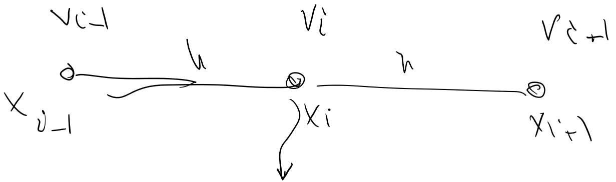
2022/09/29

Thursday, September 29, 2022 11:25 AM

Finite Difference formulas

function f takes these values



$$f'(x_i) \approx \frac{v_i - v_{i-1}}{h}$$

backward formula

$$\approx \frac{v_{i+1} - v_i}{h}$$

forward formula

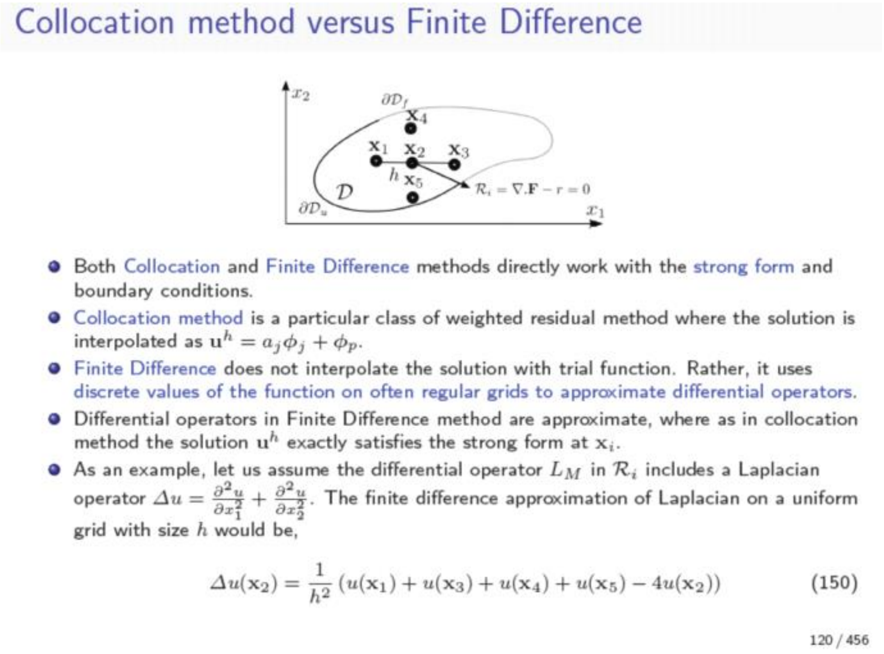
$$\approx \frac{v_{i+1} - v_{i-1}}{2h}$$

central more accurate

$$f''(x_i) = \frac{f'(x_i) - f'(x_{i-1})}{h}$$

$$= \frac{\left(\left(\frac{v_{i+1} - v_i}{h}\right) - \left(\frac{v_i - v_{i-1}}{h}\right) \right)}{h}$$

$$= \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}$$



Collocation method versus Finite Difference

- Both Collocation and Finite Difference methods directly work with the strong form and boundary conditions.
- Collocation method is a particular class of weighted residual method where the solution is interpolated as $\mathbf{u}^h = a_j\phi_j + \phi_p$.
- Finite Difference does not interpolate the solution with trial function. Rather, it uses discrete values of the function on often regular grids to approximate differential operators.
- Differential operators in Finite Difference method are approximate, where as in collocation method the solution $\mathbf{u}^h$ exactly satisfies the strong form at $\mathbf{x}_i$.
- As an example, let us assume the differential operator L_M in $\mathcal{R}_i$ includes a Laplacian operator $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$. The finite difference approximation of Laplacian on a uniform grid with size h would be,

$$\Delta u(\mathbf{x}_2) = \frac{1}{h^2}\left(u(\mathbf{x}_1) + u(\mathbf{x}_3) + u(\mathbf{x}_4) + u(\mathbf{x}_5) - 4u(\mathbf{x}_2)\right) \quad (150)$$

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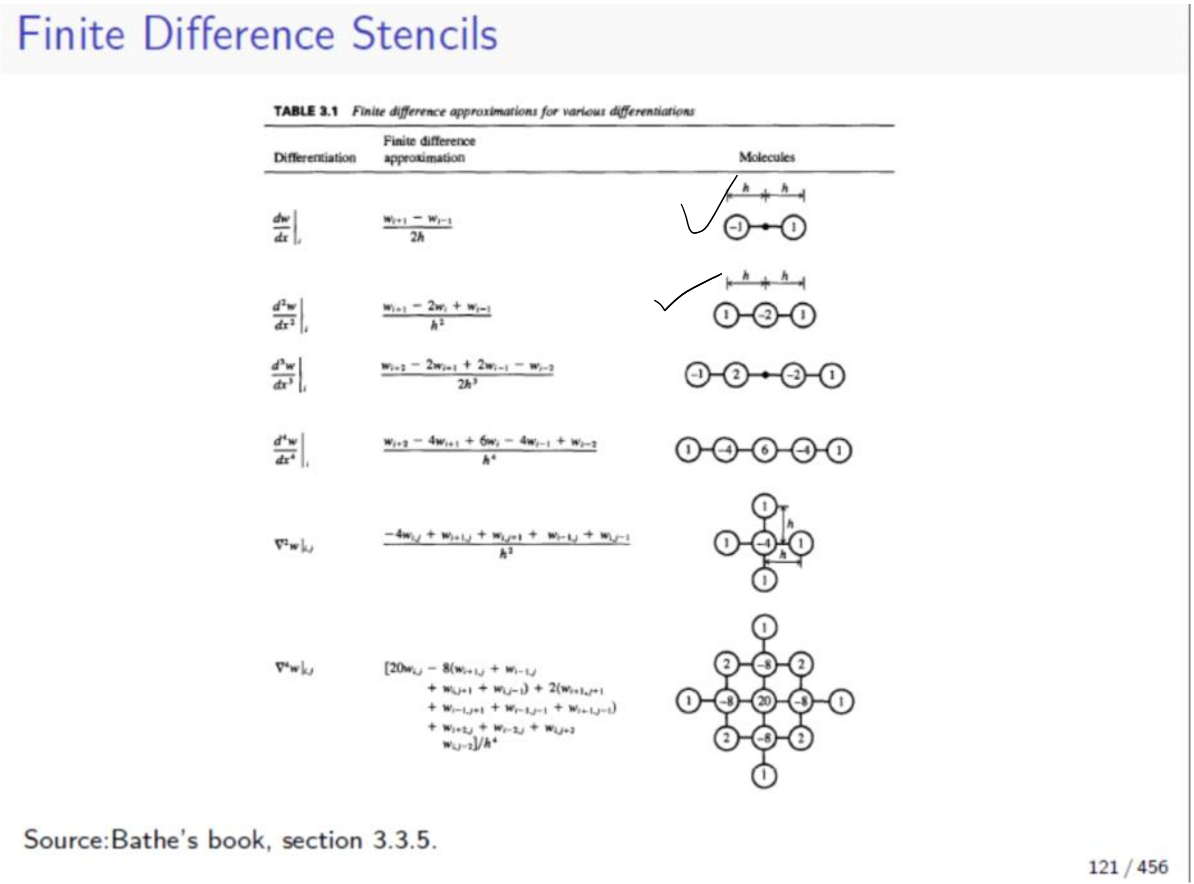


Finite Difference Stencils

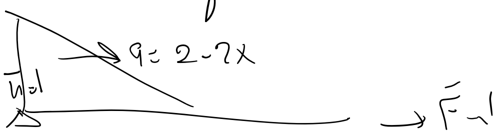
TABLE 3.1 *Finite difference approximations for various differentiations*

Differentiation	Finite difference approximation	Molecules
$\left.\frac{dw}{dx}\right\|_i$	$\frac{w_{i+1} - w_{i-1}}{2h}$	
$\left.\frac{d^2w}{dx^2}\right\|_i$	$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$	
$\left.\frac{d^3w}{dx^3}\right\|_i$	$\frac{w_{i+2} - 2w_{i+1} + 2w_{i-1} - w_{i-2}}{2h^3}$	
$\left.\frac{d^4w}{dx^4}\right\|_i$	$\frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{h^4}$	
$\nabla^2 w\|_{i,j}$	$\frac{-4w_{i,j} + w_{i+1,j} + w_{i,j+1} + w_{i-1,j} + w_{i,j-1}}{h^2}$	
$\nabla^4 w\|_{i,j}$	$[20w_{i,j} - 8(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}) + 2(w_{i+1,j+1} + w_{i-1,j+1} + w_{i-1,j-1} + w_{i+1,j-1}) + w_{i+2,j} + w_{i-2,j} + w_{i,j+2}$ $w_{i,j-2}]/h^4$	

Source:Bathe's book, section 3.3.5.

Galerkin method:

$$W = \phi$$



Recall

$$K = \int_0^2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} [0 \quad 2]\, dx - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} [1 \quad 4] \Big|_{x=2}$$

$$F = -\int_0^1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} (2-2x)\, dx - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Big|_{x=2}$$

$$u^h = \underset{=1}{\phi_p} + a_1\phi_1 + a_2\phi_2 \; \underline{= 1 + a_1 x + a_2 x^2} \qquad \phi_1 = x, \; \phi_2 = x^2$$

Galerkin $(w=\phi)$ $\quad w_1 = \phi_1 = x \;, \; w_2 = \phi_2 = x^2$

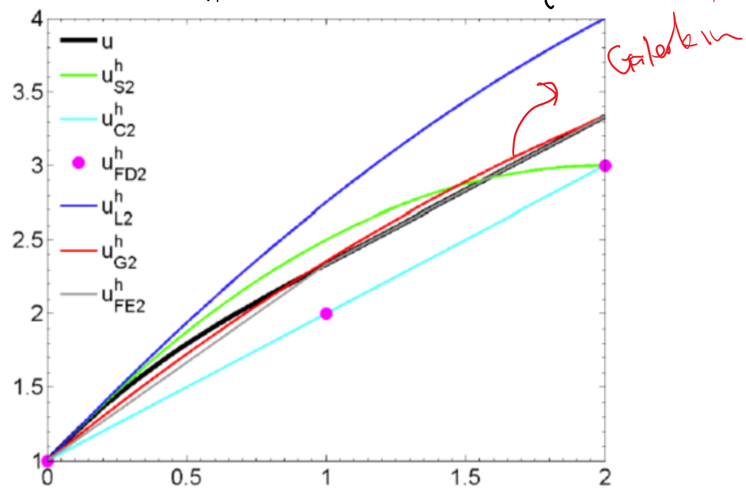
$$K = \int_0^2 \begin{bmatrix} x \\ x^2 \end{bmatrix} [0 \quad 2]\, dx - \begin{bmatrix} x \\ x^2 \end{bmatrix}\Big|_{x=2} [1 \quad 4] = \begin{bmatrix} -2 & 4 \\ -4 & -32/3 \end{bmatrix}$$

$$F = -\int^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} (2-2x)\, dx - \begin{bmatrix} x \\ x^2 \end{bmatrix}\Big|_1 - \;\rightarrow F = \begin{bmatrix} -7/3 \\ -28/ \end{bmatrix}$$

$$F = -\int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} (2-2x)\,dx - \begin{bmatrix} x \\ x^2 \end{bmatrix}\Big|_{x=2} \rightarrow F = \text{Ri}\begin{bmatrix} -7/3 \\ 25/6 \end{bmatrix}$$

$$Ka = F \quad \rightsquigarrow \quad a = \begin{bmatrix} 37/24 \\ -3/16 \end{bmatrix}$$

$$u^h_{G2} = \underset{\substack{\downarrow \\ \phi_p}}{1} + \underset{\substack{\downarrow \\ a_1}}{\tfrac{37}{24}} x - \underset{\substack{\downarrow \\ a_2}}{\tfrac{3}{16}} x^2$$



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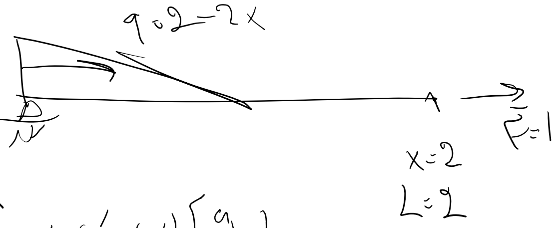
Do the same Galerkin choice but with the weak statement approach rather than WRS:

$$\int_0^{L=2} w' \underbrace{EA}_{\overset{\parallel}{1}} u'\,dx = \int_0^L w\, q(x)\,dx + w(L)\bar{F} \qquad u = \bar{u} = 1$$



$$u_h = \phi_p + [\phi_1 \;\; \phi_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow u_h' = \phi_p' + [\phi_1' \;\; \phi_2']\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$w = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad w' = \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix}$$

$$\int \underbrace{\begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix}}_{w'} EA\left(\phi_p' + [\phi_1' \;\; \phi_2']\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) dx = \int_0^L \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} q(x)\,dx + \begin{bmatrix} \phi_1(L) \\ \phi_2(L) \end{bmatrix} \bar{F}$$

$$Ka = F_r + F_N - F_D$$

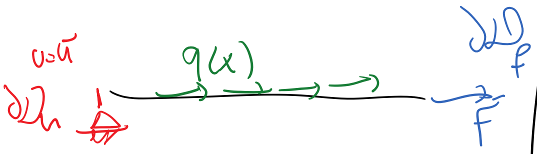
$$K = \int^L \begin{bmatrix} \phi_1' \\ \vdots \end{bmatrix} EA [\phi_1' \;\; \phi_2']\,dx = \int^L EA \begin{bmatrix} \phi_1'\phi_1' & \phi_1'\phi_2' \\ \phi_2'\phi_1' & \phi_2'\phi_2' \end{bmatrix} dx$$

$$K = \int_0^L \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} EA \begin{bmatrix} \phi_1' & \phi_2' \end{bmatrix} dx = \int_0 EA \begin{bmatrix} \phi_1'\phi_1 & \phi_1 \varphi_2 \\ \phi_2'\phi_1' & \phi_2'\phi_2' \end{bmatrix} dx$$

$$F_r = \int_0^L \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} q(x)\, dx$$ force from the source term

$$F_N = \begin{bmatrix} \phi_1(L) \\ \phi_2(L) \end{bmatrix} \bar{F}$$

$$F_D = \int_0^L \begin{bmatrix} \phi_1' \\ \varphi_2' \end{bmatrix} EA \phi_p' \, dx$$



①

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use $q(x) = \begin{cases} 2-2x & x<1 \\ 0 & x>1 \end{cases}$, $L=2$, $EA=1$, $\bar{F}=1$

$\phi_p = 1$ $\phi_1 = w_1 = x$, $\phi_2 = w_2 \cancel{=} x^2$ in ① to get

$$K = \int_0^2 \begin{bmatrix} 1 \\ 2x \end{bmatrix} \begin{bmatrix} 1 & 2x \end{bmatrix} dx = \begin{bmatrix} 2 & 4 \\ 4 & 32/3 \end{bmatrix}$$

$$F = \underbrace{\int_0^1 \underset{\downarrow \atop w=\phi}{\begin{bmatrix} x \\ x^2 \end{bmatrix}} \underset{\downarrow \atop q(x)}{(2-2x)}\, dx}_{F_r} + \underbrace{\begin{bmatrix} x \\ x^2 \end{bmatrix} \cdot 1 \Big|_{x=2}}_{F_N} - \underbrace{\int_0^2 \begin{bmatrix} 1 \\ 2x \end{bmatrix} \overset{EA}{(1)} (0)\, dx}_{F_D}$$

$\phi_p' = 1' = 0$

$$\begin{bmatrix} 1/3 \\ 1/6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 25/6 \end{bmatrix}$$

$$a = \begin{bmatrix} 37/24 \\ -3/16 \end{bmatrix} \quad u^h_{G2} = 1 + \frac{37}{24}x - \frac{3}{16}x^2$$

↓
with weak approach

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weak statement is better

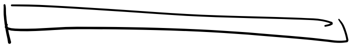
But we need to be able to evaluate the integral

$$\int_0^L w' EA u' \, dx$$

↓ we need to have this

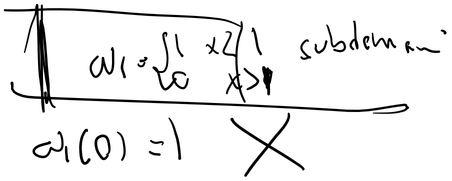
$\Downarrow$ $\downarrow$ we need to have this

$$\underline{\omega(\partial\Omega_u) = 0}$$



$\partial\Omega_u = \{x=0\}$

$\omega(0) = 0$



$\omega_1 = \begin{cases}1 & x<1 \\ 0 & x>1\end{cases}$ subdomain

$\omega_1(0) = 1$ ✗

$W = \phi$

$$\phi(\partial\Omega_u) = 0$$

is this always true

$$u^h = \phi_p + \sum_{i=1}^{n} a_i \phi_i(x)$$

$\forall x \in \partial\Omega_u$ $\phi_p(x) = \bar{u}(x)$

$\phi_i(x) = 0$ Galerkin $\omega = \phi$ so $W_i(x) = 0$

Galerkin method can always be discretized in the weak form. :)

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satisfies ess. BC

satisfy the zero of ess. BC

Ritz method

This is an older method than the FEM.
The idea:

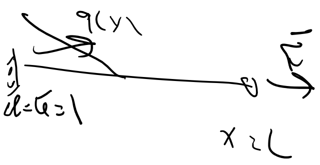
1. Discretize the problem $\rightarrow$
2. Calculate the energy
3. Minimize the energy

$$u^h(x) = \phi_p(x) + \underbrace{a_1\phi_1(x) + a_2\phi_2(x)}$$

$\phi_1 \rightarrow x$, $\phi_2 \rightarrow x^2$

choose $\phi_p(x) = 1$

$\downarrow$ strongly satisfies the "essential" BC



① $\underline{u^h(x) = 1 + a_1 x + a_2 x^2} \rightsquigarrow$ $u^{h\prime}(x) = a_1 + 2a_2 x$

② $\Pi(u^h) = V(u) - W(u) =$

$$\underbrace{\int_0^L \frac{1}{2} EA u^{h\prime 2}\, dx}_{V} - \underbrace{\left(\int_0^L u^h(x) q(x)\, dx + u^h(L)\bar{F}\right)}_{W}$$

plug u^h from ① in ②

$$\Pi = \int_0^2 \frac{1}{2}(1)\big(\underbrace{a_1 + 2a_2 x}_{u^{h\prime}}\big)^2 dx - \int_0^1 \big(\underbrace{1 + a_1 x + a_2 x^2}_{u^h}\big)\underbrace{(2-2x)}_{q(x)}\, dx$$

$$- (1 + a_1 x + a_2 x^2)\Big|_{x=2} \cdot \underset{\bar{F}}{1}$$

$$\Pi(a_1, a_2) = \left(a_1^2 + 4a_1 a_2 + \frac{16}{3} a_2^2\right) - \left(\frac{7}{3} a_1 + \frac{25}{6} a_2 + 2\right)$$

③ Minimize Π $\rightarrow$ $\nabla \Pi = \begin{bmatrix} \partial\Pi/\partial a_1 \\ \partial\Pi/\partial a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_1 + 4a_2 - 7/3 \\ 4a_1 + \frac{32}{3} a_2 - \frac{25}{6} \end{bmatrix}$

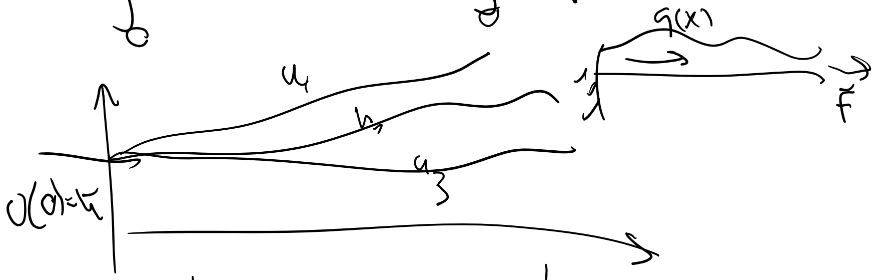
$\rightarrow$ $Ka = F$ $\quad K = \begin{bmatrix} 2 & 4 \\ 4 & 32/3 \end{bmatrix}$ $\quad F = \begin{bmatrix} 7/3 \\ 25/6 \end{bmatrix}$

$\rightarrow$ $a = \begin{bmatrix} 37/24 \\ -3/16 \end{bmatrix} \rightarrow u^h = 1 + \frac{37}{24} x - \frac{3}{16} x^2$

How did we get the discretized weak statement

① $\Pi(u^h) = \int_0^L \frac{1}{2} EA u'^2\, dx - \int_0^L u q\, dx - u(L)\bar{F}$

Continuum



② Minimized $\int_0^L \delta u' EA u'\, dx - \int_0^L (\delta u) q\, dx - \delta u(L)\bar{F} = 0$

Minimized

$$\int_0 \delta u' EAu' dx - \int_0 (\delta u) q dx - \delta u(L) F = 0$$

$\delta u \rightarrow \omega$

$u(0) = \bar{u}$ $\delta u(0) = \omega(0) = 0$

$$\underbrace{\int_0^L w' EAu' dx = \int_0^L \omega q dx + w(L) F}_{\text{weak statement}}$$

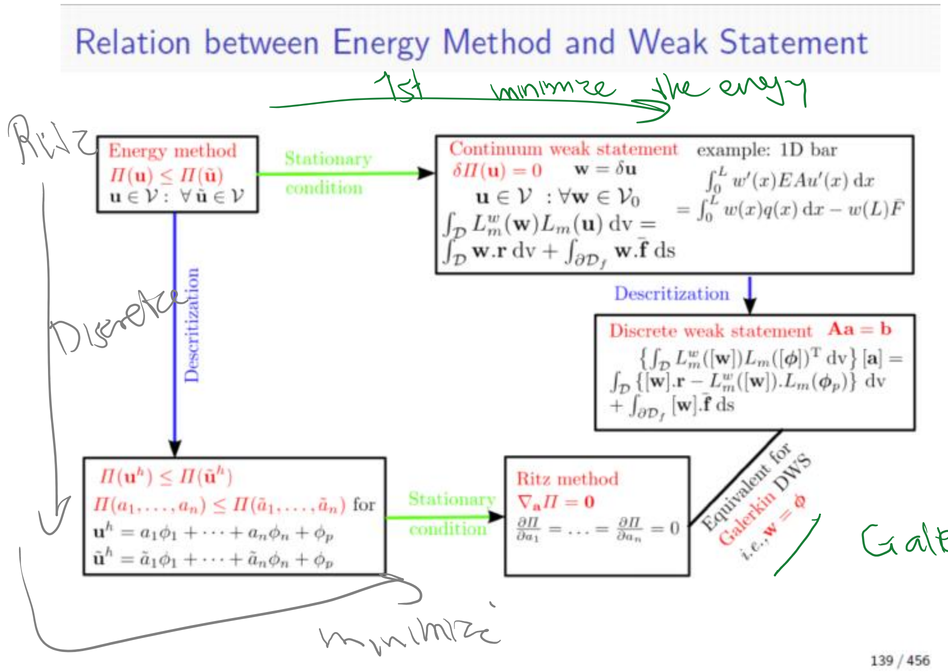
③ Discretize

$$u^h = \phi_p + \phi_1 a_1 + \phi_2 a_2 \longrightarrow 1 + a_1 x + a_2 x^2$$

$\omega_1 = \phi_1 = x$

$\omega_2 = \phi_2 = x^2$

this was the weak statement + Galerkin solution earlier



Relation between Energy Method and Weak Statement

1st minimize the energy

Ritz

Energy method
$\Pi(\mathbf{u}) \le \Pi(\tilde{\mathbf{u}})$
$\mathbf{u} \in \mathcal{V}: \forall \tilde{\mathbf{u}} \in \mathcal{V}$

Stationary condition

Continuum weak statement example: 1D bar
$\delta\Pi(\mathbf{u}) = 0$ $\mathbf{w} = \delta\mathbf{u}$
$\mathbf{u} \in \mathcal{V} : \forall \mathbf{w} \in \mathcal{V}_0$
$\int_{\mathcal{D}} L_m^w(\mathbf{w}) L_m(\mathbf{u})\, \mathrm{dv} =$
$\int_{\mathcal{D}} \mathbf{w}.\mathbf{r}\, \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{f}}\, \mathrm{ds}$

$\int_0^L w'(x) EAu'(x)\, dx$
$= \int_0^L w(x) q(x)\, dx - w(L)\bar{F}$

Descritization

Discretize

Descritization

Discrete weak statement $\mathbf{Aa} = \mathbf{b}$
$\{\int_{\mathcal{D}} L_m^w([\mathbf{w}]) L_m([\boldsymbol{\phi}])^{\mathrm{T}}\, \mathrm{dv}\}\, [\mathbf{a}] =$
$\int_{\mathcal{D}} \{[\mathbf{w}].\mathbf{r} - L_m^w([\mathbf{w}]).L_m(\boldsymbol{\phi}_p)\}\, \mathrm{dv}$
$+ \int_{\partial\mathcal{D}_f} [\mathbf{w}].\bar{\mathbf{f}}\, \mathrm{ds}$

$\Pi(\mathbf{u}^h) \le \Pi(\tilde{\mathbf{u}}^h)$
$\Pi(a_1, \ldots, a_n) \le \Pi(\tilde{a}_1, \ldots, \tilde{a}_n)$ for
$\mathbf{u}^h = a_1\phi_1 + \cdots + a_n\phi_n + \phi_p$
$\tilde{\mathbf{u}}^h = \tilde{a}_1\phi_1 + \cdots + \tilde{a}_n\phi_n + \phi_p$

Stationary condition

Ritz method
$\nabla_{\mathbf{a}}\Pi = \mathbf{0}$
$\frac{\partial\Pi}{\partial a_1} = \ldots = \frac{\partial\Pi}{\partial a_n} = 0$

Equivalent for Galerkin DWS i.e., $\mathbf{w} = \boldsymbol{\phi}$

minimize

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2nd discretized

Galerkin (① minimize ② Discretize)

Finite Element Method

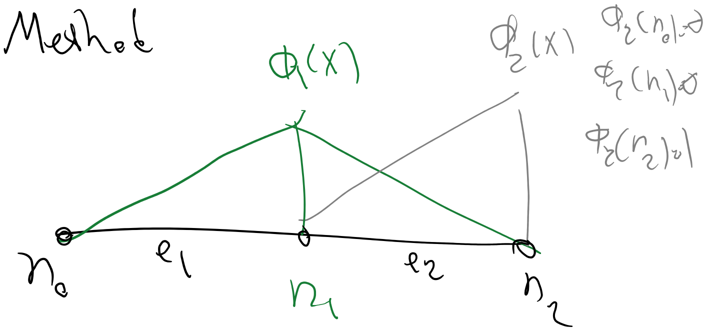
Finite element method is a Galerkin method with particular form of basis functions

$\downarrow$

$\omega = \phi$

$n = 2$

$W_1 = \Phi_1$



$W_1 = \Phi_1$

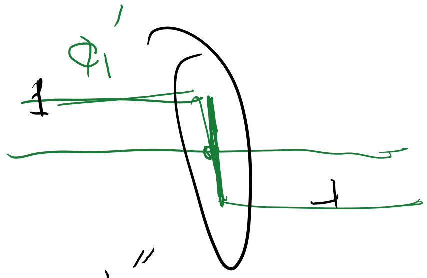
$W_2 = \Phi_2$

$\phi_1(n_1) = 1 \quad \phi_1(n_2) = 0 \quad \phi_1(n_3) = 0$

Can we use FEM basis functions in the WRS

$$\int w EA u'' dx$$

$$u^{h''} = \phi_p'' + a_1 \phi_1'' + a_2 \phi_2$$



$$\phi_1'' = -2\delta(x-1)$$

Generally, we cannot use the WRS for FEM, because the solution is not smooth enough
We can however use the weak statement (which is always better if available)
Because FEM is a Galerkin method, we readily can use the weak statement

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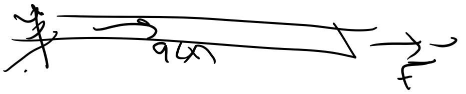
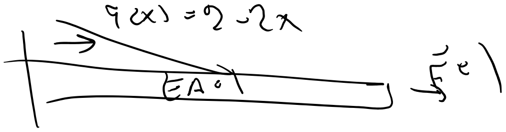
FEM + weak statement



$$K = \int \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} EA [\phi_1' \ \phi_2'] dx$$

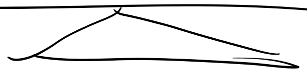
$$F = \int_0^L \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} q(x) dx + \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{x=L} F$$



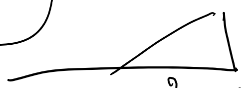
$$K = \int_0^L \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} [\phi_1' \ \phi_2'] dx$$

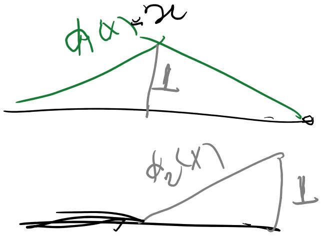
$$F = \int_0^1 \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} (2-2x) dx + \begin{bmatrix} \phi_1(2) \\ \phi_2(2) \end{bmatrix}$$

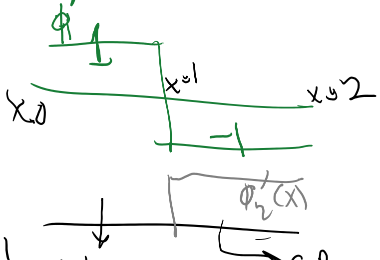
$\Phi_1 =$



$, \Phi_2 =$







$$K = \int_0^2 \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} [\phi_1' \ \phi_2'] dx$$

$$= \int_0^1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \ 0] dx + \int_1^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} [-1 \ 1] dx = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \int_0 \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}dx + \int_1 \begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}dx = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$K = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$$

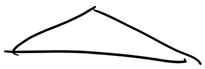
$$+\begin{bmatrix}\phi_1\\\phi_2\end{bmatrix}_{1\times 2}$$

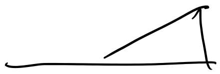
$$F = \int_0^1 \begin{bmatrix}\phi_1\\\phi_2\end{bmatrix}(2-2x)\,dx = \int_0^1 \begin{bmatrix}x\\0\end{bmatrix}(2-2x)\,dx + \begin{bmatrix}0\\1\end{bmatrix}$$

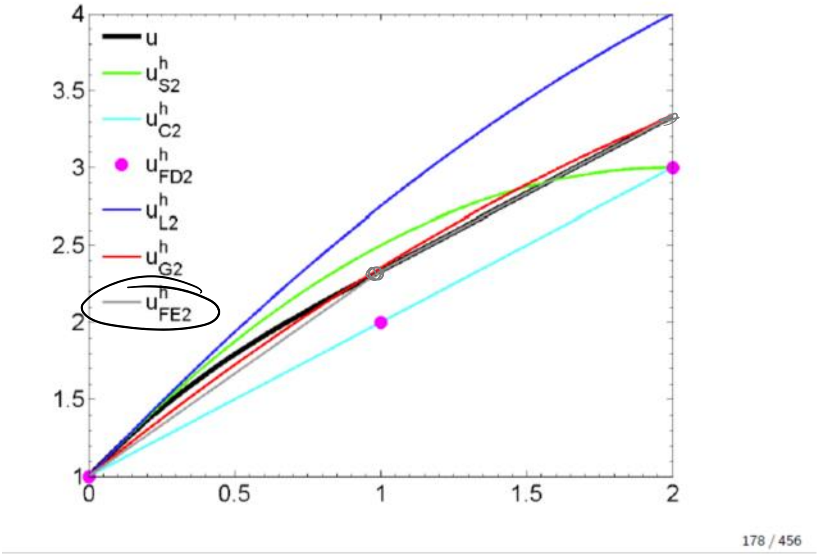
$$= \begin{bmatrix}1/3\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1/3\\1\end{bmatrix} \qquad Ka = F$$

$$\rightarrow \quad a = \begin{bmatrix}4/3\\7/3\end{bmatrix} \qquad u^h = \cancel{1 + \frac{4}{3}x + \frac{7}{3}x^2}$$

$$= 1 + \frac{4}{3}\phi_1(x) + \frac{7}{3}\phi_2(x)$$

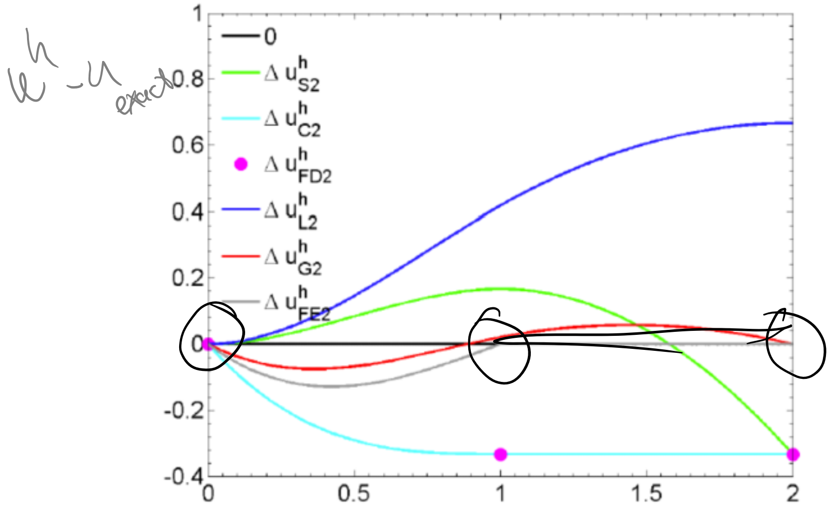






Solution

Bar example, $n = 2$, Comparison of solutions



FEM gets the exact solution @ nodes for 1D

Hughes' book for the proof