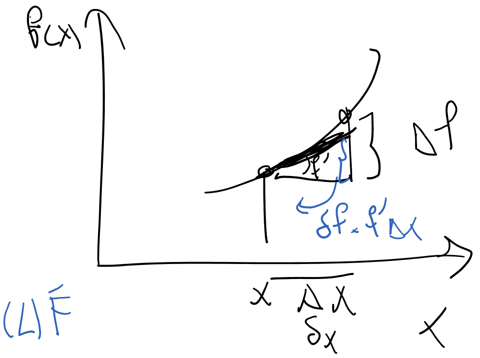
2022/09/22

Thursday, September 22, 2022 11:12 AM

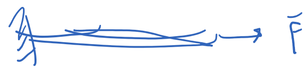
How to easily find the first increment?

We need to calculate the first increment & set it to zero for the energy method.



$$\Pi(u,u') = \int_0^L \frac{1}{2}EA\,u'^2\,dx - \int_0^L u(x)q(x)\,dx - u(L)\bar{F}$$

$u \to z$, $u' \to y$, z



$$\delta\Pi(u,u') = \int_0^L \Bigg[\underbrace{\frac{\partial \frac{1}{2}EAu'^2}{\partial u'}\delta(u')}_{\text{like } \delta f = \frac{\partial f}{\partial x}\delta x} + \underbrace{\frac{\partial \frac{1}{2}EAu'^2}{\partial u}}_{0}\delta u\Bigg]dx$$

$$-\int\Big(\underbrace{\frac{\partial u}{\partial u}}_{1}\delta u\Big)q\,dx \quad - \underbrace{\frac{\partial u}{\partial u}}_{1}\delta u\Big|_L \bar{F}$$

$$\frac{\partial \frac{1}{2}EAu'^2}{\partial u'} = \frac{1}{2}EA(2u') \qquad \text{like } \frac{\partial z^2}{\partial z} = 2z$$

$$= \int_0^L EA\,u'\,\underbrace{\delta(u')}_{=(\delta u)'}\,dx - \int_0^l \delta u\,q\,dx - \delta u(L)\bar{F}$$

$(\delta u)'$ see below

this is simply written as $\delta u'$

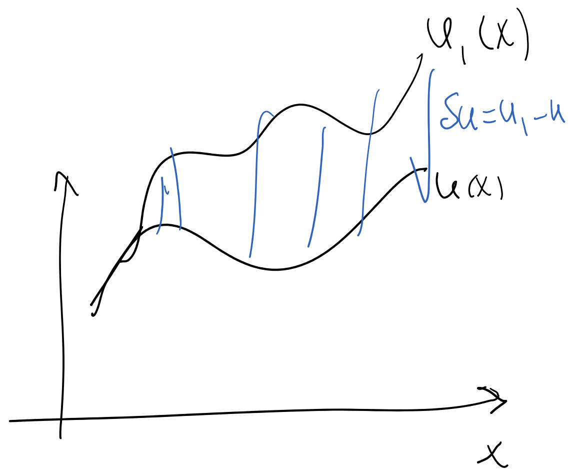
$\delta(u')$: increment of u'

$(\delta u)'$: derivative of increment of u

$\delta u = u_1 - u \rightarrow (\delta u)' = (u_1 - u)' =$

$u_1' - u' = \delta(u')$

$(\delta u)' = \delta(u')$



Final statement → satisfies essential BC

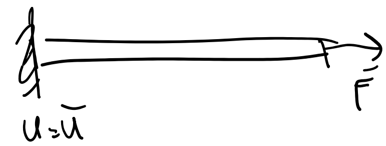
→ satisfies BC

Final statement

Find $u \in \mathcal{V} = \{ f \in C^1(0,L) \mid f(0) = \bar{u} \}$

such that $\forall w \in \mathcal{W} = \{ f \in C^1(0,L) \mid f(0) = 0 \}$

$$\int_0^L w' EA u' \, dx = \int_0^L w q \, dx + w(L)\bar{F}$$



$w = \delta u$

Weak statement from the energy approaches matches the weak statement we derived from Balance law -> WRS -> weak statement but much more directly.

Side note (not needed for FEM)

Why is it called natural BC?

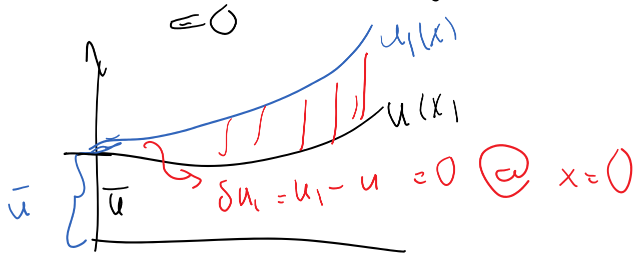
$$\int_0^L (\delta u)' EAu' \, dx - \int_0^L \delta u \, q \, dx - \delta u(L)\bar{F} = 0$$

$(\delta u)' \to U'$, $EAu' \to V = EAu'$

$U = \delta u$

$$= UV\Big|_0^L - \int UV' \, dx - \int_0^L \delta u \, q \, dx - \delta u(L)\bar{F} =$$

$$\delta u(L) EAu'(L) - \underbrace{\delta u(0) EAu'(0)}_{=0} - \int_0^L \delta u \, (EAu')' \, dx - \int_0^L \delta u \, q \, dx - \delta u(L)\bar{F} = 0$$

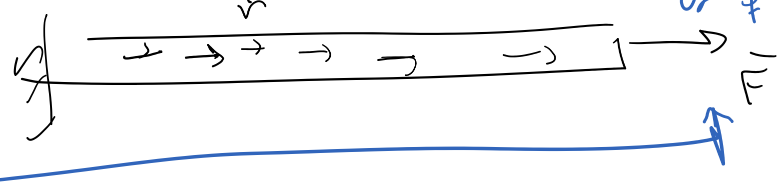


$$\delta u(L)\left(EAu'(L) - \bar{F}\right) - \int^L \delta u\left((EAu')' + q\right) dx = 0$$

$$\boxed{\underbrace{\delta u(L)}_{w(L)}\underbrace{\left(EAu'(L)-\bar F\right)}_{\text{natural BC}} - \int_0^L \underbrace{\delta u}_{w(x)}\left((EAu')'+q\right)dx = 0}$$

$- w(L) R_f(u)$

DE $\quad R_i = (EAu')'+q$



∂D_f

$R_f = \bar F - F = \bar F - EAu' = 0$

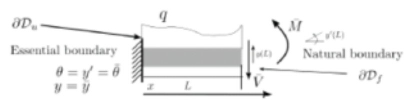
Clearly, the integration by parts of the weak statement gives us the WRS and from which we derive the DE and NATURAL BC

For more complex problems (e.g. gradient elasticity / higher order beam and shell models the energy approach provides complex "natural" boundaries.

For a more interesting problem about this see

Energy method to Strong Form and Boundary Conditions

We realized the convenience of energy methods in deriving the weak form in one step. They can also be used to strong form and boundary conditions by the common approach of integration by part (divergence theorem in $D > 1$).
The weak form from (107) is:



$$\delta\Pi = \int_0^L \delta y''(x)EIy''(x)\,\mathrm{d}x - \int_0^L \delta y(x)q(x)\,\mathrm{d}x + \delta y(L)\bar V - \delta y'(L)\bar M = 0$$

Two consecutive integration by parts yield:

$$\int_0^L -\frac{\mathrm{d}\delta y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2})\,\mathrm{d}x - \int_0^L \delta y(x)q(x)\,\mathrm{d}x + \delta y(L)\bar V - \delta y'(L)\bar M + \delta y'(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})\Big|_0^L = 0 \Rightarrow$$

$$\int_0^L \delta y\left(\frac{\mathrm{d}^2EI}{\mathrm{d}x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - q\right)\mathrm{d}x - \delta y'(L)\left(\bar M - EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(L)\right) - \delta y'(0)EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(0)$$

$$+\,\delta y(L)\bar V - \{\delta y\frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2})\}\Big|_0^L = 0$$

$$\int_0^L \delta y\underbrace{\left(\frac{\mathrm{d}^2EI}{\mathrm{d}x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - q\right)}_{\text{DE}}\mathrm{d}x - \delta y'(L)\left(\underbrace{\bar M - EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(L)}_{\text{natural BCs}}\right) + \delta y(L)\left(\underbrace{\bar V - \frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2})}_{\text{natural BCs}}\right)$$

$$+\,\delta y(0)\frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2})(0) = 0$$

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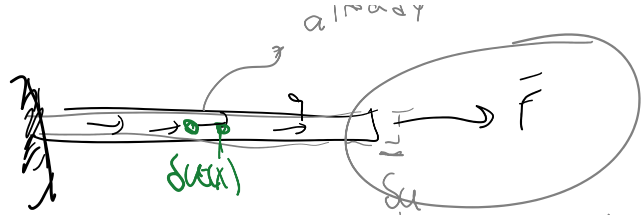
Energy method vs. Principle of Virtual Work

Virtual Work is even easier than energy method to derive the weak statement

In 1 line we already write the weak statement already deformed

Lastly virtual work from stress & virtual strain

$$\int \underbrace{\delta\varepsilon}_{(\delta u)'} \; \sigma(x) \underbrace{(A\,dx)}_{dV}$$



virtual displacement

$dF = q\,dx$

virtual work
force * virtual displacement

$(q\,dx)\,\delta u$

$\delta u(L)\,\bar{F}$
virtual work here

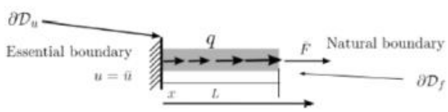
$q: \quad \int_0^L \delta u\, q\, dx$

$$\underbrace{\int_0^L (\delta u)' \, A E \overbrace{u'(x)}^{\sigma(x)} dx}_{\text{internal virtual work}} = \underbrace{\int_0^L \delta u\, q\, dx + \delta u(L)\,\bar{F}}_{\text{external virtual work}}$$

Energy Method vs. Principle of Virtual Work

- Principle of virtual work or virtual displacement in solid mechanics states that if $\mathbf{u}$ is the solution to a boundary value problem, the virtual internal and external works produces by admissible virtual displacements $\delta\mathbf{u}$ are equal.
- Virtual displacements $\delta\mathbf{u}$ refer to displacements that are zero at essential boundary values (so that solution displacement plus virtual displacement $\tilde{\mathbf{u}} = \mathbf{u} + \delta\mathbf{u}$ (*cf.* (79)) as another admissible trial function also satisfies essential boundary conditions).
- Virtual Displacement/Virtual work is basically the equation we obtain by minimizing the energy function $\delta\Pi = 0$.
- Similar principles (virtual temperature for heat flow in solids and virtual velocities for fluid flow) are also directly derived from $\delta\Pi = 0$.
- While principle of virtual work can be obtained from $\delta\Pi = 0$, it is often quite easy to directly write and equate internal and external works for a given problem.

Virtual work: 1D solid bar



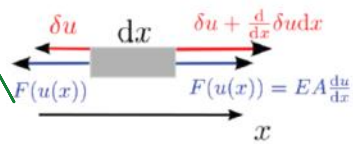
Equation (98) can be written as,

Find $u \in \mathcal{V} = \{v \in C^1([0, L]) \mid v(0) = \bar{u}\}$, such that,
$\forall \delta u \in \mathcal{W} = \{v \in C^1([0, L]) \mid v(0) = 0\}$

$$\underbrace{\int_0^L \overbrace{\delta u'(x)}^{\frac{\mathrm{d}}{\mathrm{d}x}\delta u}\,\overbrace{EAu'(x)}^{F(u(x))}\,\mathrm{d}x}_{\text{Virtual Internal Work}} = \underbrace{\int_0^L \delta u(x) q(x)\,\mathrm{d}x + \delta u(L)\bar{F}}_{\text{Virtual External Work}} \tag{109}$$

Note that the internal work differential is:

$$\mathrm{d}V = F(u(x)).(\delta u + \frac{\mathrm{d}}{\mathrm{d}x}\delta u \mathrm{d}x) - F(u(x)).\delta u$$
$$= \frac{\mathrm{d}\delta u}{\mathrm{d}x}.F(u(x))\mathrm{d}x \tag{110}$$



stress & virtual strain

DISCRETIZATION

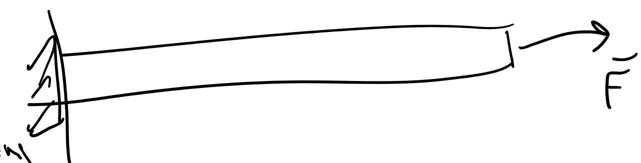
Discretization: Going from continuum (infinite unknown) statement to "discrete" statement wherein we have a finite number of unknowns

$$u^h = \phi_p(x) + \sum_{i=1}^{n} \phi_i(x) a_i$$

u^h: discretized solution

ϕ_i: shape functions "we choose these"

unknown $(a_1, \ldots, a_n)$



ϕ_p satisfies ALL essential BC's

ϕ_i's " zero version of ALL " "

$\rightarrow u^h$ satisfies all essential BC

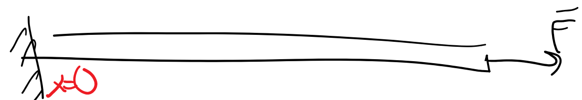
$\Rightarrow$ that's why we didn't add essential BC to WRS

$$\cancel{\int_{\partial\mathcal{D}_u} f(w)(\bar{u}-u)\,ds}$$

Examples

problem 1

$u = \bar{u}$ eg: $\bar{u} = 1$

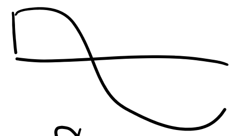


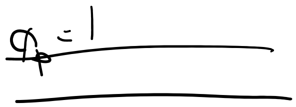
$\partial\Omega_u = \{0\}$

ϕ_p : $\phi_p(0) = \bar{u} = 1$

ϕ_i : $\phi_i(0) = 0$

ϕ_p cos x



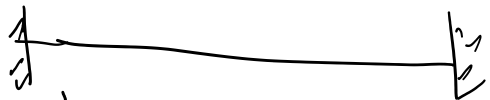


Sin x, Sin 2x, Sin 3x, ...

x, x^2, x^3, x^4

Example

$u=0$

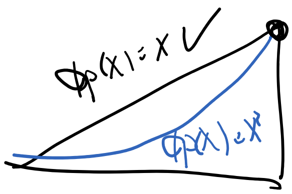


$u(1) = 0$ $\xrightarrow{=1}$ class

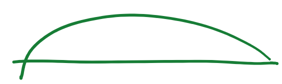
HW

$\partial\Omega_u = \{0, 1\}$

$\phi_p(0) = 0$ $\phi_p(1) = 1$

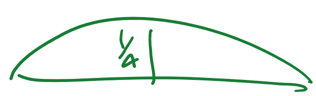


ϕ_i $\phi_i(0) = 0$ $\phi_i(1) = 0$ examples $\sin \pi x$



use x & x^2 to form a decent ϕ_1

$\phi_1 = x - x^2$



$$\phi = ax + bx^2 + c \qquad \phi_1 = \cancel{a}(x - x^2)$$

$$\phi(0) = c = 0$$

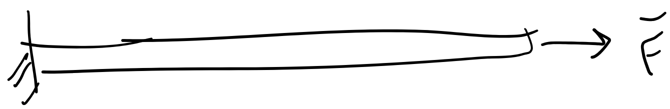
$$\phi(1) = a + b + \cancel{c}^{\,0} = 0 \rightarrow a = -b$$

Note ϕ_i's are zero on essential BC $\Longrightarrow$

They can be used as weight functions for weak statement

Some notations: bar DE: $R_i = (EAu')' + q = 0$

Example



$\tilde{F}$

order of DE

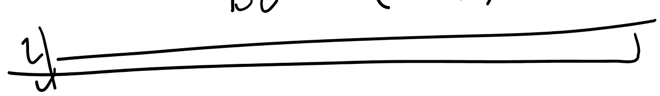
$$R_i = \underbrace{\mathcal{L}_M}_{\text{Differential operator}}(u) - \underbrace{r}_{\text{source term}}$$

Here $\mathcal{L}_M = (EA(\ )')'$ $\qquad M = 2$

$r = -q$

$M = 4$

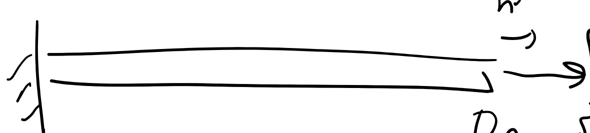
beam problem DE $(EIy'')'' + q$



$$\mathcal{L}_M(y) = (EIy'')''$$

$$\mathcal{L}_M = (EI(\ )'')''$$

$\mathcal{L}_f$: Differential operator on natural Boundary



$n = 1$ $\tilde{F}$

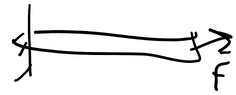
$$R_f = \tilde{F} - \mathcal{L}_f(u)\cdot n$$

$$= \tilde{F} - (EAu')\cdot n$$

$$\mathcal{L}_f = EA(\ )'$$

$\mathcal{L}_m$

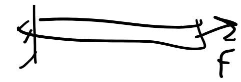
$$\int_0^L w\Big[\underbrace{(EAu')'}_{\mathcal{L}_M(u)} + q\Big]dx + w(L)(\tilde{F} - \underbrace{EAu'(L)}_{\mathcal{L}_f(u)}) = 0$$



$\tilde{F}$

- for 2D

am $\int_0 \underbrace{\quad}_{\mathcal{L}_M(u)}$ ... $\mathcal{L}_f(u)$



after IBP

weak $m=1$ $\displaystyle\int \underset{\mathcal{L}_m(w)}{w'}\, \underbrace{EA}_{D}\, \underset{\mathcal{L}_m(u)}{u'}\, dx = \int_0^L w q\, dx + w(L)\bar{F}$

$m = \frac{M}{2}$

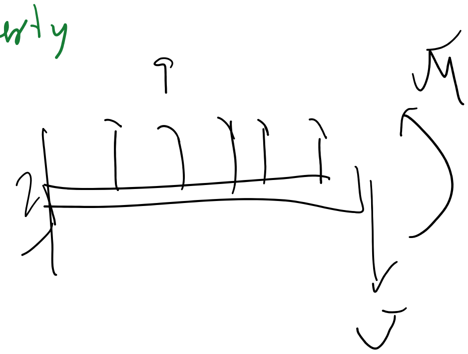
$D \rightarrow$ section - material property

Beam problem

$$\int_0^L w'' \underbrace{EI}_{D} y''\, dx = \int_0^L w q\, dx - y(L)\bar{V} + y'(L)\bar{M}$$

$\mathcal{L}_m(w) = w''$

$\mathcal{L}_m = (\ )''$



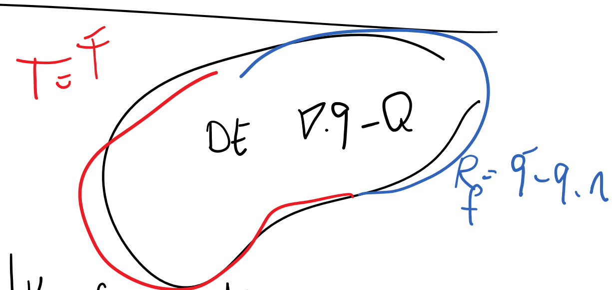
HW 2

weak statement

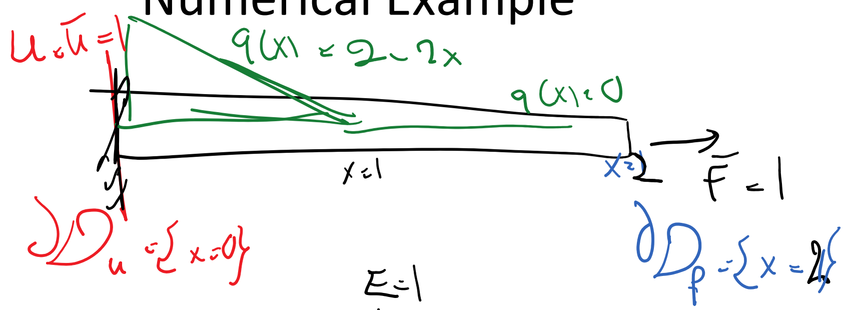
$m=1$

$$\int_{\mathcal{D}} \underbrace{\nabla w}_{\mathcal{L}_m w = \nabla w}\, \underbrace{k}_{D = k}\, \nabla T\, dV = \int_{\mathcal{D}} w Q\, dV - \int_{\partial\mathcal{D}_f} w \bar{q}\, d\Gamma$$

$\mathcal{L}_m = \nabla$



Numerical Example



WRS

$\Sigma_u \; e \; x \ldots$ $\quad E=1$ $\quad \Sigma_p = \{x = 4\}$

$A=1$

WRS

$$\int_0^2 w(x)\,\big(\underbrace{(\overset{1}{EA}u')' + q(x)}_{R_i}\big)\,dx \;+\; \underbrace{w_f}(L)\,\big(\bar{F} - \underset{1}{EA}\,u'\big)$$

this weight is equal to u for all the methods we'll solve except the least square method

$\downarrow$

$$\int_0^2 w(x)\,\big(\underbrace{u'' + q(x)}_{R_i}\big)\,dx + w_f(2)\,\big(\underbrace{\bar{F} - u'}_{R_f}\big) = 0$$