} f'(x)\,dx = f(b)\cdot 1 + f(a)\cdot(-1)$$



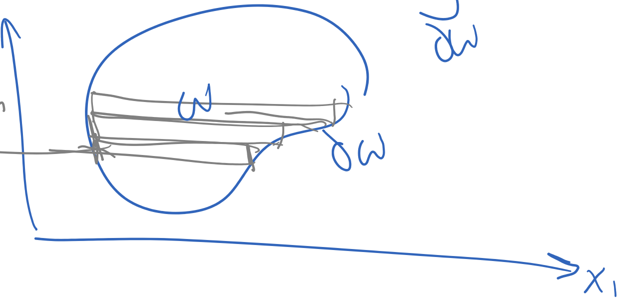
3 (★) $\int_\omega f'(x)\,dx = f(a)\,n_a + f(b)\,n_b = \int_{\partial\omega} f(x)\cdot n(x)\,ds$ like

$\nabla\cdot f$

Compare this with divergence theorem

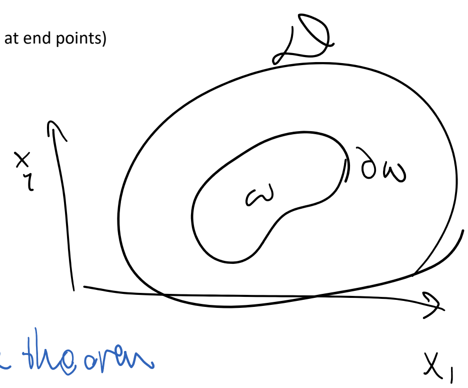
$$\int_\omega \nabla\cdot f(x)\,dV = \int_{\partial\omega} f(x)\cdot n\,dS$$

for any d there (★) applies



Divergence theorem is multi-dimension version of fundamental theorem of calculus (turning line integral to values at end points)

Which one is more general



Motivation:

$$\sum F = \int_\omega \rho b\,dV + \int_{\partial\omega} \sigma\cdot n\,dS$$

using divergence theorem

$$\int_\omega \nabla\cdot\sigma\,dV$$

we have derivative :(

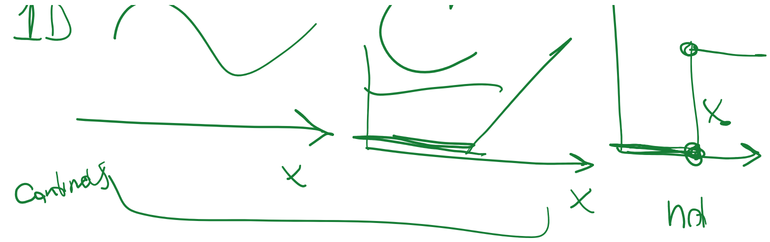
For the balance law, all we need is that the spatial flux (sigma here) to be integrable over the boundary of omega.

However, for getting to DE, ie. the blue term divergence of spatial flux should exist and in fact should be continuous.

C^0 function is continuous 1D

C^0 function is continuous f:

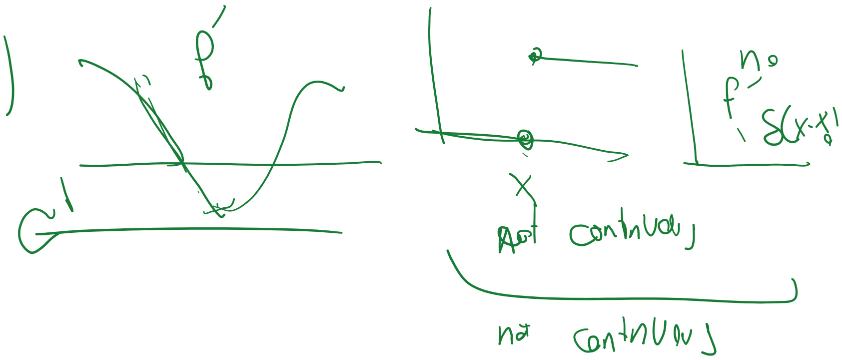
1D



C^1 if f & f' are continuous

C^2 if f, f', f'' are continuous

f'



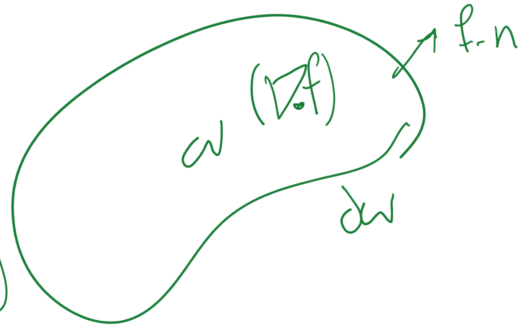
C^n means $f, f', \ldots, f^{(n)}$ derivates exist and all are continuous

1D

2D $\quad f, \ \dfrac{\partial f}{\partial x_i}, \ \dfrac{\partial^2 f}{\partial x_i \partial x_j}, \ \ldots\ldots$ n partial derivatives

Divergence theorem

$$\int_{\partial\omega} f \cdot n \, dS = \int_{\omega} (\nabla \cdot f) \, dV$$



Divergence theorem requires $\nabla \cdot f$ to exist & to be continuous $\boxed{f \in C^1(D)}$

this is restrictive. For balance law, f should be integrable this is restrictive. For balance law, f should be integrable

2022/09/08

Thursday, September 8, 2022 11:23 AM

Refer to slides 19-20 for divergence theorem

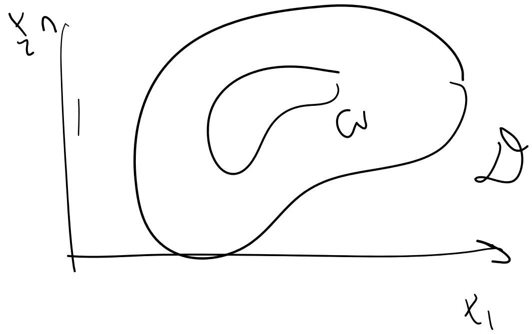
Second point from last time

Localization theorem:

$\forall \omega \subseteq D \quad \int_\omega g(\vec{x})\,dV = 0$

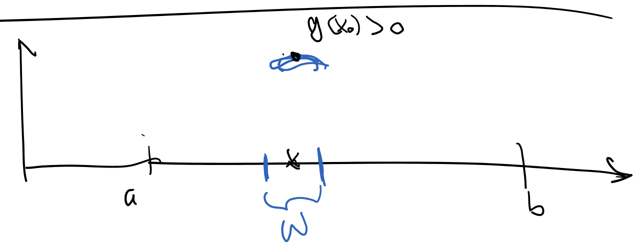
& g is continuous

$\Longrightarrow \forall \vec{x} \in D \quad g(\vec{x}) = 0$



Proof: 1D

$D = (a, b)$



$\exists x_0 \in D$ such that $g(x_0) > 0$

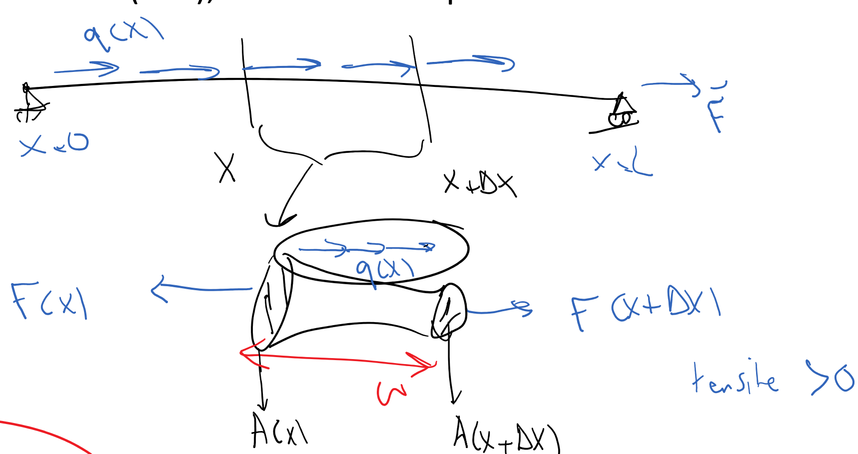
there is a neighborhood ω of $x_0 \Rightarrow g(x) > 0$ from continuity

$\Longrightarrow \int_\omega \underbrace{g(x)}_{>0}\,dV > 0$ negates $\forall \omega \subseteq D \int_\omega g(x)\,dV = 0$

$\Longrightarrow$ false

$\forall x \in D \quad g(x) = 0$

1D bar problem: Balance law (balance of forces), Differential Equation, Boundary Conditions (BCs), constitutive equation



Balance of forces

"Balance law"

$F(x + \Delta x) - F(x) + q(x)\,\Delta x = 0$

$A(x)$ $A(x+\Delta x)$

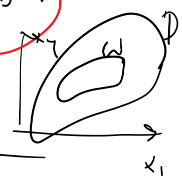
$F(x+\Delta x) - F(x) + q(x)\Delta x = 0$ Δx small

divide by Δx

comprision with 3D

$\Sigma F = 0$

Balance law $\int_{\partial\omega} \sigma \cdot n \, ds + \int_{\omega} \rho b \, dV = 0$



let $\Delta x \to 0$ $\dfrac{F(x+\Delta x) - F(x)}{\Delta x} = q(x)$

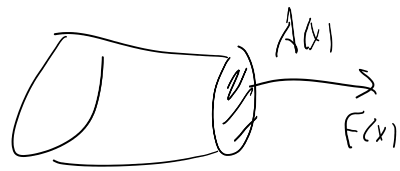
DE $F'(x) = \dfrac{dF(x)}{dx} + q(x) = 0$

PDE $\nabla \cdot \sigma + \rho b = 0$

$\nabla\cdot\sigma \to F(x)$, $\rho b \to q(x)$

(1D) DE $\dfrac{dF(x)}{dx} + q(x) = 0$

Closing the system



stress $\sigma(x) = \dfrac{F(x)}{A(x)}$

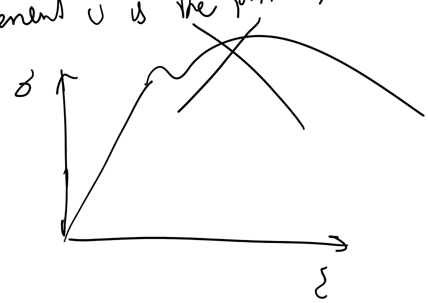
① $\boxed{\dfrac{d\, A(x)\sigma(x)}{dx} + q(x) = 0}$ Strong form 1D bar problem

Some BC's are in terms of u & displacement u is the primary field

we need to express σ in terms of u:

② $\boxed{\sigma(x) = E(x)\,\varepsilon(x)}$

$E(x)$: elastic modulus, $\varepsilon(x)$: strain



Constitutive eqns are empirical eqns

③ $\boxed{\varepsilon(x) = \dfrac{du(x)}{dx}}$ first class Compatibility eqn

①, ②, ③ $\longrightarrow$

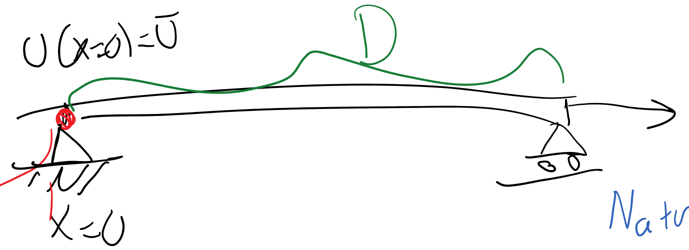
DE

④ $\dfrac{d\, A(x)\sigma(x)}{dx} + q(x) = \dfrac{d\left(E(x)A(x)\frac{du}{dx}\right)}{dx} + q(x) = 0$

$(A(x)\sigma(x))' + q(x) = (EA(x)u'(x))' + q(x) = 0$

$AE(x) = A(x)E(x)$

$U(x=0)=\bar{U}$ $\qquad D \qquad$ $u(x=L)=\bar{F}$



∂D_u = essential boundary of D

Essential BC
" specify the primary field u

Here $\partial D_u = \{0\}$

Natural boundary
"specify the spatial flux

$\partial D = \{0, L\}$

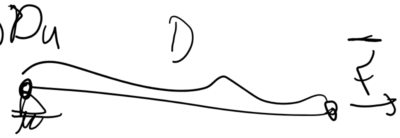
∂D_f = natural boundary

$\partial D_f = \{L\}$

$\partial D_u \cup \partial D_f = \partial D$

$\partial D_u \cap \partial D_f = \emptyset \longrightarrow$ at one point we can specify only 1 BC

Boundary value Problem (BVP)

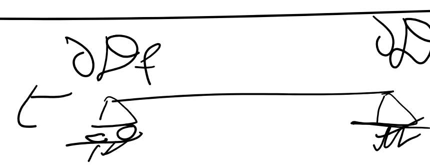


DE $\forall x \in D \quad (EAu')' + q = 0$

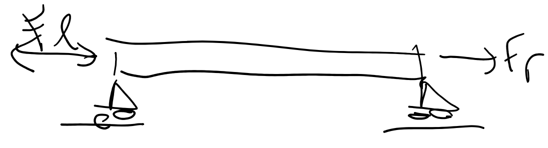
BCs: $\begin{cases} u(x=0) = \bar{u} & @\ \partial D_u = \{0\} \\ F(x=L) = AEu'(x=L) = \bar{F} & @\ \partial D_f = \{L\} \end{cases}$

(2)

Other BCs in 1D



not valid for statics

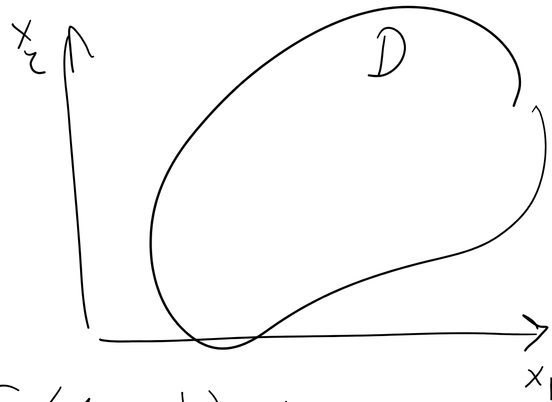


$\partial D_u = \emptyset$

$\partial D_f = \{0, L\}$

2D Balance law

$$\int_{\partial \omega} \sigma \cdot n \, dS + \int_{\omega} \rho b \, dV = 0$$

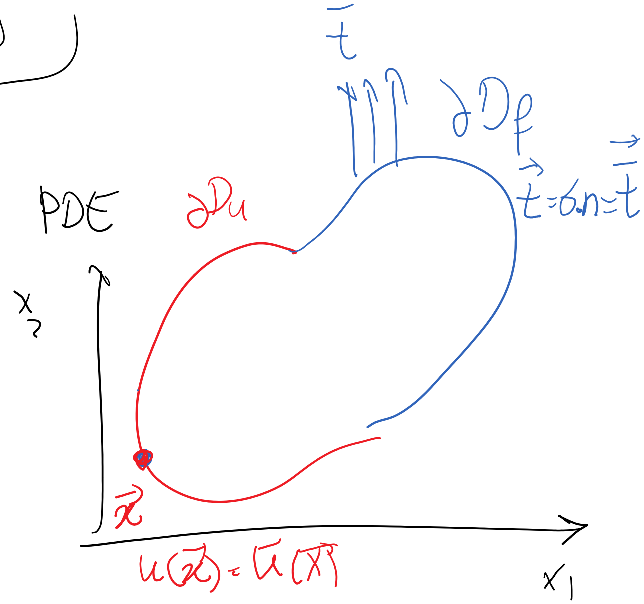


divergence theorem

$$\int_{\omega} \nabla \cdot \sigma \, dV + \int \rho b \, dV = 0 \rightarrow \int_{\omega} (\nabla \cdot \sigma + \rho b) \, dV = 0$$

localization problem

(3) $\boxed{\nabla \cdot \sigma + \rho b = 0}$ Strong form PDE



Can I solve this? No.

we need to close the system

I'll do this in 2D

(3) $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$

$\sigma_{21} = \sigma_{12}$ sym

$\nabla \cdot \sigma = \begin{bmatrix} \sigma_{11,1} + \sigma_{12,2} \\ \sigma_{12,1} + \sigma_{22,2} \end{bmatrix}$ $\quad \rho b = \begin{bmatrix} \rho b_1 \\ \rho b_2 \end{bmatrix}$

$$\nabla \cdot \sigma + \rho b = 0 \longrightarrow \boxed{\begin{array}{ll} \text{Eq1)} & \sigma_{11,1} + \sigma_{12,2} + \rho b_1 = 0 \\ \text{Eq2)} & \sigma_{12,1} + \sigma_{22,2} + \rho b_2 = 0 \end{array}}$$

2 eqn's 3 unknowns $\sigma_{11}, \sigma_{12}, \sigma_{22}$:(

σ is a linear function of strain

2D & 3D $\qquad \sigma$ is a linear function of strain

2D

$$\begin{array}{l} \text{Eq 3} \\ \text{Eq 4} \\ \text{Eq 5} \end{array} \underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}}_{\text{Voigt stress notation}} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}}_{\text{Voigt stiffness}} \underbrace{\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}}_{\text{Voigt strain notation}}$$

$2\epsilon_{12} \to \gamma_{12}$ ↗ Engineering shear strain

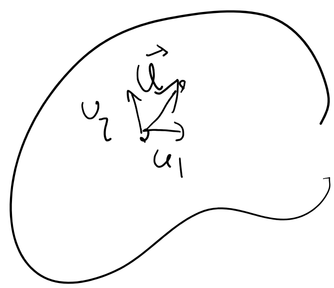
2D version of $\sigma = E\varepsilon$ Constitutive eqn

I added 3 more equations

I add 3 unknowns $(\epsilon_{11}, \epsilon_{22}, \epsilon_{12})$

$\varepsilon \xrightarrow[\text{expressed in terms of}]{} U$ $\qquad$ Compatibility equation

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \qquad \bar{\bar{\varepsilon}} = \frac{\nabla u + (\nabla u)^T}{2}$$



$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{bmatrix} + \begin{bmatrix} u_{1,1} & u_{2,1} \\ u_{1,2} & u_{2,2} \end{bmatrix} \right)$$

independent strain components

$$\epsilon_{12} \leftarrow \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} = \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) \\ \text{same} & u_{2,2} \end{bmatrix}$$

$$\boxed{\begin{array}{ll} \text{Eq 6} & \epsilon_{11} = u_{1,1} \\ \text{Eq 7} & \epsilon_{22} = u_{2,2} \\ \text{Eq 8} & \epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) \end{array}}$$

Added 3 more eqns

All unknowns

$\sigma = [\sigma_{11} \; \sigma_{12}]$

Added 3 more eqns

(2) , unknowns (u_1, u_2)

$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$

$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix}$

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Summary

$$\nabla \cdot \sigma + \rho b = 0 \quad \text{PDE} \qquad i$$

Constitutive equ

$$\sigma = \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}}_{\text{2 indices}} = \mathbb{C} \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix} \qquad ii$$

4th Elasticity tensor — 2 indices

2+2 = 4 indices

Compatibility

$$\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T) \qquad iii$$

i, ii $\quad \nabla \cdot \sigma + \rho b = \nabla \cdot (\mathbb{C}\,\varepsilon) + \rho b = 0$

iii $\quad \varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$

$$\boxed{\nabla \cdot \left(\mathbb{C}\left(\frac{1}{2}\left[\nabla u + \nabla u^T\right]\right)\right) + \rho b = 0}$$

2 equs

2 unknowns $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

2nd order differential equation

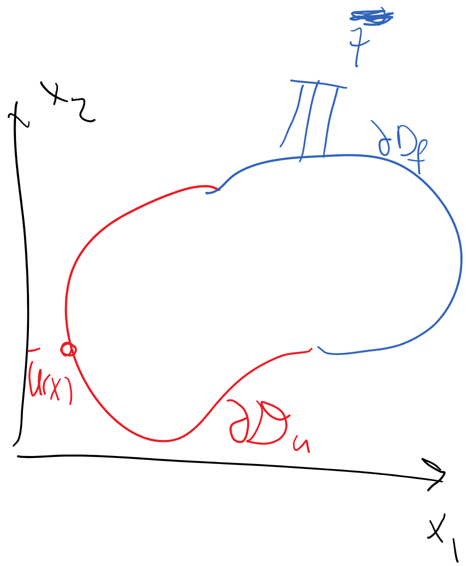
2D version of

$$(EAu')' + q = 0$$

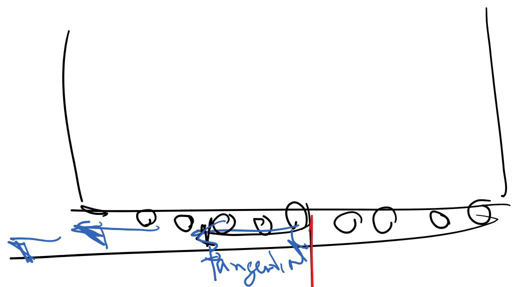
BVP for 2D elasticity

1) PDE $\forall x \in D$: $\nabla\cdot\sigma + \rho b = \nabla\cdot\mathcal{C}(\nabla u) + \rho b = 0$

2) BCs
- Essential BCs $\forall x \in \partial D_u$: $u(x) = \bar{u}(x)$
- Natural BC $\forall x \in \partial D_f$: $t = \sigma\cdot n = \bar{t}$



Note:



$t_{tangential} = \bar{t}_{tan}$ $\qquad$ $u_n = 0$ or sth else

Closing the system of equations (Statics)

Strong form (23) of balance of linear momentum for statics is:

$$\nabla.(-\sigma) - \rho\mathbf{b} = \mathbf{0}, \quad \Rightarrow \nabla.\sigma + \rho\mathbf{b} = \mathbf{0} \quad \Rightarrow \quad \sigma_{ij,j} + \rho b_i = 0 \qquad (24)$$

where $\mathbf{f} = -\sigma$, $\mathbf{r} = \rho\mathbf{b}$, and $\nabla(.) = \frac{\partial(.)}{\partial x_1} + \frac{\partial(.)}{\partial x_2} + \frac{\partial(.)}{\partial x_3}$.

Type	Equation	n_e	new unknowns	n_u	$N_e - N_u$
Balance law	$\sigma_{ij,j} + \rho b_i = 0$	3	$\sigma_{ij} = \sigma_{ji}$, $i, j \in \{1, 2, 3\}$	6	3
Constitutive equation	$\sigma_{ij} = C_{ijkl}E_{kl}$	6	$E_{kl} = E_{lk}$	6	3
kinematic compatibility	$E_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$	6	u_k	3	0

n_e = number of new equations $\quad$ n_u = number of new unknowns
N_e = total number of equations $\quad$ N_u = total number of unknowns

- We need other equations (constitutive equations and kinematic compatibility equations) to balance the number of unknowns and equations.

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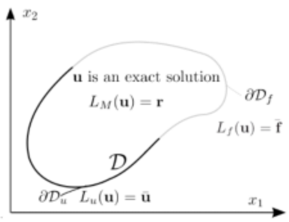
Slide 26, FYI

Different types of spatial boundary conditions (BC)

- $L_M(\mathbf{u}) = \mathbf{r}$ is the strong form after incorporating the "constitutive" and "compatibility" conditions.
- $L_u(\mathbf{u}) = \bar{\mathbf{u}}$: Dirichlet BC, order M_u.
- $L_f(\mathbf{u}) = \bar{\mathbf{f}}$: Neumann BC, order M_f.
- $\mathbf{u}$ is a primary field, (*e.g.*, displacement for solid mechanics; temperature for heat conduction)
- M is typically even (*e.g.*, $M = 2m$)



$\partial\mathcal{D}_u$	$\partial\mathcal{D}_f$
Dirichlet BC	Neumann BC
Essential BC (typically strongly enforced)	Natural BC ("naturally" derived from balance law fluxes)
"primary" or "kinematic" BC	"flux" or "force" BC

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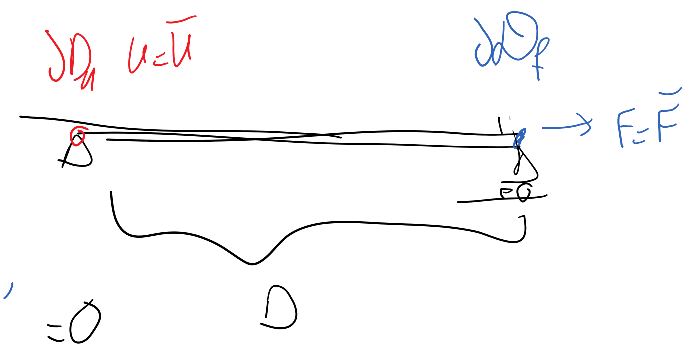
Next step

1D

DE: $\forall x \in \mathcal{D}$ $\quad R_i = \sigma' + q = (EAu')' + q = 0$

BC: $\partial\mathcal{D}_u \quad R_u = \bar{u} - u = 0$

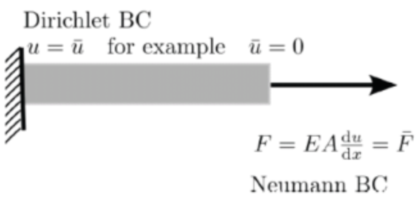
$\partial\mathcal{D}_f \quad R_f = \bar{F} - F = \bar{F} - AEu' = 0$



2022/09/13

Tuesday, September 13, 2022 11:09 AM

Bar problem



$(EAu')' - q = 0$

Operator	Sample	1D bar	operator order
$L_{2m}(\mathbf{u}) = \mathbf{r}$	$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = -q$	$L_{2m} = \frac{d}{dx}\left(EA\frac{d(.)}{dx}\right)$	$m = 1 (M = 2)$
$L_u(\mathbf{u}) = \bar{\mathbf{u}}$	$u = \bar{u}$	$L_u = (.)$	$M_u = 0$
$L_f(\mathbf{u}) = \bar{\mathbf{f}}$	$EA\frac{du}{dx} = \bar{f}$	$L_f = EA\frac{d(.)}{dx}$	$M_f = 1$

$M = 2$ ← order of DE

$m = \frac{M}{2} = 1$

BC's are order 0 to $M-1$

$M = 2$ → BC's 0th order $u = \bar{u}$ Essential $u = \bar{u}$

1st order $AEu' = \bar{F}$ Natural $F = AEu' = AEu' = \bar{F}$

BC's 0 lower half Essential

$M-1$ upper half Natural

Slides 32 to 34 provide the formulation of the beam problem. I'll cover it later, but it's good to read it and see how the essential and natural BCs are divided

WRS and Weak statement:

Weighted Residual Statement (WRS)

$R_i = (EAu')' + q$ in Ω

$\partial\Omega_u$ $u = \bar{u}$ Ω $F = EAu' = \bar{F}$

$R_u = \bar{u} - u$ essential BC residual on $\partial\Omega_u$

DE: $(EAu')' + q = 0$ $\partial\Omega_f$

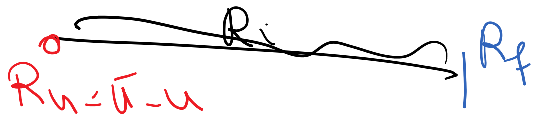
$R_f = \bar{F} - F = \bar{F} - EAu'$ on $\partial\Omega_f$

Multiply by weights

some function of ω e.g. ω'

$\int \omega R_i \, dV$

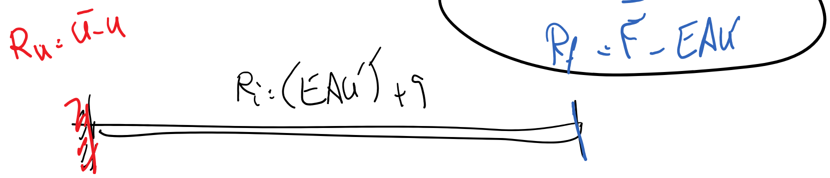
$$\int_{\mathcal{D}} \underbrace{w}_{\text{weight function}} R_i \, dV + \int_{\partial \mathcal{D}_u} f(w) R_u \, dS + \int_{\partial \mathcal{D}_f} w R_f \, dS = 0$$



1D case, this problem: $f(w) R_u \big|_{x=0}$

1D, this problem: $w R_f \big|_{x=L} = 0$

In continuous FEM (this course), I'll show that we don't have the middle term (integral of residual on essential BC)



$$\int_0^L w R_i \, dx + w R_f \big|_{x=L}$$

$$= \int_0^L \underbrace{w}_{0^{th}\text{ order}} \big(\underbrace{(EAu')'}_{2nd\text{ order}} + q \big) dx + w\left(\bar{F} - EAu'\right)\Big|_{x=L} = 0$$

$$\int_0^L U V' \, dx = \int_0^L \left((UV)' - U'V \right) dx$$

$$= UV\Big|_0^L - \int U' V \, dx$$

IBP

$$\int_0^L \underbrace{w}_{U} \underbrace{(EAu')'}_{V'} dx, \quad V = EAu'$$

$$= w\,EAu'\Big|_0^L - \int \big(\underbrace{w'}_{U'} \underbrace{EAu'}_{V} \big) dx$$

$$\int_0^L w (EAu')' \, dx = w\,EAu'\Big|_{x=L} - w\,EAu'\Big|_{x=0} - \int_0^L w' EAu' \, dx$$

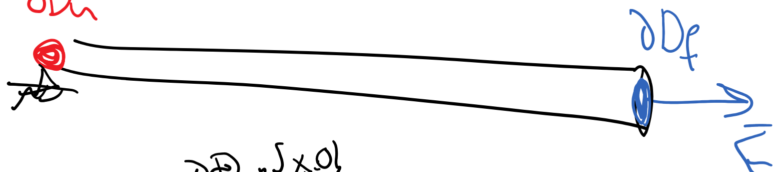
$$|_{x=L} \qquad |_{x=0} \qquad \int_0 w \ldots dx$$

1st order

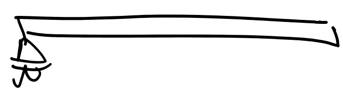
$$\underbrace{\int_0^L w(EAu')'\,dx + \int_0^L wq\,dx}_{\int_0^L wR_I\,dx} + \underbrace{w(\bar F - EAu')|_{x=L}}_{wR_f} = 0$$

$$\underbrace{wEAu'|_{x=L} - \overbrace{wEAu'}^{F(x=0)}|_{x=0} - \int_0^L w'EAu'\,dx}_{\int_0^L w(EAu')'\,dx} + \int_0^L wq\,dx + w\bar F|_{x=L} - wEAu'|_{x=L}$$

$=0$

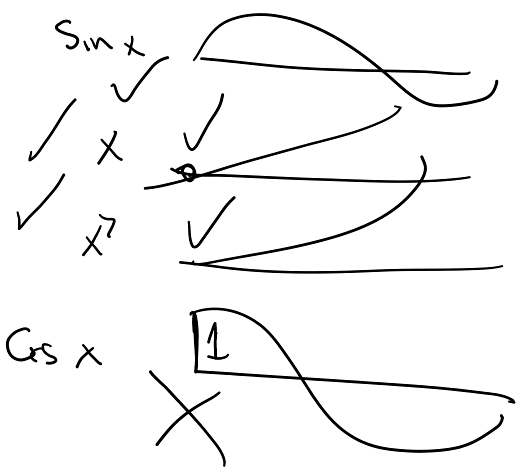


$\partial D_u = \{x=0\}$



When we go from WRS to weak statement we only limit ourselves to weight functions that are zero on essential BC.

w:

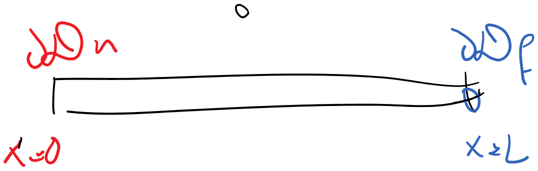


good for WRS

bad for weak statement

$$\cancel{-w(x=0)EAu'(x=0)} - \int_0^L w'EAu'\,dx + \int_0^L wq\,dx + w(L)\bar F|_{x=L} = 0$$

0



1 derivative

$$\int_0^L w'EAu'\,dx = \int_0^L wq\,dx + w\bar F|_{x=L} \qquad \text{Weak statement}$$

$$\int_0^L w(EAu')'dx + w(\bar{F} - EAu')\big|_{x=L} = 0 \quad \text{WRS}$$

0 der 2 derivatives

WRS

$\mathcal{D} = [0,L]$

Find $u(x) \in \mathcal{V} = \{ f \in C^2(\mathcal{D}) \mid \forall x \in \partial\mathcal{D}_u \;\; u(x) = \bar{u} \}$

space of solution

here for this 1D problem $u(0) = \bar{u}$

such that for all $w(x) \in \mathcal{W} = \{ f \in C^0(\mathcal{D}) \mid \text{———} \}$

we have

$$\int_0^L w(\underbrace{(EAu')'}_{\text{2 derivatives}} + q)\,dx + w(\bar{F} - EAu')\big|_{x=L} = 0$$

0th order

☆ Because we did not add weight time residual essential boundary to the WRS

$\cancel{\int_{\partial\mathcal{D}_u} w R_e \, ds}$

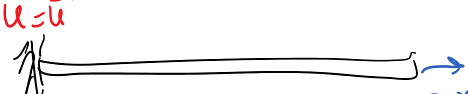
WRS

Weak statement

Find $u(x) \in \mathcal{V} = \{ f \in C^1(\mathcal{D}) \mid f(0) = \bar{u} \}$

such that for all weight functions $w \in \mathcal{W} = \{ f \in C^1(\mathcal{D}) \mid f(0) = 0 \}$

$$\int_0^L w' EA u' \, dx = \int_0^L wq \, dx + w\bar{F}\big|_{x=L}$$

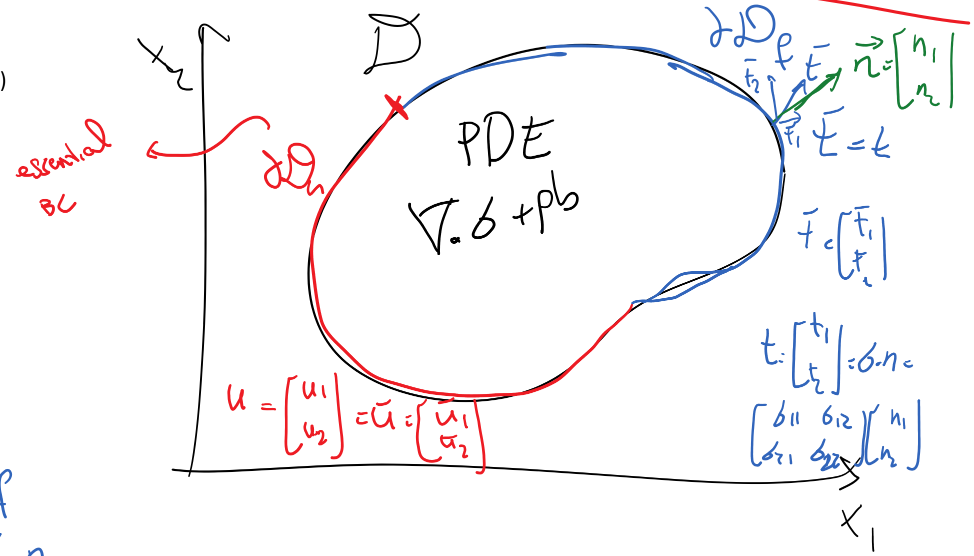


Key points for the weak statement:

- Derivative orders are balanced for weight and solution
- Both w and u satisfy the essential BC.
  - For solution the actual essential BC
  - For the weight, the homogenous (e.g. 0) version of that.

2D examples:

1. Elastostatics (in class)
2. Heat conduction (HW2)



$R_i : \forall x \in \mathcal{D}$

$R_i = \nabla\cdot\sigma + \rho b$

$R_f : \forall x \in \partial\mathcal{D}_f$

$R_f = \bar{t} - \sigma\cdot n$

$R_i : \forall x \in \partial\mathcal{D}_u \quad R_i = \bar{u} - u$

Weighted residual statement — we strongly satisfy essential BC

Find $u \in \mathcal{V} = \{ f \in C^2(\mathcal{D}) \mid \forall x \in \partial\mathcal{D}_u \; u(x) = \bar{u} \}$

such that $\forall\, w \in \mathcal{W} = \{ f \in C^0(\mathcal{D}) \mid \quad \}$

$$\int_{\mathcal{D}} w R_i \, dv + \int_{\partial\mathcal{D}_f} w R_f \, ds + \cancel{\int_{\partial\mathcal{D}_u} h(w) R_u \, ds} = 0$$

$R_i \to \nabla\cdot\sigma + \rho b$

$h(w)$: some function of weight

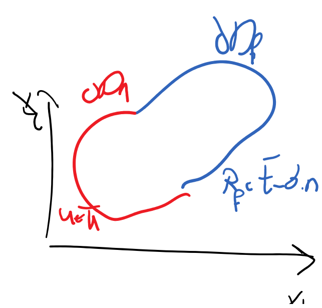
2 der of u

we cross this term as we satisfy it strongly

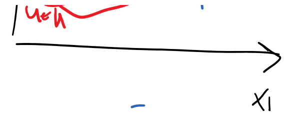
The process of deriving the weak statement

$$\int_{\mathcal{D}} w(\nabla\cdot\sigma + \rho b)\, dV + \int_{\partial\mathcal{D}_f} w(\bar{t} - \sigma\cdot n)\, dS = 0$$

$(EAu')'$



$\mathcal{D}$ $\quad (EAu')' \quad \partial\mathcal{D}_f$



similar to 1D $\int \omega (AG)' + f \quad + \quad \omega(\bar{F} - EAu')$

$\bar{F} - F = \bar{F} - EAu'$

$$\int_{\mathcal{D}} \omega \underbrace{(\nabla \cdot \sigma)}_{2 \text{ derivatives for } u} dV$$

2D version of IBP

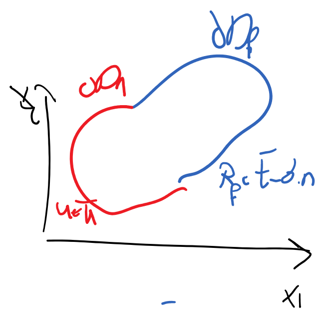
$$\int_{\mathcal{D}} w(\nabla \cdot \sigma) dV = \int_{\partial\mathcal{D}} w\, \sigma \cdot n\, dS - \int_{\mathcal{D}} \nabla w \cdot \sigma\, dV$$

WRS $\displaystyle \int_{\mathcal{D}} \omega(\nabla \cdot \sigma) dV + \int_{\mathcal{D}} \omega(\rho b) dV + \int_{\partial\mathcal{D}_f} \omega(\bar{t} - \sigma \cdot n) dS$

$$\left(\int_{\partial\mathcal{D}} w(\sigma \cdot n) dS - \int \nabla w \cdot \sigma\, dV \right) + \int_{\mathcal{D}} \omega\, \rho b\, dV$$

$$+ \int_{\partial\mathcal{D}_f} \omega \bar{t}\, dS - \int_{\partial\mathcal{D}_f} \omega\, \sigma \cdot n\, dS = 0$$

$$\int_{\partial\mathcal{D}_u \cup \partial\mathcal{D}_f} w(\sigma \cdot n) dS - \int_{\partial\mathcal{D}_f} \omega\, \sigma \cdot n\, dS$$



$$= \int_{\partial\mathcal{D}_u} \omega\, \sigma \cdot n\, dS$$

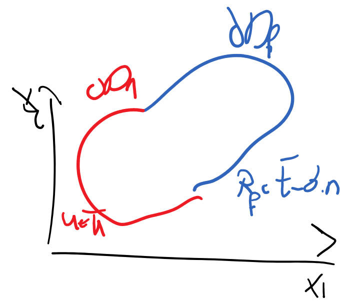
$\omega = 0$ on $\partial\mathcal{D}_u$

2022/09/15

Thursday, September 15, 2022 11:10 AM

The process of deriving the weak statement

$$\int_{D} w(\nabla\cdot\sigma+\rho b)\,dV+\int_{\partial D_f} w(\bar{t}-\sigma\cdot n)\,dS=0$$

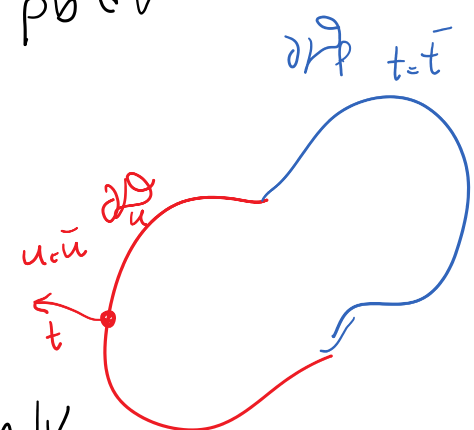


Note

$$\int_{D} w(\nabla\cdot\sigma)\,dV=\int_{\partial D} w\,\sigma\cdot n\,dS-\int_{D}\nabla w\cdot\sigma\,dV$$

By plugging the RHS we obtained:

$$\left(\int_{\partial D} w(\sigma\cdot n)\,dS-\int_{D}\nabla w\cdot\sigma\,dV\right)+\int_{D} w\rho b\,dV+\int_{\partial D} w\bar{t}\,dS-\int_{\partial D_f} w\,\sigma\cdot n\,dS=0$$



$$\int_{\partial D_f} w(\sigma\cdot n)\,dS+\int_{\partial D_u} w(\sigma\cdot n)\,dS-\int_{D}\nabla w\cdot\sigma\,dV+\int_{D} w\rho b\,dV+\int_{\partial D} w\bar{t}\,ds-\int_{\partial D_f} w\,\sigma\cdot n\,ds=0$$

now we want to get rid of this

We will choose weight functions that are zero on Essential BC

$\forall x\in\partial D_u,\ w(x)=0$

$\sigma(u)=C\,\varepsilon(u)$

1D

$\sigma=E\varepsilon$

weak statement

$$\int_{D} \underbrace{(\nabla w)}_{\varepsilon(w)}\,C\,\varepsilon(u)\,dV=\int_{D} w\rho b\,dV+\int w\bar{t}\,ds$$

$$\int_{\mathcal{D}} (\nabla w \cdot) C\, \varepsilon(u)\, dV = \int_{\mathcal{D}} w \rho b\, dV + \int_{\partial \mathcal{D}_f} w \bar{t}\, ds$$

$$\varepsilon(u) = \frac{\nabla u + \nabla u^T}{2}$$

1D $\varepsilon(u) = u'$

$$\varepsilon(w) = \frac{\nabla w + \nabla w^T}{2}$$

we can show $\nabla w \cdot C\, \varepsilon(u) = \underbrace{\left(\frac{\nabla w + \nabla w^T}{2}\right)}_{\varepsilon(w)} . C\, \varepsilon(u)$ from symmetries of C

Final form of the weak statement:

because we strongly satisfy essential BC

Find $u \in \mathcal{V} = \{ f \in C^1(\mathcal{D}) \mid \forall x \in \partial \mathcal{D}_u \quad f(x) = \bar{u}(x) \}$

such that $\forall w \in \mathcal{W} = \{ f \in C^1(\mathcal{D}) \mid \forall x \in \partial \mathcal{D}_u \quad f(x) = 0 \}$

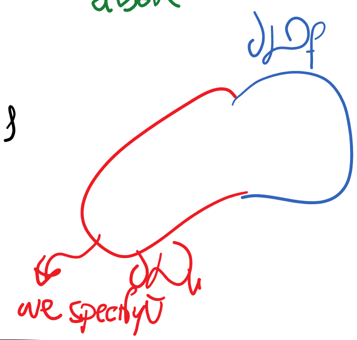
funct.

because we got rid of term above

$$\int_{\mathcal{D}} \varepsilon(w) : C\, \varepsilon(u)\, dV = \int_{\mathcal{D}} w \rho b + \int_{\partial \mathcal{D}_f} w \bar{t}\, ds$$

$\varepsilon(w) = \frac{\nabla w + \nabla w^T}{2}$, $\varepsilon(u) = \frac{\nabla u + \nabla u^T}{2}$

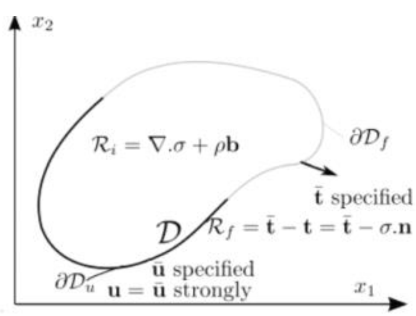
weak statement



$\partial \mathcal{D}_f$

$\partial \mathcal{D}_u$ we specify

Compare this with the WRS:



The Weighted Residual Statement reads as,

Find $\mathbf{u} \in \mathcal{V}^{\mathrm{WRS}} = \{\mathbf{v} \in C^2(\mathcal{D}) \mid \forall \mathbf{x} \in \partial \mathcal{D}_u \ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}\}$, such that, (66a)

$\forall w \in \mathcal{W}^{\mathrm{WRS}} = C^0(\mathcal{D})$ no need to enforce the homogeneous essential BCs for WRS (66b)

$$0 = \int_{\mathcal{D}} \mathbf{w}.(\underbrace{\nabla.\sigma}_{C_{ijkl} u_{k,lj}} + \rho \mathbf{b})\, \mathrm{d}v + \int_{\partial \mathcal{D}_f} \mathbf{w}.(\bar{\mathbf{t}} - \mathbf{t})\, \mathrm{d}s \quad (66c)$$

$$0 = \int_{\mathcal{D}} \mathbf{w}.(\underbrace{\nabla.\sigma}_{C_{ijkl}\,u_{k,lj}} + \rho \mathbf{b})\, \mathrm{dv} + \int_{\partial \mathcal{D}_f} \mathbf{w}.(\bar{\mathbf{t}} - \mathbf{t})\, \mathrm{ds} \qquad (66c)$$

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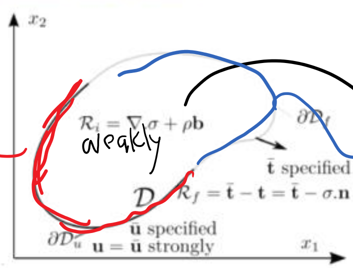
2 derivatives for u

Q: distinction between weak and strong

FEM

∂D_u strongly



weakly

WRS:

$R_i = \nabla.\sigma + \rho b$ in D

$R_f = \bar{t} - t = \bar{t} - \sigma.n$ on ∂D_f

$R_u = \bar{u} - u$ on ∂D_u

The Weighted Residual Statement reads as,

we need to satisfy this a priori

Find $\mathbf{u} \in \mathcal{V}^{\mathrm{WRS}} = \{\mathbf{v} \in C^2(\mathcal{D})\ |\forall \mathbf{x} \in \partial\mathcal{D}_u\ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}\}$, such that, (66a)

$\forall w \in \mathcal{W}^{\mathrm{WRS}} = C^0(\mathcal{D})$ no need to enforce the homogeneous essential BCs for WRS (66b)

$$0 = \int_{\mathcal{D}} \mathbf{w}.(\underbrace{\nabla.\sigma}_{C_{ijkl}\,u_{k,lj}} + \rho \mathbf{b})\, \mathrm{dv} + \int_{\partial \mathcal{D}_f} \mathbf{w}.(\bar{\mathbf{t}} - \mathbf{t})\, \mathrm{ds} \qquad (66c)$$

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WRS

$$\int_D w R_i + \int_{\partial D_f} w R_f\, ds + \cancel{\int_{\partial D_u} w R_u\, ds} = 0$$

"weak"

FEM

we decide to satisfy the last term strongly

Strong -> the equation is satisfied at all points
Weak -> the equation is satisfied in "integral" form, where a weight function multiplies the equation

Specific meaning of weak statement -> The great looking :) equation we get after "integtration by part" of the WRS

Weak statement is much better than WRS because the solution and the weight have the same regularity requirement and this enables continuous FE formulation.

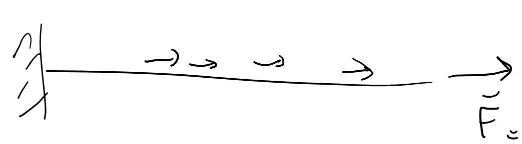
A brief note on how to satisfy the essential boundary condition for the solution and the homogeneous version of that for the weight when dealing with the Weak Statement.

1D Example

the exact soln is

$u(x) \longrightarrow$ is discretized to

$u = \bar{u} = 1$



$\bar{F} = 5$

$u(x)$ $\rightarrow$ is discretized to

n unknowns

$$\overset{\text{discretized}}{\longleftarrow} u^h(x) = \sum_{i=1}^{n} \underbrace{\phi_i(x)}_{\text{the functions we choose}} \underbrace{a_i}_{\text{unknowns}} + \phi_p(x)$$

Motivation from DE

$\dot{x} + 5x = 10$

$x(0) = 0$

$x_p(t) = \frac{10}{5} = 2$ satisfies non-homogeneous $\neq 0$

source term

$x = \phi_1(t) a_1 + \phi_p(t)$

$\phi_1(t) = e^{-5t}$ → satisfies homogeneous DE

$\dot{\phi}_p + 5\phi_p = 10$

$\dot{\phi}_1 \; 5\phi_1 = 0$

$$\dot{x} + 5x = \overbrace{(\dot{\phi}_p + 5\phi_p)}^{10} + \overbrace{(\dot{\phi}_1 + 5\phi_1)}^{0} a_1 = 10$$

We do the same trick to satisfy the essential BCs

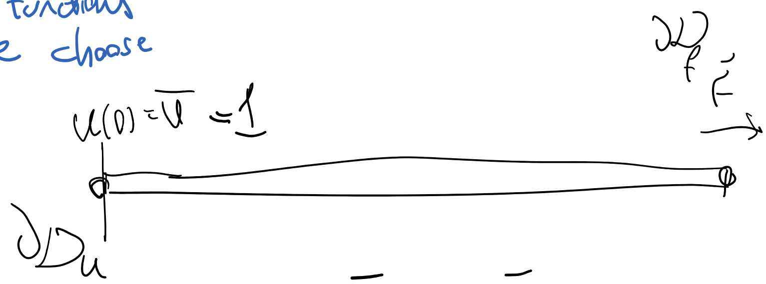
$$\overset{\text{discretized}}{\longleftarrow} u^h(x) = \sum_{i=1}^{n} \underbrace{\phi_i(x)}_{\text{the functions we choose}} \underbrace{a_i}_{\text{unknowns}} + \phi_p(x)$$

were

$\partial\Omega_u : x = 0$

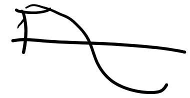
$\phi_p(x=0) = \bar{u}$

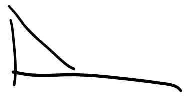
$\forall_{i=1..n} \phi_i(x=0) = 0$



$$u^h(x=0) = \sum_{i=1}^{n} \phi_i(x=0) a_i + \phi_p(0) = \sum_{i=1}^{n} 0 \cdot a_i + \bar{u} = \bar{u}$$

Examples of ϕ_p : $\phi_p = 1$

Examples of ϕ_p : $\phi_p = 1$  $\phi_p = \cos x$ 

$1 - x$ 

ϕ_i's	1,	x,	x^2,	x^3,	$\sin x$,	$\cos x$
$\phi_i(0) = 0$	✗	✓	✓	✓	✓	✗

I can choose $\phi_1 = x$ $\phi_2 = \sin x$

$\phi_p = 1$

$u^h = \phi_1 a_1 + \phi_2 a_2 + \phi_p = a_1 x + a_2 \sin x + 1$ this satisfies essential BC a priori

---

Since ϕ_i's are already zero on Essential BC they can readily be used as weight functions in weak statement.

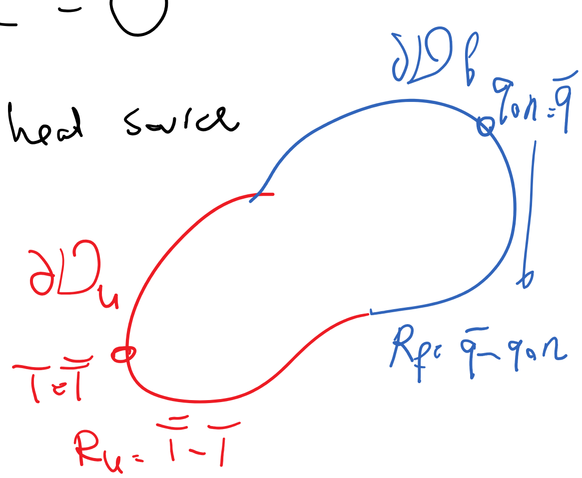
HW hint

PDE $+\nabla \cdot q - Q = 0$

$\nabla \cdot q$: heat flux, Q: heat source

WRS

$$\int_{\Omega} w(\underbrace{\nabla \cdot q - Q}_{R_i})\, dV + \int_{\partial\Omega_q} w(\underbrace{\bar{q} - q \cdot n}_{R_f})\, ds = 0$$



$$\int w \nabla \cdot q \, dV$$

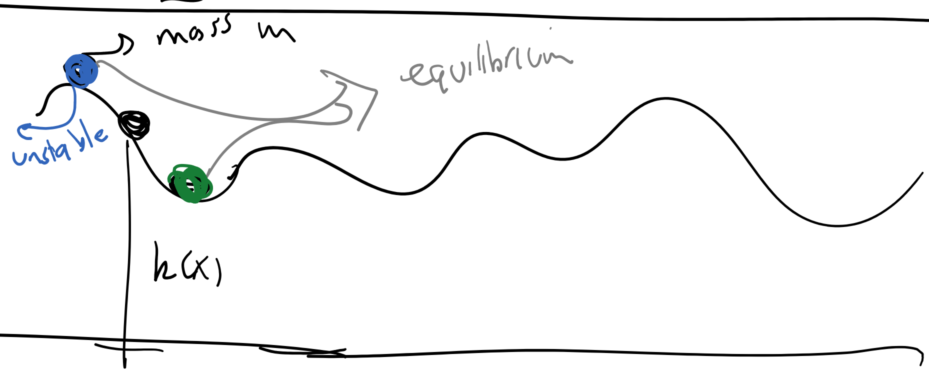
bad

$$\int w(\nabla\cdot q)\,dV$$
bad

Use $$\int_D \underbrace{w\,\nabla\cdot q}_{\text{bad}}\,dV = \int_D \nabla\cdot(wq)\,dV - \int_D \overbrace{\nabla w\cdot q}^{\text{good}}\,dV =$$

$$\int_{\partial D} w\,q\cdot n\,ds - \int \nabla w\cdot q\,dV$$

Energy method



$$\Pi(x) = m g h(x)$$
mass

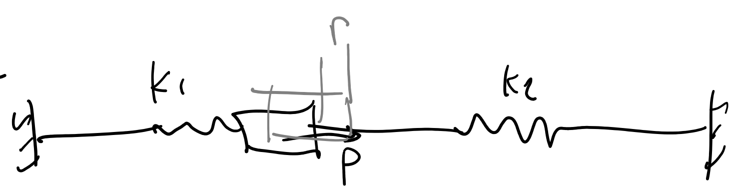
Equilibrium $\frac{d\Pi}{dx} = 0$ Π is an extremum (min or max)

unstable local max $\frac{d\Pi}{dx} > 0$, ...

stable; local min $\frac{\partial \pi}{\partial x} < 0$

Stable solution → we want to find a local minimum

Discrete system → Ex 1
Ex 2

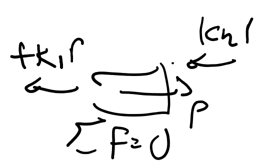


internal energy

$$\Pi = V - W$$
external work

$$V = \frac{k_1 r^2}{2} + \frac{k_2 r^2}{2} \qquad W = Pr$$

$$\Pi(r) = \frac{k_1 r^2 + k_2 r^2}{2} - Pr$$

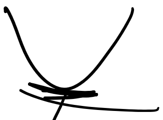


$d\Pi$ ... equilibrium P

$$\frac{d\Pi}{dr} = (k_1+k_2)r - P = 0 \quad \text{equilibrium} \qquad r = \frac{P}{k_1+k_2}$$

matches our force approach

$$\frac{d^2\Pi}{dr^2} = k_1+k_2 > 0$$



local minimum — stable equilibrium

Continuum version

Energy Method for Solid Mechanics

The total energy in solid mechanics is,

$$\Pi = (V - W) - T = \text{Total energy} \tag{85a}$$

(V → internal, W → external)

$$T = \int_{\mathcal{D}} \frac{1}{2}\rho \mathbf{v}.\mathbf{v}\,\mathrm{dv} = \text{Kinetic energy} \tag{85b}$$

$= 0$ for 'static' problems

$$\boxed{V = \int_{\mathcal{D}} e(\epsilon)\,\mathrm{dv} = \text{Internal energy}} \tag{85c}$$

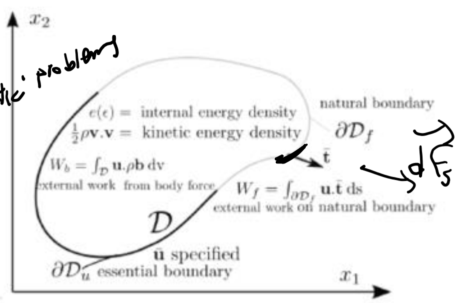
$$W = W_b + W_f = \text{External work} \tag{85d}$$

$$W_b = \int_{\mathcal{D}} \mathbf{u}.\rho\mathbf{b}\,\mathrm{dv} \tag{85e}$$

(u → displacement; $\rho\mathbf{b}\,\mathrm{dv}$ → dF_r)

$$W_f = \int_{\partial\mathcal{D}_f} \mathbf{u}.\bar{\mathbf{t}}\,\mathrm{ds} \tag{85f}$$

($\bar{\mathbf{t}}\,\mathrm{ds}$ → dF_s)



- For static problems $T = 0$.
- Internal energy density, $e(\epsilon) = \frac{1}{2}\epsilon : \sigma(\epsilon) = \frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl}$ for linear solid.
- Natural boundary forces are naturally incorporated into the energy (W_f).
- Essential boundary conditions are incorporated into function space:

$$\mathbf{u} \in \mathcal{V} = \{\mathbf{v}\ |\mathbf{v} \in C^1(\mathcal{D}) : \ \forall \mathbf{x} \in \partial\mathcal{D}_u\ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x})\}, \text{ is a solution if}$$
$$\forall \tilde{\mathbf{u}} \in \mathcal{V}, \quad \Pi(\mathbf{u}) \le \Pi(\tilde{\mathbf{u}}). \tag{86}$$

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1D

$$e = \frac{\sigma \cdot \varepsilon}{2} = \frac{1}{2}E\varepsilon^2$$

internal energy density

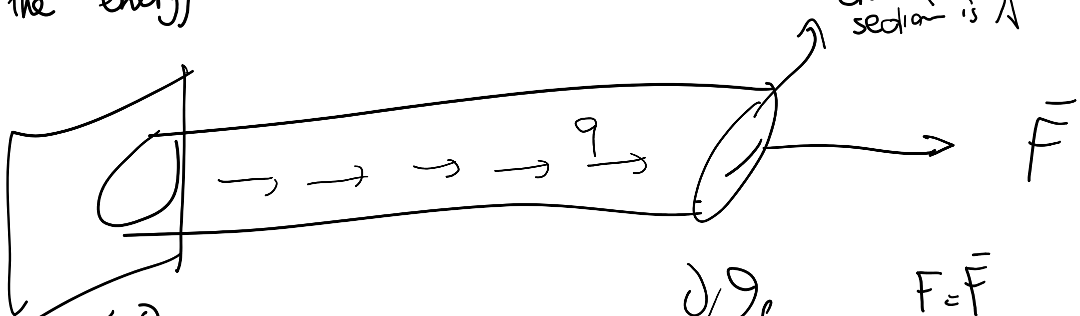
2D, 3D linear elasticity

$$\frac{1}{2}\varepsilon : \sigma = \frac{1}{2}\varepsilon : C : \varepsilon$$

(C → stiffness matrix)

For the problems we'll do (static) $T = 0$

Derive the energy statement for a bar



$x = 0$, $\partial\mathcal{D}_u$, $u = \bar{u}$

$\partial\mathcal{D}_f$, $x = L$, $F = \bar{F}$

$\Pi = V - W$ ... work | $V = \int e(\varepsilon)\,dV = \int^L (e(\varepsilon)\,dA\,dx$

$u=\bar{u}$

$\Pi = V - W$ — V: internal energy, W: external work

$$V = \int_V e(\varepsilon)\, dV = \int_{x=0}^{L}\int_A e(\varepsilon)\, \underbrace{dA\,dx}_{dV}$$

$e(\varepsilon) \rightarrow \frac{E\varepsilon^2}{2} \rightarrow$ constant across the section

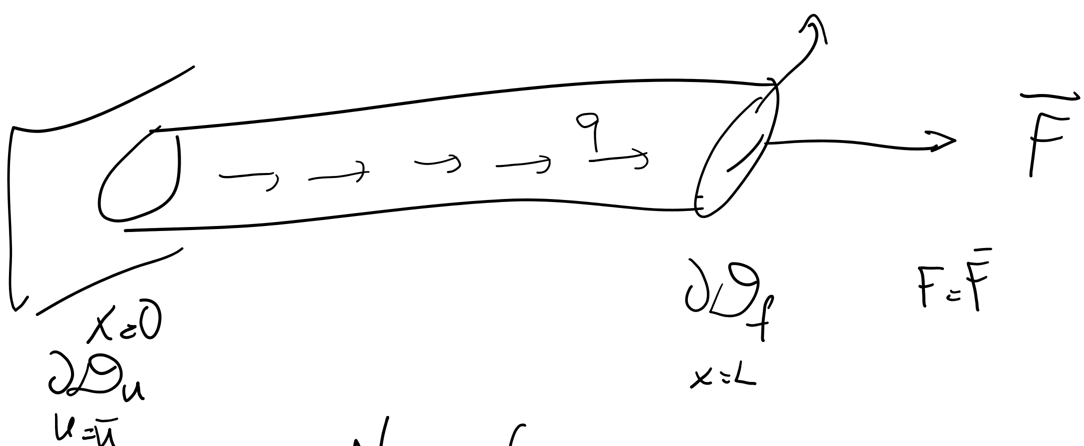
$$= \int_0^L \int_A \frac{E}{2}\varepsilon^2\, dA\, dx$$

constant across area for each x

$\sigma(x)$ constant so is $\varepsilon(x) = \frac{\sigma(x)}{A}$

$$= \frac{1}{2}\int_0^L E(x)A(x)\,\varepsilon^2\, dx \qquad \varepsilon = \frac{du}{dx} = u'$$

$$\boxed{V = \frac{1}{2}\int EA(x)\, u'^2(x)\, dx} \quad (i)$$

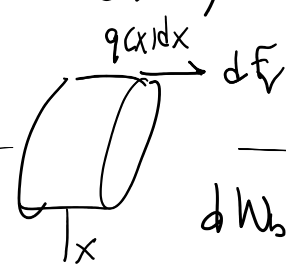


$W_f = \bar{F}\,u(L)$ (iii)

$$W = W_b + W_f \quad (ii)$$

W_b: body/bulk; $W_f \rightarrow$ natural boundary

$$W_b = \int_0^L u(x)\,q(x)\, dx \quad (iv)$$

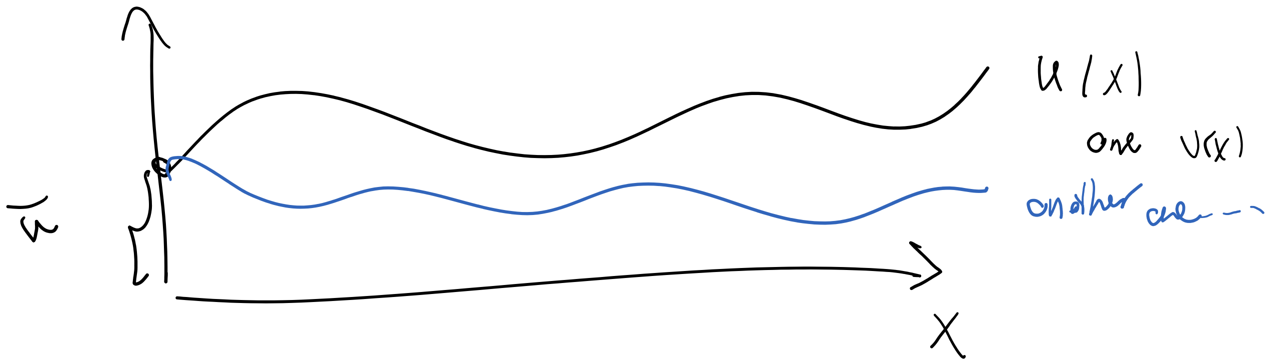


$dW_b = u(x)\, dF_v = u(x)\, q(x)\, dx$

$$\Pi(u(x)) = V(u(x)) - W_b - W_f$$

$$\rightarrow \boxed{\Pi(u(x)) = \frac{1}{2}\int_0^L EA(x)\, u'^2(x)\, dx - \int_0^L u(x)\,q(x)\, dx - u(L)\bar{F}}$$

$$\rightarrow \quad \Pi(u(x)) = \frac{1}{2}\int_0 EA(x)\,u'(x)^2\,dx - \int_0 \dots u(x)\,dx - \dots$$



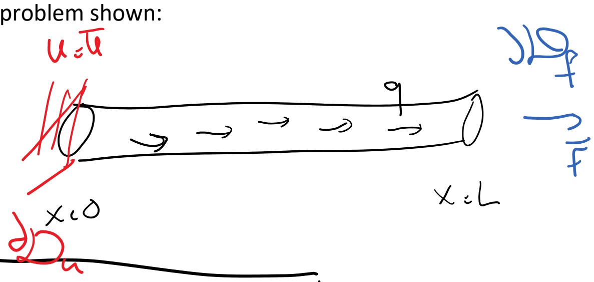
A function of a function is called a functional.

1. Useful links for energy method (not necessary to apply energy approach in the derivation of weak statement) – link Functional optimization: How an equation for first variation of a functional (e.g. equations 93, 95 on slide 78) can be derived. You clearly do not need to read this document for this course and this is only provided as a related material for students that want to understand the logic behind the derivation of equations 93, 95. – link Exact calculation of total, first, and second variations for a simple example: In this document the total variation of the energy functional for the bar problem is directly calculated. The first and second variations are directly obtained and higher variations are zero for this simple functional. It is observed that the first variation is exactly the same as what we would have obtained by equation 96 on slide 78.

2022/09/20

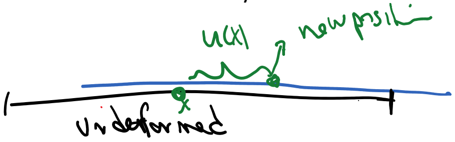
Tuesday, September 20, 2022 11:24 AM

From the last time, we had the potential energy statement for the bar problem shown:



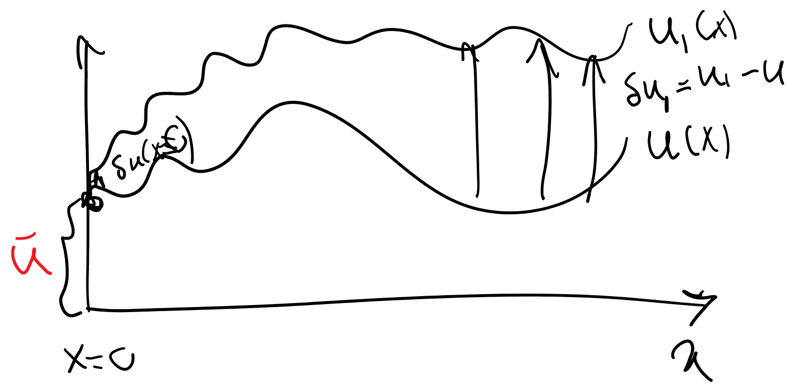
$$\boxed{\Pi(u,u') = \frac{1}{2}\int_0^L EA(x)\, u'^2\, dx - \int_0^L u(x)\, q(x)\, dx - u(L)\bar{F}} \quad \text{(I)}$$

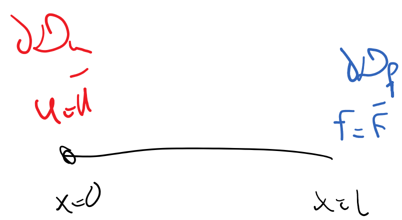
1) Look for minimum potential energy condition.
2) How the essential boundary condition is treated



we plot u on the vertical axis

Let u be the exact solution, that is it minimizes Π





Let $u_1(x)$ be another "trial function"

$$\Pi(u_1,u_1') = \int \frac{1}{2} EA\, u_1'^2 - \ldots$$

$$\Pi(u,u') \leq \Pi(u_1,u_1')$$

exact solution — another trial function

$$\delta u_1 = u_1 - u \longrightarrow \boxed{u_1 = u + \delta u_1}$$

δu_1: increment of u_1; u: exact solution

$$\Pi(u_1,u_1') = \frac{1}{2}\int_0^L EA(x)\,(u_1')^2\, dx - \int_0^L u_1(x)\, q(x)\, dx - u_1(L)\bar{F} \longrightarrow$$

$\Pi(u_1, u_1') = \frac{1}{2}\int_0 EA(u_1')^2 \ldots \int_0 u_1(x)q(x) \ldots - u_1(L)F$

$u_1' = (u + \delta u_1)' = u' + (\delta u_1')$

$$\Pi(u_1, u_1') = \frac{1}{2}\int_0^L EA\left(u'^2 + 2u'\delta u_1' + (\delta u_1')^2\right)dx - \int_0^L \left(u(x) + \delta u_1(x)\right)q(x)\,dx - \left(u(L) + \delta u_1(L)\right)F$$

$$= \frac{1}{2}\int_0^L EAu'^2 dx - \int_0^L u(x)q(x)dx - u(L)F$$

$\Pi(u, u')$ ↓ exact solution

$$+ \left(\int_0^L EA\,\delta u_1'(x)\,u'(x) - \int_0^L \delta u_1(x)q(x)dx - \delta u_1(L)F\right)$$

$\underbrace{\delta\Pi}$ → 1st order terms in δu

$$+ \frac{1}{2}\int EA(\delta u_1')^2 dx$$

$\delta^2\Pi$ → 2nd order terms in δu

Summary

$$\Pi(u_1, u_1') = \Pi(u, \delta u) + \delta\Pi + \delta^2\Pi + \ldots$$

these are zero here

$$\Pi(u, \delta u) = \frac{1}{2}\int_0^L EAu'^2 dx - \int_0^L u(x)q(x)dx - u(L)F$$

$\delta u \to -\delta u$

$\delta\Pi \to -\delta\Pi$

$$\delta\Pi = \int_0^L \delta u' EAu' dx - \int_0^L \delta u\, q\, dx - \delta u(L)F = 0$$

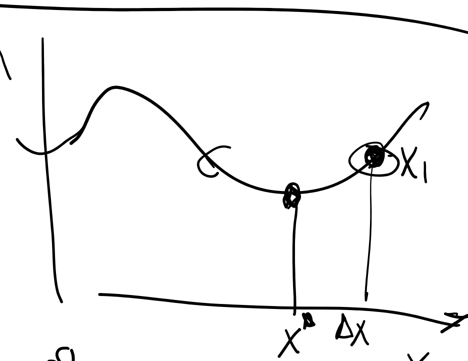
$$\delta^2\Pi = \int_0^L \frac{1}{2}EA(\delta u')^2 dx > 0$$

$\Pi(u_1, u_1') \geq \Pi(u, \delta u)$

↓ minimizes the energy

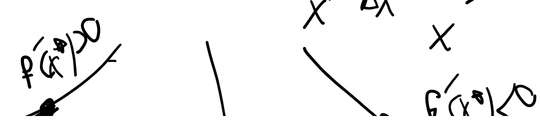
$$f(x_1) = f(x^* + \Delta x) = f(x^*) + \Delta x f'(x^*) + \frac{1}{2}\Delta x^2 f''(x^*)$$

$+ \ldots$ HOT

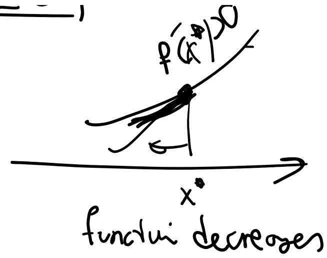


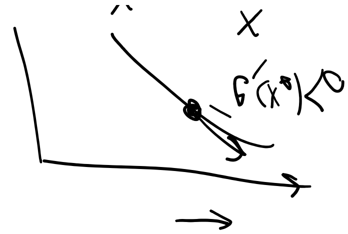
$$\boxed{0 \leq f(x_1) - f(x^*) = \underbrace{\Delta x f'(x^*)}_{\delta f < 0} + \underbrace{\frac{1}{2}\Delta x^2 f''(x^*)}_{\delta^2 f \geq 0}}$$

i) $f'(x^*) = 0$ why



i) $f'(x^*) = 0$ why





after knowing $f'(x^*)<0$

ii) $0 \le f(x) - f(x^*) = \underbrace{\frac{1}{2}\Delta x^2 f''(x^*)}_{\ge 0} \equiv f''(x^*)>0$

For functions x^* minimizes f $\rightarrow$ $\delta f(x^*) = \Delta x\, f'(x^*) = 0 \equiv f'(x^*) = 0$

$\delta^2 f(x^*) > 0 \equiv f''(x^*) > 0$

note if $\delta^2 f = 0$ we need to look @ higher order derivatives

Similar to functions of real variable above, if a functional is mimimized for solution u, we have the following conditions:

$$\Pi(u_1, u_1') = \Pi(u+\delta u_1, u'+\delta u_1') = \Pi(u, u') + \delta\Pi + \delta^2\Pi + \ldots$$

$\delta\Pi = 0$

$\delta^2\Pi > 0$ minimum condition for functionals

here $\delta\Pi = \int_0^L \delta u'(x)\, EA(x)\, u'(x)\, dx - \int_0^L \delta u(x)\, q(x)\, dx - \delta u(L)\bar{F} = 0$

$\delta^2\Pi = \int_0^L \frac{1}{2} EA(x) (\delta u')^2 dx > 0$ this is already satisfied :)

so we only need to worry about this one.

$$\int_0^L \underbrace{\delta u'(x)}_{w'(x)} EA u'(x)\, dx = \int_0^L \underbrace{\delta u(x)}_{w(x)} q(x)\, dx + \underbrace{\delta u(L)}_{w(L)} \bar{F}$$

this is our weak statement

$$\int_0 \underbrace{\delta u'(x)}_{w'(x)} \ldots \, dx = \int_0 \underbrace{\delta u(x)}_{\delta w(x)} \ldots \; \underbrace{\ldots}_{w(L)}$$ is weak statement

Recall the weak statement

$$\int_0^L w'(x)\, EA\, u'(x)\, dx = \int_0^L w(x)\, q(x)\, dx + w(L)\bar{F}$$

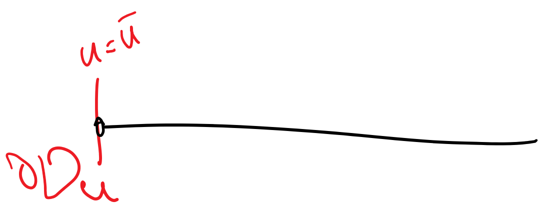
Recall: In the derivation of the weak statement (from WRS) we needed the weight function to be ZERO AT ALL ESSENTIAL BC

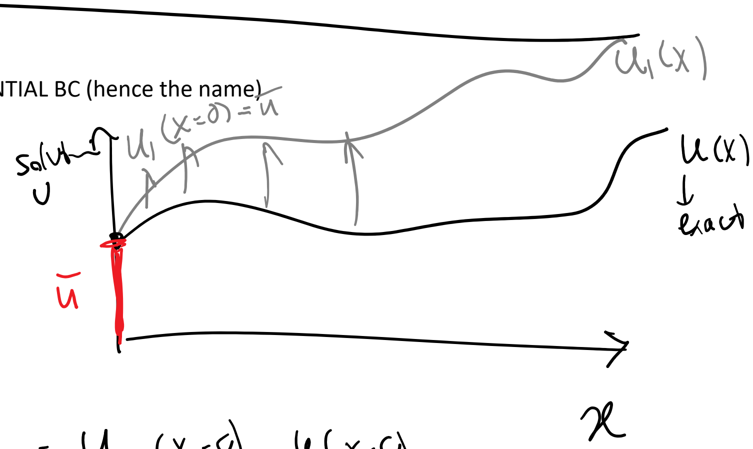
Do we have the same condition here?

$\delta u = w$ means Do we need $\delta u(\partial\Omega_u) = 0$

$w = 0$ on $\partial\Omega_u$

In energy methods, ALL TRIAL FUNCTIONS must satisfy the ESSENTIAL BC (hence the name)

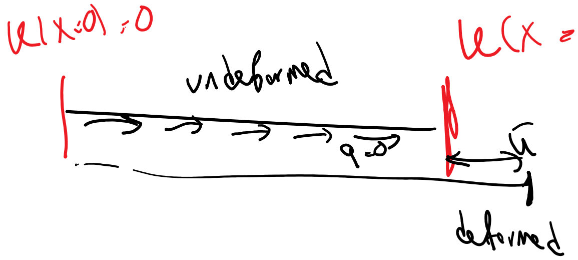




$$\left.\begin{aligned} u_1(x=0) &= \bar{u} \\ u(x=0) &= \bar{u} \end{aligned}\right\} \rightarrow \delta u_1(x=0) = u_1(x=0) - u(x=0) = \bar{u} - \bar{u} = 0$$

$$\boxed{\delta u = 0 \text{ on } \partial\Omega_u}$$

Why we must satisfy essential BC?



$$\varepsilon = \frac{du}{dx} = \frac{\bar{u}}{L}$$ constant for the whole bar

I choose $q = 0$

$\Pi(u) = \int^L \ldots$

deformed $\quad$ I choose $q = 0$

$$\Pi(u,\bar{u}) = \int_0^L \frac{1}{2} EA \underbrace{u'^2}_{u' = \varepsilon} dx - \int u(x) \overset{\nearrow 0}{q(x)} dx$$

no natural Boundary term

$\partial D_u = \{0, L\}$

$\partial D_p = \phi$

For the exact solution $u' = \varepsilon = \left(\frac{\bar{u}}{L}\right)$

$$\Pi(u,u') = \int_0^L \frac{1}{2} EA \left(\frac{\bar{u}}{L}\right)^2 dx = \frac{EA}{2}\frac{\bar{u}^2}{L}$$

energy of the exact solution

How about trial function $u_1 = 0$

$$\Pi(u_1, u_1') = 0$$

$$\underbrace{\Pi(u_1, u_1')}_{\text{another function}} < \underbrace{\Pi(u, u')}_{\text{exact solution}} \quad !$$

It must have been $>$

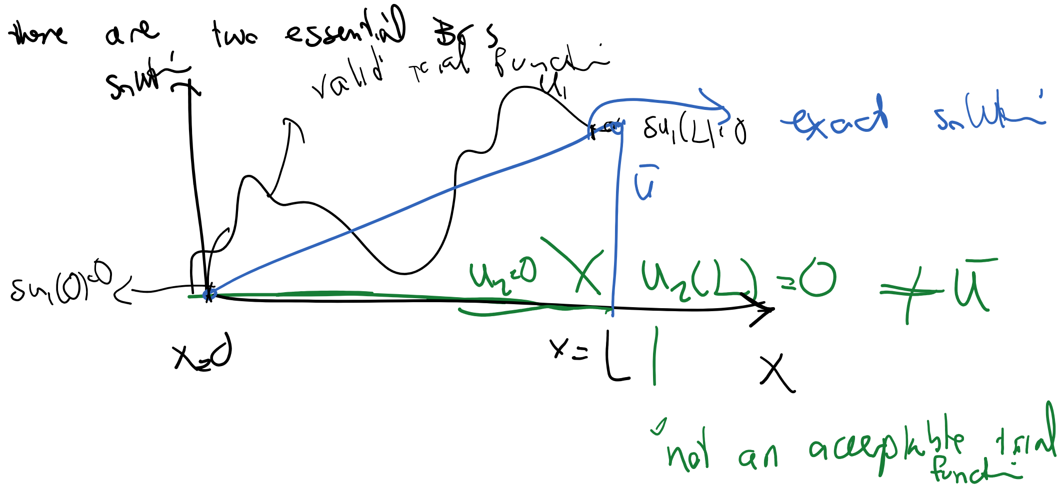
The problem here is that u1 is not acceptable.

Acceptable trial functions MUST satisfy all essential BCs. Otherwise the exact solution does not minimize the potential energy.

For this problem there are two essential BCs

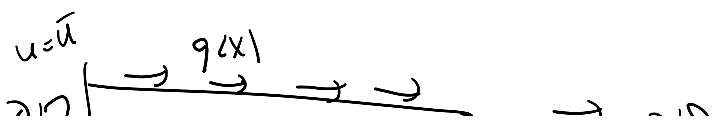
$u(0) = 0$

$u(L) = \bar{u}$

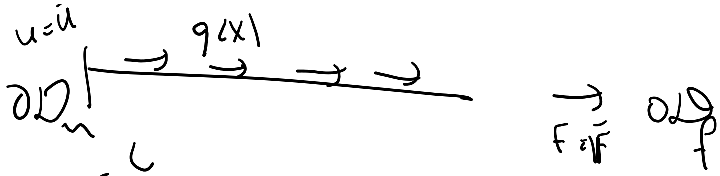


Summary:

For this problem



Summary:



For this problem

$$\delta\Pi = \int_0^L \delta u' \, EA \, u' \, dx - \int_0^L \delta u \, q \, dx - \delta u(L)\bar{F} = 0$$

and $\delta u(0) = 0$ (because trial functions satisfy essential BC $\rightarrow \delta u = 0$ on $\partial\Omega_u$)

Replace $\delta u \rightarrow w$

Find $u \in \mathcal{D} = \{ f \in C^1([0,L]) \mid f(0) = \bar{u} \}$ → must satisfy essential BC

such that $\forall w \in \mathcal{V} = \{ f \in C^1([0,L]) \mid f(0) = 0 \}$ → increment is zero

$$\int_0^L \underbrace{w'(x)}_{(\delta u)'} EA \, u'(x) \, dx = \int_0^L \underbrace{w(x)}_{\delta u} q(x) \, dx + \underbrace{w(L)}_{\delta u} \bar{F}$$

Automated way of calculating the first increment -> So we can easily calculate it and find the weak statement.

$$\underbrace{\Delta f}_{\text{total increment}} = f(x+\Delta x, y+\Delta y) - f(x,y)$$

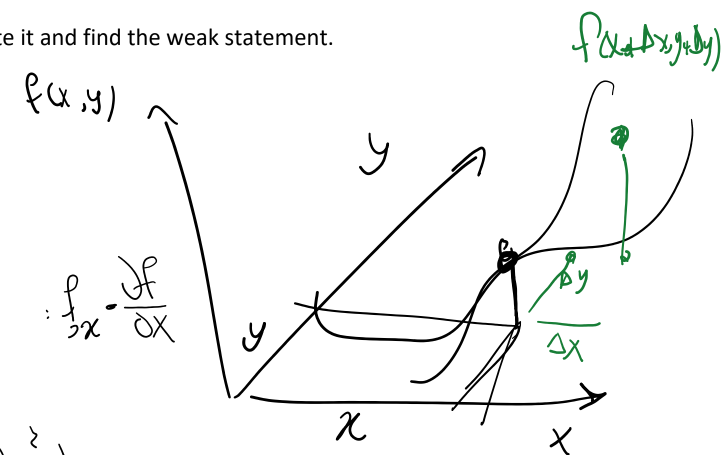
$$= \underbrace{\left(f_{,x}\Delta x + f_{,y}\Delta y \right)}_{\delta f}$$

$$+ \underbrace{\left(\tfrac{1}{2} f_{,xx}\Delta x^2 + f_{,xy}\Delta x \Delta y + \tfrac{1}{2} f_{,yy}\Delta y^2 \right)}_{\delta^2 f}$$

Hessian term

$+$ HOTs

$f_{,x} = \frac{\partial f}{\partial x}$



1D version of this from slide 76

First variation of a function / extremum condition

Let $f(x)$ be a function from $\mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is the real number set). We are interested in finding the increment to the function value due to change in the function argument x_0:

$$x_0 \to x_0 + \Delta x \; : \; f(x_0) \to ?$$

We adopt the following definitions:

- Total variation: $\Delta f(x_0, \Delta x) = f(x_0 + \Delta x) - f(x_0)$
- First variation: $\delta f(x_0, \Delta x) = \frac{\mathrm{d}f}{\mathrm{d}x}(x_0)\Delta x$

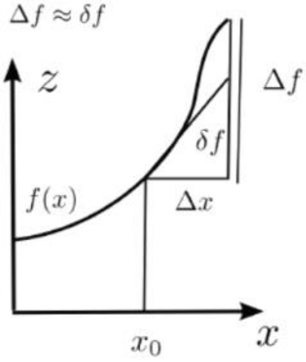
We often drop the arguments x_0 and Δx as shown. For a differentiable function we expect:

$$\Delta f \approx \delta f \text{ for "small" } |\Delta x|$$

$\delta f = f'(x_0)\Delta x$

$\Delta f = f(x_0 + \Delta x) - f(x_0)$

$\Delta f \approx \delta f$



$f(x,y)$; $\quad \delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$

$$\Pi(\underbrace{u}_{x}, \underbrace{u'}_{y}) = \frac{1}{2}\int_0^L EA\, u'^2\, dx - \int_0^L uq\, dx - u(L)\, q$$

$$\delta \Pi = \frac{\partial \Pi}{\partial u}\delta u + \frac{\partial \Pi}{\partial u'}\delta u' = \underbrace{\int \frac{1}{2} EA \left(\frac{\partial u'^2}{\partial u'}\right)\delta u'\, dx}_{\int \frac{1}{2} EA (2u'\delta u')} - \int \delta u\, q\, dx - \delta u(L) F$$

$\frac{\partial y^2}{\partial y} = 2y$

$$= \int_0^L EA\, u' \delta u' - \int_0 \delta u\, q\, dx - \delta u(L) F = 0$$