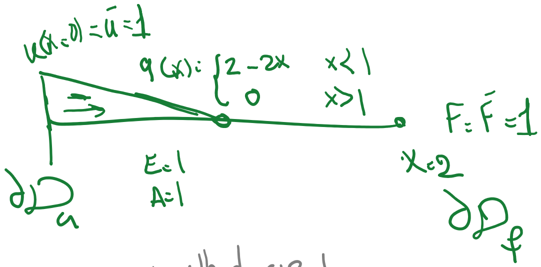
2022/09/27

Tuesday, September 27, 2022 11:18 AM

Thursday 9/29: 12:40 - 2:20 pm ET, I'll post the zoom link

We are going to solve the following problem with 2 unknowns in class



$$\int_0^L w R_i \, dx + w_f R_f = 0 \qquad (1)$$

$\int_0^L w R_i \, dx$ → interior residual

w_f → we'll see that for the Least square method we have a different weight on $\partial\Omega_f$

$$R_i = (EAu')' + q = u'' + q$$

$$R_f = \bar{F} - F = \bar{F} - AEu' = 1 - u' \qquad (2)$$

we satisfy the essential BC strongly

$$u^h = \phi_p + a_1\phi_1 + a_2\phi_2 = \phi_p + \underbrace{[\phi_1 \;\; \phi_2]}_{\phi}\begin{bmatrix} a_1 \\ a_2\end{bmatrix} \qquad (3)$$

ϕ ← basis vector; $\begin{bmatrix} a_1 \\ a_2\end{bmatrix}$ → vector of unknowns

unknowns → $n=2$

Since we have two unknowns (n = 2), we need two weights to solve the problem.

We'll discuss the choice of w's later as each one corresponds to a different method.

$$w = \begin{bmatrix} w_1 \\ w_2\end{bmatrix}, \quad w_f = \begin{bmatrix} w_{f_1} \\ w_{f_2}\end{bmatrix} \qquad (4)$$

plug (2) to (4) in WRS (1) to get

$$\int_0^2 \begin{bmatrix} w_1 \\ w_2\end{bmatrix}(u'' + q)\,dx + \begin{bmatrix} w_{f_1} \\ w_{f_2}\end{bmatrix}_{x=2}(1 - u')\Big|_{x=2} = \begin{bmatrix} 0 \\ 0\end{bmatrix}$$

$$u(x) = [\phi_1(x) \;\; \phi_2(x)]\begin{bmatrix} a_1 \\ a_2\end{bmatrix} + \phi_p(x) \rightarrow u'(x) = [\phi_1'(x) \;\; \phi_2'(x)]\begin{bmatrix} a_1 \\ a_2\end{bmatrix} + \phi_p'(x)$$

$$\rightarrow u''(x) = [\phi_1''(x) \;\; \phi_2''(x)]\begin{bmatrix} a_1 \\ a_2\end{bmatrix} + \phi_p''(x)$$

$$\int_0^2 \begin{bmatrix} w_1 \\ w_2\end{bmatrix}\left(\phi_p'' + [\phi_1'' \;\; \phi_2'']\begin{bmatrix} a_1 \\ a_2\end{bmatrix}\right)dx + \int_0^2 \begin{bmatrix} w_1 \\ w_2\end{bmatrix} q(x)\,dx + \begin{bmatrix} w_{f_1}(2) \\ w_{f_2}(2)\end{bmatrix}\left(1 - \phi_p'(2) - [\phi_1'(2) \;\; \phi_2'(2)]\begin{bmatrix} a_1 \\ a_2\end{bmatrix}\right) = 0$$

$F=1$

keep them on the LHS

$$K_{2\times2}\, a_{2\times1} = F_{2\times1} = F_r + F_N - F_D$$

F_r ↳ source term

$\phi_p(0) = \bar{u} = 1$

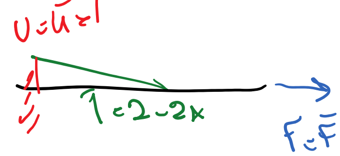
$F = 1$

$$K = \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \underbrace{\begin{bmatrix} \phi_1'' & \phi_2'' \end{bmatrix}}_{0 \quad 2} dx - \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix} \underbrace{\begin{bmatrix} \phi_1'(2) & \phi_2'(2) \end{bmatrix}}_{\begin{bmatrix} 1 & 2x \end{bmatrix}_2 = \begin{bmatrix} 1 & 4 \end{bmatrix}}$$

$$F_r = -\int_0^{1} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} q(x)\, dx$$

$$F_N = -\begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix} \bar{F} \quad \bar{F} = 1$$

$\bar{F} = 1$



$$F_D = \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \phi_p''\, dx - \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix} \phi_p'(2) \qquad (*)$$

Except the finite element method, I'll use monomials for the solution

ϕ_i's satisfy zero version of essential BC $\quad \phi_i(0) = 0$

$\{x, x^2, x^3, \dots\}$ $\quad n = 2 \rightarrow \phi = [x \quad x^2]$ $\quad \phi' = [1 \quad 2x]$

$\phi'' = [0 \quad 2]$

ϕ_p satisfies the essential BC. Let's choose $\phi_p(x) = 1$

$\phi_p' = 0 \quad \phi_p'' = 0 \rightarrow F_D = 0$

We are seeking a solution in the form $u^h(x) = \phi_p(x) + [\phi_1(x)\ \phi_2(x)]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 1 + a_1 x + a_2 x^2$

Plug $[\phi_1 \quad \phi_2] = [x \quad x^2]$ & $\phi_p = 1$ in $*$ to get

$$K = \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} [\,\underbrace{0 \quad 2}_{\phi''}\,]\, dx - \begin{bmatrix} \omega_{f_1} \\ \omega_{f_2} \end{bmatrix} [\,\underbrace{1 \quad 4}_{\phi'(2)}\,] \qquad (**)$$

$$F = -\int_0^1 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} (2 - 2x)\, dx - \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix}$$

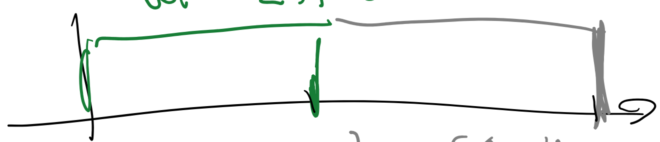
$$\underbrace{\int w_2 (2-2x)\,dx}_{F_G} \qquad \underbrace{\left[w_{f_2}(2) \right]}_{F_N}$$

(**) $Ka = F \rightsquigarrow$ find $a \to u^h = 1 + a_1 x + a_2 x^2$

Different weight functions correspond to different methods

1. Subdomain method

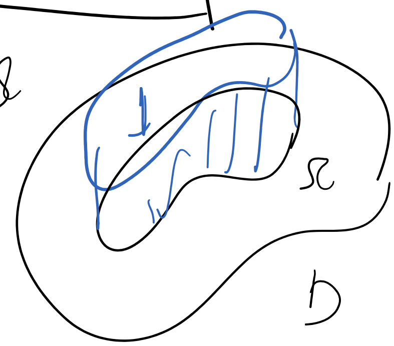
$w_1 = \chi_{[0,1]} = \begin{cases} 1 & x \le 1 \\ 0 & x > 1 \end{cases}$



$x = 0$ $\qquad$ $x = 2$

$w_2 = \chi_{[1,2]} = \begin{cases} 0 & x < 1 \\ 1 & x \ge 1 \end{cases}$

$\chi_\Omega(x) = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$



$w_f = w$

$$\underset{\text{row } i}{K_i} = \int_0^2 w_i [0 \;\; 2]\,dx - w_i(2)[1 \;\; 4]$$

row i of stiffness

$$K_1 = \int_0^1 (1)[0 \;\; 2]\,dx + \int_1^2 \cancel{(0)[0 \;\; 2]\,dx}^{\,0} - (0)[1 \;\; 4] = [0 \;\; 2]$$

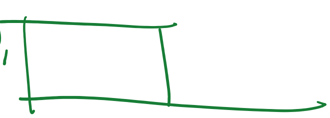
$$K_2 = \int_0^1 (0)[0 \;\; 2]\,dx + \int_1^2 (1)[0 \;\; 2]\,dx - (1)[1 \;\; 4] = [0 \;\; 2] - [1 \;\; 4] = [-1 \;\; -2]$$

$$K = \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix}, \quad F = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

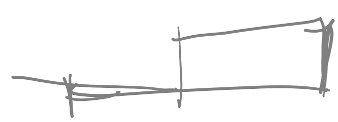
$$F_i = -\int_0^1 w_i (2 - 2x)\,dx - w_i(2)$$

$$F_1 = -\int_0^1 (1)(2-2x)\,dx - 0 = -1$$

w_1



$$F_2 = -\int_0^1 (0)(2-2x)\,dx - 1 = -1$$



$Ka = F$

$Ka = F$

$$Q = \begin{bmatrix} 2 \\ -\tfrac{1}{2} \end{bmatrix}$$

$$\underset{\text{discrete}}{u^h_{S2}} = \phi_p + [\phi_1 \;\; \phi_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 1 + [x \;\; x^2]\begin{bmatrix} 2 \\ -\tfrac{1}{2} \end{bmatrix}$$

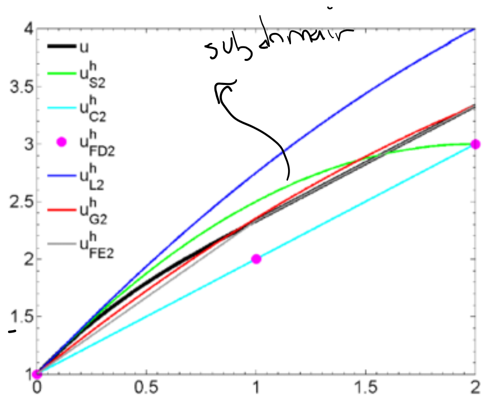
$$= 1 + 2x - \frac{x^2}{2}$$

$S2$: subdomain, $n=2$

- The exact solution can be summarized as,

$$u(x) = \begin{cases} \frac{x^3}{3} - x^2 + 2x + 1 & 0 \le x \le 1 \\ x + \frac{4}{3} & 1 < x \le 2 \end{cases} \qquad (179)$$

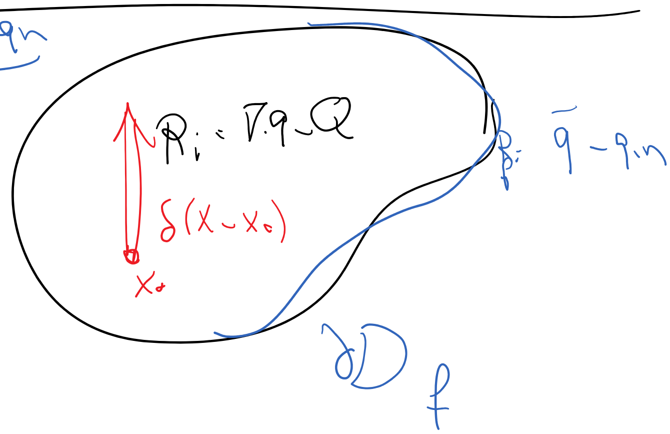
Bar example, $n = 2$, Comparison of solutions



2. Subdomain:

$$\int_D \omega R_i \, dV + \int_{\partial D_f} \omega R_f \, ds = 0$$

heat eqn



$$\int_D \delta(x - x_0) f(x) \, dV = f(x_0)$$

choose $\omega = \delta(x - x_0)$

$$\rightarrow \int_D \delta(x - x_0) R_i(x) \, dx + \int_{\partial D_f} \delta(x - x_0) R_f \, ds = 0$$

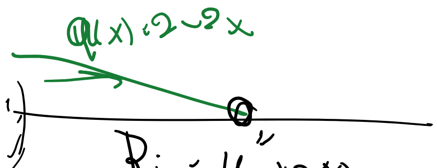
$$\longrightarrow \underbrace{\int_{\Omega} \delta(x - x_0)\, R_i(x)\,dx}_{R_i(x_0) \,=\, 0} + \int_{\partial\Omega_f} \ldots$$

we're satisfying the PDE @ x_0

Collocation means we satisfy a given equation at a few sample points.

Advantage to other WRS methods:
  - We don't need to do any integration.

Disadvantage:
  - We may miss out the key actions (where the loads are applied etc) as we'll see below



$R_i = u'' + q(x)$ for $x \in [0, 2]$

$\longrightarrow F = EAu' = \bar{F} = 1$

$R_f = 1 - u'$, $x = 2$

$u^h = 1 + a_1 x + a_2 x^2$

Eq 1 $\quad R_i(x=1) = 0$

Eq 2 $\quad R_f = 0 \quad @ \quad x = 2$

Eq 1 $\quad u'' + q(x=1) = 0 \longrightarrow \underbrace{\phi_p''(1)}_{0} + \left[\underbrace{\phi_1''(1)}_{0} \quad \underbrace{\phi_2''(1)}_{2}\right] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$

$\boxed{2a_2 = 0} \rightsquigarrow \boxed{\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = [0]}$

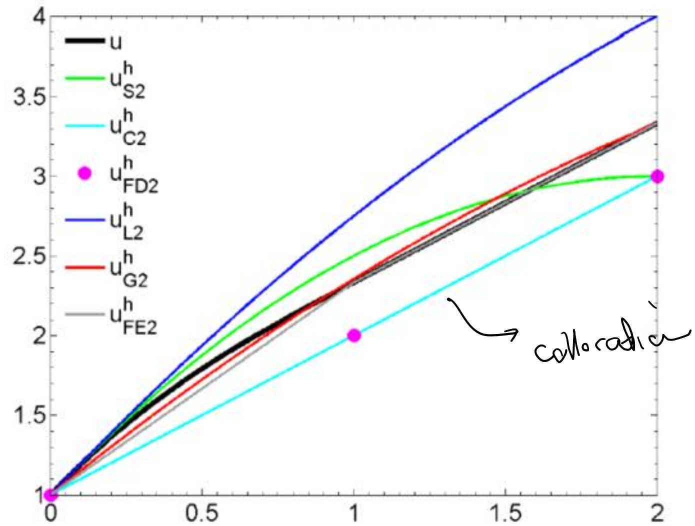
Eq 2 $\quad R_f = \bar{F} - EAu' = 1 - u' = 1 - \begin{bmatrix} \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\Bigg|_{x=2} = 1 - \begin{bmatrix} 1 & 2x \end{bmatrix}\Big|_{x=2} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$

$\begin{bmatrix} -1 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = [-1]$

$$\begin{bmatrix} 0 & 2 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \longrightarrow \boxed{\begin{aligned} u^h_{C_2} &= 1 + 1\,x + 0\,x^2 \\ &= 1 + x \end{aligned}}$$

also $u^h_{C_1}$ I have in the notes

Bar example, $n = 2$, Comparison of solutions



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Finite Difference

$u(x=0) = \bar{u} = 1$

$F = \bar{F}$

We need 2 equations

Choose the equations like collocation method

$u_0 = 1$ ✓ $\quad u_1 = ? \quad u_2 = ?$

$x = 0 \quad x = 1 \quad x = 2$

Eq 1) $R_i(x=1) = 0$

Eq 2 $R_e = 0$ @ $x = 2$

Eq 1 $\quad u'' + q(1) = 0 \quad x = 1$ $\quad q(1) \to 0$

Eq 2 $\quad 1 - u' = 0$

$u_0 \quad u_1 \quad u_2$

$u_0 \quad u_1 \quad u_2$

$u_0 \overset{h}{—} u_1 \overset{h}{—} u_2$

Eq 1 $\quad u''(x_1) = \dfrac{u_2 + u_0 - 2u_1}{h^2} = 0 \quad u_0 = 1$

$\boxed{h = 1}$

Eq 2

h

solution: $u_0 \quad u_1 \quad u_2$

position: $x_0 = 0 \quad x_1 = 1 \quad x_2 = 2$

$$u' \approx \frac{u_2 - u_1}{h} = 1$$

$$\begin{cases} 1 + u_2 - 2u_1 = 0 & \text{eq 1} \\ u_1 - u_2 = -1 & \text{eq 2} \end{cases}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$u_1 = 2$ Recall $u_0 = 1$

$u_2 = 3$