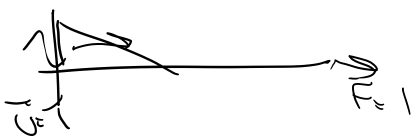
2022/10/11

Tuesday, October 11, 2022 11:21 AM

Spectral methods are ideal for simple geometries (circle, sphere, rectangle, …) and homogeneous material properties as one can take advantage of their exponential convergence properties and even very nice, e.g. diagonal, stiffness matrix form by using special polynomial spaces.

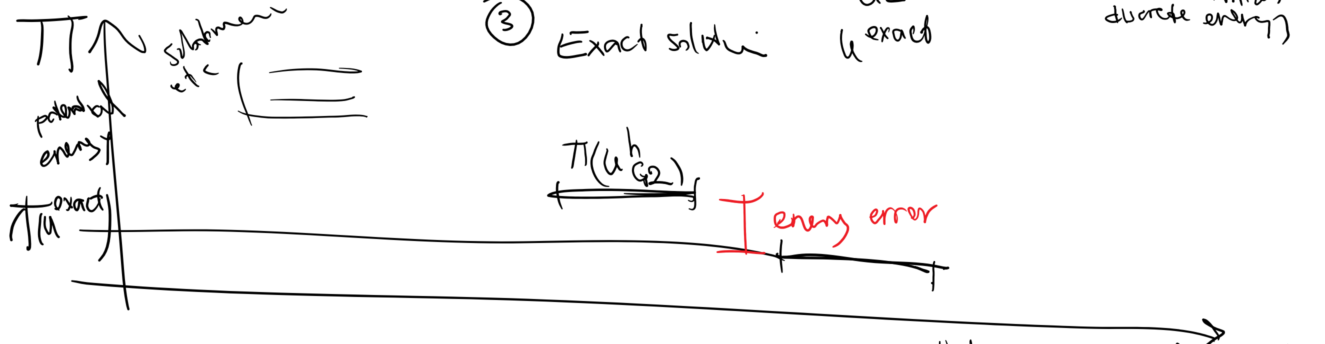


$n=2$ solutions $u^h = \phi_1 a_1 + \phi_2 a_2$

① Subdomain → u^h_{S2}

② Galerkin u^h_{G2} ⟹ Ritz solution which minimizes discrete energy

③ Exact solution u^{exact}

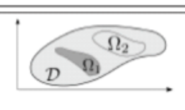
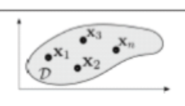
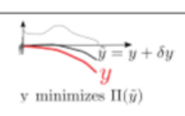


Galerkin method has the lowest energy in discrete methods

In fact the energy error is minimum for Galerkin methods

$\Pi(u^h) - \Pi(u)$ is minimum

energy of the error $\Pi(u^h - u)$ is also minimum

Approach	Equation	Figure	Discretization	Discretization method
Balance Law (20)	$\forall\Omega \subset \mathcal{D}: \int_{\partial\Omega}(\mathbf{f}.\mathbf{n})ds - \int_{\Omega} \mathbf{r}\, dv = 0$		Change $\forall\Omega$ to $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$	Similar to subdomain method in WRM
Strong Form (23)	$\forall \mathbf{x} \in \mathcal{D}: \nabla.\mathbf{f} - \mathbf{r} = \mathbf{0}$		Change $\forall\mathbf{x}$ to $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$	Collocation method in WRM. Also FD & FV.
Energy Method (80)	$\forall\tilde{y} \in \mathcal{V}: \ \Pi(y) \le \Pi(\tilde{y})$		$\forall\{\tilde{a}_1, \ldots, \tilde{a}_n\}$: $\Pi(a_1, \ldots, a_n) \le \Pi(\tilde{a}_1, \ldots, \tilde{a}_n) \Rightarrow \frac{\partial\Pi}{\partial a_1} = \cdots = \frac{\partial\Pi}{\partial a_n} = 0$	Ritz Energy Method. Also yields Weak Form.

Subdomain

Galerkin

Ritz

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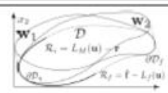
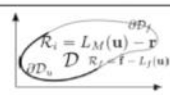
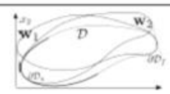
Approach	Equation	Figure	Discretization	Discretization method
Weighted Residual Method (45)	$\forall \mathrm{w} \in \mathcal{W}$: $\int_{\mathcal{D}} \mathrm{w}.\mathcal{R}_i \; \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathrm{w}^f.\mathcal{R}_f \; \mathrm{ds} = 0$		Change $\forall \mathrm{w}$ to $\{\mathrm{w}_1, \mathrm{w}_2, \dots, \mathrm{w}_n\}$	Weighted Residual Method (WRM)
Least Square (51)	$R^2 = \int_{\mathcal{D}} \mathcal{R}_i^2 \; \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathcal{R}_f^2 \; \mathrm{ds} = 0$		Change $R^2 = 0$ to $\forall\{\tilde{a}_1, \dots, \tilde{a}_n\}$: $R^2(a_1, \dots, a_n) \le R^2(\tilde{a}_1, \dots, \tilde{a}_n) \Rightarrow \frac{\partial R^2}{\partial a_1} = \dots = \frac{\partial R^2}{\partial a_n} = 0$	Least Square method, a WRM for linear L_M (& L_f).
Weak Form (74)	$\forall \mathrm{w} \in \mathcal{W}$ $\int_{\mathcal{D}} L_m^w(\mathrm{w}) L_m(\mathrm{u}) \; \mathrm{dv} = \int_{\mathcal{D}} \mathrm{w}.\mathrm{r}\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathrm{w}.\bar{\mathrm{f}}\mathrm{ds}$		Change $\forall \mathrm{w}$ to $\{\mathrm{w}_1, \mathrm{w}_2, \dots, \mathrm{w}_n\}$	Weak Formulation

→ Minimizes the error is PDE & natural BC

→ Galerkin methods

FYI section

Appendix: Function spaces (optional)

- We define the function spaces

$$C^k(\mathcal{D}) = \{f \mid f \text{ and } \frac{\partial^i f}{\partial \mathrm{x}^i} \text{ exist and are continuus } \forall 0 < i \le k \;\wedge \mathrm{x} \in \mathcal{D}\} \tag{274}$$

$$C^0(\mathcal{D}) = \text{ continuus functions on } \mathcal{D}$$

$$C^1(\mathcal{D}) \models \text{ functions with continuus derivative on } \mathcal{D}$$

$$C^\infty(\mathcal{D}) = \text{ infinitely differentiable function on } \mathcal{D}$$

We have the following in the weak statement of a bar

$$\int_0^L w' EA u' \, dx = - -$$

$$\left|\int_0^L w' EA u' \, dx\right| \le \max_{x\in[0,L]}(EA) \left|\int_0^L w' u' \, dx\right| \le \sqrt{\int_0^L (w')^2 dx} \sqrt{\int_0^L (u')^2 dx}$$

Cauchy-Schwarz Inequality

All needed to be able to evaluate the weak statement integral above is to have

$$\sqrt{\int_0^L w'^2 dx} < \infty$$

$$\sqrt{\int_0^L u'^2 dx} < \infty$$

All needed is that total function be H^1

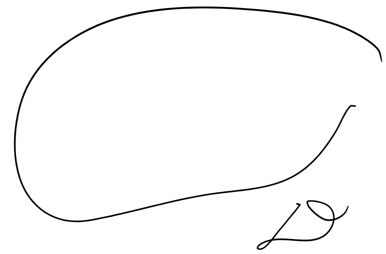
$\sqrt{\int_0^L u'^2 \, dx} < \infty$ be H^1

L^2 functions : function absolute value is integrable

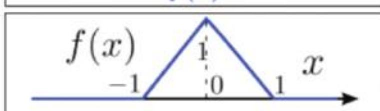
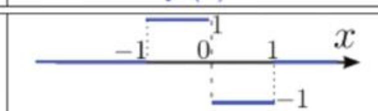
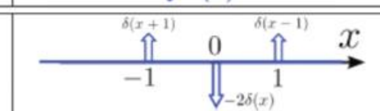
$\int_\Omega |f|^2 \, dx < \infty$

H^1 function if f & ∇f are square integrable

$$\int_\Omega \left(|f|^2 + \left|\frac{\partial f}{\partial x_1}\right|^2 + \left|\frac{\partial f}{\partial x_2}\right|^2 \right) dV < \infty$$



Comparison of C^k and Sobolev spaces

$f(x)$	$f'(x)$	$f''(x)$
		
$C^0(\mathbb{R})$	$C^1(\mathbb{R})$	$C^2(\mathbb{R})$
Yes	No no derivatives at $\{-1, 0, 1\}$	No not a C^0
$H^0(\mathbb{R}) = L^2(\mathbb{R})$	$H^1(\mathbb{R})$	$H^2(\mathbb{R})$
Yes $\int_{-\infty}^{\infty} (f(x))^2 \, dx = \frac{2}{3} < \infty$	Yes $\int_{-\infty}^{\infty} (f(x))^2 \, dx = \frac{2}{3} < \infty$	No $\int_{-\infty}^{\infty} (f(x))^2 \, dx = \frac{2}{3} < \infty$
	$\int_{-\infty}^{\infty} (f'(x))^2 \, dx = 2 < \infty$	$\int_{-\infty}^{\infty} (f'(x))^2 \, dx = 2 < \infty$
		$\int_{-\infty}^{\infty} (f''(x))^2 \, dx =$ $\int_{-\infty}^{\infty} (\delta(x+1))^2 \, dx +$ $\int_{-\infty}^{\infty} (2\delta(x))^2 \, dx +$ $\int_{-\infty}^{\infty} (\delta(x-1))^2 \, dx$ Not Defined

Previously, I mentioned that in the Weak statement because of 1 derivative we needed C1 functions for weight and solution

$\int w' E A u' \, dx$

However in finite element, the basis functions are C0! And still the method works

Conventional (continuus) finite element methods:

Strong Form order $M = 2m$ $\Rightarrow$
Trial functions are C^{m-1}

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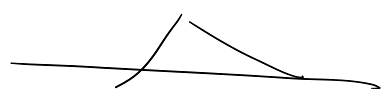
$(EAu')' + q = 0$
bar problem

$M = 2$

$m = \frac{M}{2} = 1$

$C^{1-1} = C^0$ functions are good for basis functions



Beam problem

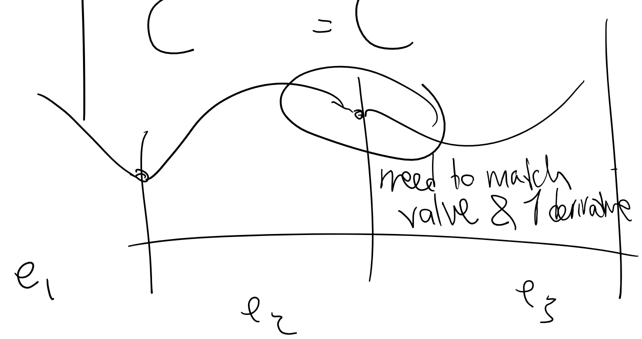
$(EIy'')'' + q = 0$

$M = 4$

$m = \frac{M}{2} = 2$

$\cancel{C^2}$ previously

$C^{2-1} = C^1$



1D elements

Element types:

1. 1D solid bar element.
2. Truss element.

Concepts:

1. Global (weighted residual) vs local (element level) perspectives.
2. Stiffness matrix.
3. Forces: 1.Source term; 2.Natural BC; 3.Essential BC, 4.Nodal.
4. Nodes, elements, shape function, dof.
5. Nodes with more than one dof (truss).
6. Element local coordinate system ξ (bar).
7. Rotation of element local coordinate system (truss).
8. Full stiffness $\mathbf{K}$ (free + prescribed dofs) vs (free only dofs) $\mathbf{K}_{ff}$.
9. High order differential equations (*e.g.*, C^1 beam elements).
10. Multiphysics coupling (beams: axial, bending, & torsional coupling).

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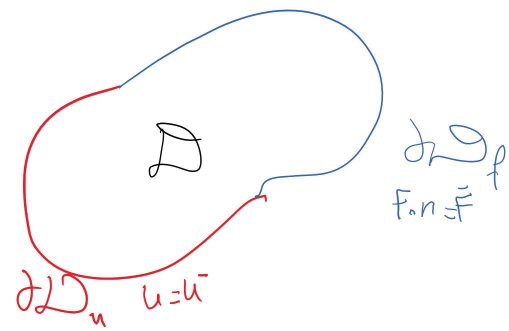
In this section we learn concepts such as elements, nodes, etc, and a general formulation of finite element stiffness matrix and force assembly

$$u^h = \phi_p + \sum_{i=1}^{n} a_i \phi_i(x)$$

this break down ensures we satisfy essential BC



PDE + nat. BC $\rightarrow$ WRS $\xrightarrow{IPB}$

$$\boxed{\int_{\mathcal{D}} L_m(w)\, D\, L_m(u)\, dv = \int_{\mathcal{D}} w\, r\, dV + \int_{\partial\mathcal{D}_f} w\, \bar{F}\, ds} \quad (1)$$

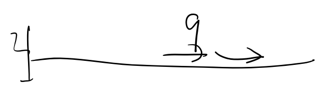
L_m: Differential operator for weak statement

D: material / section property

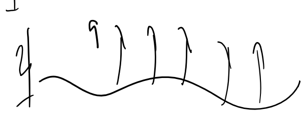
r: source term

Examples

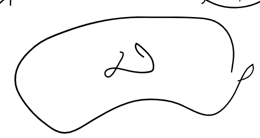
(1) Bar: $L_m = (\ )'$, $D = EA$ $\quad r = q \quad$ LHS: $\int w' EA u'\, dx$



(2) Beam $L_m = (\ )''$, $D = EI$ $\quad$ " $\quad$ LHS $= \int w'' EI y''\, dx$



(3) 2D heat conduction $\quad L_m = \nabla \quad D = k \quad r = Q \quad$ LHS $= \int \nabla w\, k\, \nabla T\, dv$



(4) 2D elasticity $\quad L_m = \underbrace{\frac{\nabla + \nabla^T}{2}}_{u \to \varepsilon} \quad D = \mathbb{C}$ (elasticity stiffness) $\quad r = \rho b \quad$ LHS $= \int \varepsilon(w) : \mathbb{C}\, \varepsilon(u)\, dv$

Equation (1) holds for a general weak statement. From this equation, we want to obtain formulas for the stiffness matrix and all possible force vectors

(1) $$\int_{\mathcal{D}} L_m(w)\, D L_m(u)\, dv = \int_{\mathcal{D}} w\, r\, dV + \int_{\partial\mathcal{D}_f} w\, \bar{F}\, ds$$

(2) $$\boxed{\underline{\mathcal{A}(w, u)} = (w, r) + (w, \bar{F})_N}$$

N: referring to Neumann Boundary

Properties of bilinear form $\mathcal{A}(w, u)$

$$\mathcal{A}(w_1 + w_2, u) = \int_{\mathcal{D}} L_m(w_1 + w_2)\, D L_m(u)\, dv \overset{?}{=} \int \underbrace{\left(L_m(w_1) + L_m(w_2)\right)}_{L_m \text{ is a linear differential operator}} D L_m(u)\, dv$$

$$= \int L_m(w_1) D L_m(u) dV + \int L_m(w_2) D L_m(u) dv = \mathcal{A}(w_1, u) + \mathcal{A}(w_2, u) \quad (i)$$

$$= \int_{\mathcal{D}} L_m(w_1) D L_m(u) dv + \lambda \int_{\mathcal{D}} L_m(w_2) D L_m(u) dv = \mathcal{A}(w_1, u) + \lambda \mathcal{A}(w_2, u) \quad (i)$$

Similarly we can show $\mathcal{A}(w, u_1 + \lambda u_2) = \mathcal{A}(w, u_1) + \lambda \mathcal{A}(w, u_2)$ (ii)

This is why we call operation A, bilinear, because it's linear with respect to both arguments w and u.

Side note: If we solve a nonlinear problem like large deformation or plasticity solid mechanics or many fluid mechanics problems the operator on the left hand side (A) is ALWAYS linear in weight w but not linear in solution u.

We also note the RHS operators are linear in w

$$(w, r) = \int_{\mathcal{D}} w r \, dv$$

$$(w_1 + w_2, r) = \int_{\mathcal{D}} (w_1 + w_2) r \, dv = \int_{\mathcal{D}} w_1 r \, dv + \int_{\mathcal{D}} w_2 r \, dv$$

$$= (w_1, r) + (w_2, r)$$

Same with $(w, \bar{F})_N = \int_{\partial \mathcal{D}_f} w \bar{F} \, ds$

$$(w_1 + w_2, \bar{F}) = (w_1, \bar{F}) + (w_2, \bar{F})$$

In general for any linear or even nonlinear problem all operators in FEM weak statement are linear in weight function. ->
We can satisfy the weak statement ONLY for n weights and this implies that the weak statement is in fact satisfied for any linear combination of these weights. ->
We are going to satisfy the weak statement for n weights and obtain the formulas for the stiffness and forces

$$\left\{ \begin{array}{l} \mathcal{A}(w, u) = (w, r) + (w, \bar{F})_N \\ u^h = \phi_p + \sum_{j=1}^{n} \phi_j(x) a_j \quad \text{(to satisfy essential BC)} \\ \underbrace{w = \phi_i}_{\text{Galerkin}} \quad \text{for } i = 1, \dots n \end{array} \right\} \rightarrow$$

$$\forall i \in \{1, \dots n\} \; \mathcal{A}\Big(\phi_i, \underbrace{\phi_p + \sum_{j=1}^{n} a_j \phi_j}_{\text{open this, use the linear property of } \mathcal{A}}\Big) = (\phi_i, r) + (\phi_i, \bar{F})_N$$

open this, use the linear property of $\mathcal{A}$

$\forall i = 1 \dots n$

$$\mathcal{A}(\phi_i,\phi_p) + \sum_{j=1}^{n} \mathcal{A}(\phi_i,\phi_j)\,a_j = (\phi_i, r) + (\phi_i, \bar{F})_N$$

$$\underbrace{\begin{bmatrix} \mathcal{A}(\phi_1,\phi_1) & \mathcal{A}(\phi_1,\phi_2) & \mathcal{A} \dots & \mathcal{A}(\phi_1,\phi_n) \\ \mathcal{A}(\phi_i,\phi_1) & \dots & & \mathcal{A}(\phi_i,\phi_n) \\ & & & \mathcal{A}(\phi_n,\phi_n) \end{bmatrix}}_{K} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \overbrace{\begin{bmatrix} (\phi_1, r) \\ (\phi_2, r) \\ \vdots \\ (\phi_i, r) \\ \vdots \\ (\phi_n, r) \end{bmatrix}}^{F_r} + \overbrace{\begin{bmatrix} (\phi_1, \bar{F})_N \\ \vdots \\ (\phi_i, \bar{F})_N \\ \vdots \\ (\phi_n, \bar{F})_N \end{bmatrix}}^{F_N}$$

row i

$F_N \rightarrow$ Neumann BC

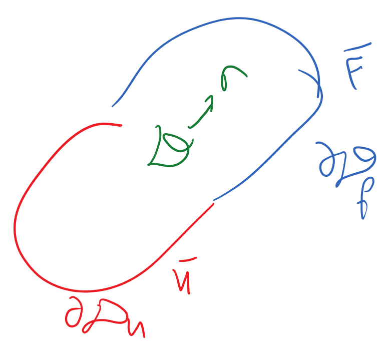
$$\underbrace{\begin{bmatrix} \mathcal{A}(\phi_1,\phi_p) \\ \vdots \\ \mathcal{A}(\phi_i,\phi_p) \\ \vdots \\ \mathcal{A}(\phi_n,\phi_p) \end{bmatrix}}_{F_D}$$

Force from Dirichlet BC

Summary

For any Galerkin method (Spectral FEM ...)

we have the following system of equation



$$Ka = F, \quad F = F_r + F_N - F_D$$

$$K_{ij} = \mathcal{A}(\phi_i,\phi_j) = \int_{\mathcal{D}} \mathcal{L}_m(\phi_i)\, D\, \mathcal{L}_m(\phi_j)\, dV$$

$$K = \int_{\mathcal{D}} \mathcal{L}_m\left(\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix}\right) D\, \mathcal{L}_m([\phi_1, \dots, \phi_n])\, dV$$

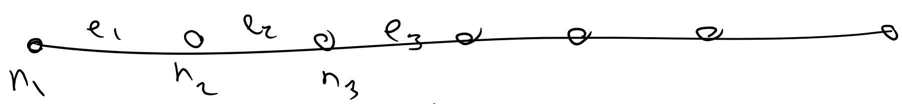
$$(F_r)_i = \int_{\mathcal{D}} \phi_i\, r\, dV \qquad \left(F_r = \int_{\mathcal{D}} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} r\, dV \right)$$

$$(F_N)_i = \int_{\partial\mathcal{D}_f} \phi_i \bar{F}\, ds \qquad \left(F_N = \int_{\partial\mathcal{D}_f} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} \bar{F}\, ds \right)$$

$$(F_D)_i = \mathcal{A}(\phi_i,\phi_p) = \int_{\mathcal{D}} \mathcal{L}_m(\phi_i)\, D\, \mathcal{L}_m(\phi_p)\, dV \qquad F_D = \int_{\mathcal{D}} \mathcal{L}_m\left(\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix}\right) D\, \mathcal{L}_m(\phi_p)\, dV$$

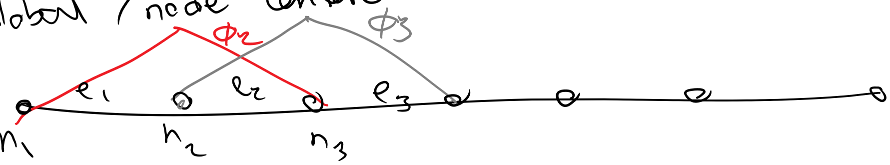
$\star$

From this point, we only focus on FEM and even further simplify equation *

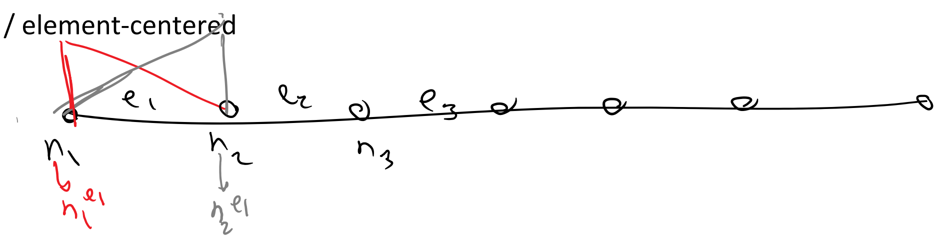


2 Approaches will be discussed

1 Global / node centered



2. Local / element-centered



Approach 2 is what everyone does (FEM codes, …)
Approach 1 is what the weak statement provides. Need to work with this to turn it to more favorable approach 2 that will be covered later.

For your HW assignments, as soon as we cover approach 2, PLEASE don't use approach 1 anymore

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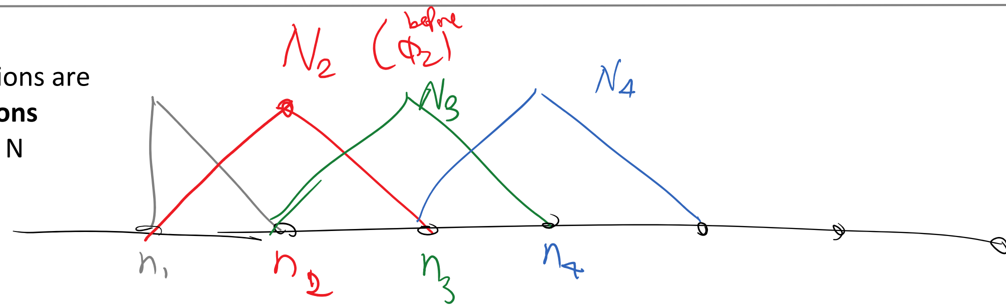
In FEM, the basis functions are
- Called **shape functions**
- and are denoted by N



$$u^h = \sum_{i=1}^{n} a_i N_i(x)$$

$$u^h(n_3) = a_1 \underbrace{N_1(n_3)}_{0} + a_2 \underbrace{N_2(n_3)}_{0} + a_3 \underbrace{N_3(n_3)}_{1} + a_4 \underbrace{N_4(n_3)}_{0} + \cdots$$

$$= a_3$$

$$\boxed{a_i = u^h(n_i)}$$

FEM, unknown #i is

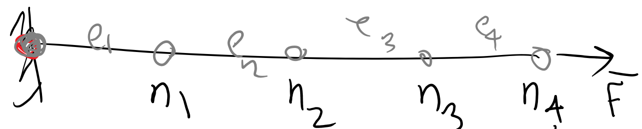
$u_i = u(n_i)$

FEM, unknown # i is the solution value at that corresponding degree of freedom (dof) / i.e. node in 1D bar problem

this is a consequence of FEM shape function delta property

$$\boxed{N_i(n_j) = \delta_{ij}} \qquad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

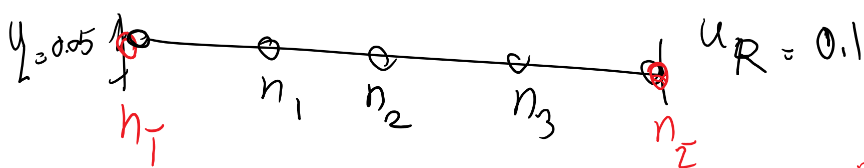
Definitions of nf and np:



→ free dofs $\quad n_f = 4$

n_T ← prescribed dof

$n_p = 1$



$n_f = 3$

$n_p = 2$