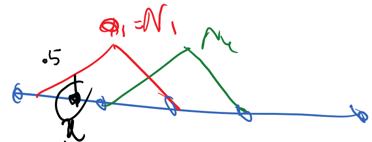
2022/10/13

Thursday, October 13, 2022 11:10 AM

Last time we obtained K, Fr, FN, FD for general Galerkin methods. We want to specialize those to FEMs, using FEM basis functions N

A. Stiffness matrix

$$K = \int \begin{bmatrix} \phi_1' \\ | \\ | \\ \phi_n' \end{bmatrix} EA [\phi_1' \; — \; \phi_n'] dx$$



$\hookrightarrow N = [.5, 0, \; — \;]$

FEM

$$K = \int \begin{bmatrix} N_1 \\ N_{nf} \end{bmatrix} EA [N_1 \; -- N_{nf}]' dx$$

For bar problem

$$u^h_{(x)} = \cancel{\phi_p} + \sum_{j=1}^{nf} a_j N_j = \underbrace{[N_1(x) \; -- \; N_{nf}(x)]}_{} \begin{bmatrix} a_1 \\ | \\ | \\ a_{nf} \end{bmatrix}$$

ignore this for now

N maps $\begin{bmatrix} a_1 \\ | \\ a_{nf} \end{bmatrix}$ nodal displacements to displacement $u^h(x)$ at a given position x.

$$u^{h\prime}(x) = \varepsilon(x) = \underbrace{[N_1'(x) \; --- \; N_{nf}'(x)]}_{} \begin{bmatrix} a_1 \\ | \\ | \\ a_n \end{bmatrix}$$

this vector maps $\begin{bmatrix} a_1 \\ | \\ | \\ a_n \end{bmatrix}$ (nodal displacements)

$\longrightarrow$ position x, strain

$B(x) = [N_1'(x) \; N_2'(x) --- \; N_{nf}'(x)]$

is called <u>displacement to strain map</u>

$$K = \int_0^L \begin{bmatrix} B_1 \\ | \\ | \\ B_{nf} \end{bmatrix} D [B_1 \; --- \; B_{nf}] dx$$

bar problem

$D = EA$

$B = N'$

$\boxed{B = \mathcal{L}_m(N)}$

$$\int \underbrace{w'}_{\mathcal{L}_m(w)} \underbrace{EA}_{D} \underbrace{u'}_{\mathcal{L}_m(u)} dx$$

other examples

1D beam

$$\int \underbrace{w''}_{(\;)''} \overbrace{EI}^{D = EI} y'' dx$$

$B = \mathcal{L}_m N = N''$

1D beam $\int w'' EI y'' \, dx$

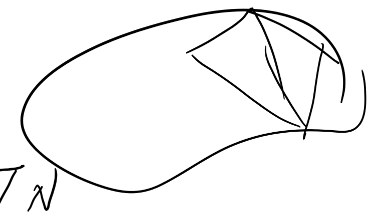
$L_m = (\ )''$

$B = L_m N = N''$

2D/3D heat conduction

$\int \nabla w \, \kappa \, \nabla T \, dx$

$D \quad L_m(w) \quad D$



$B = L_m N = \nabla N$

2D/3D solid mechanics

$\int \varepsilon(w) \, C \, \varepsilon(u) \, dV$

$D \quad L_m = \frac{\nabla + \nabla^T}{2} \quad D$

$B = L_m N = \varepsilon(N) \rightsquigarrow \frac{\nabla N + \nabla^T N}{2}$

For FEM K is

$$K = \int B^T D B \, dV$$

$B = L_m(N)$, D = section/material property

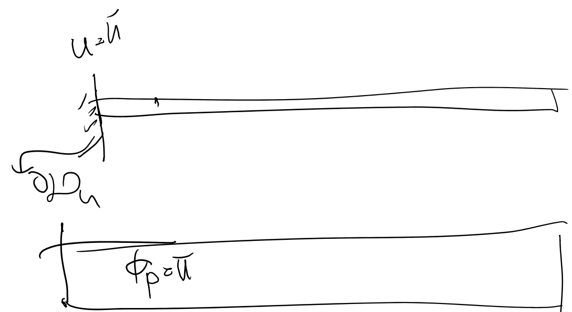
	L_m	D
bar	$(\ )'$	EA
beam	$(\ )''$	EI
heat conduction	∇	κ
2D/3 elas.	$\frac{\nabla + \nabla^T}{2}$	C

1

B. Essential BC

$u^h = \phi_p + \sum a_i N_i$

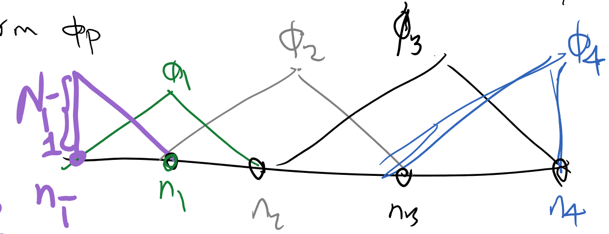
Previously we used



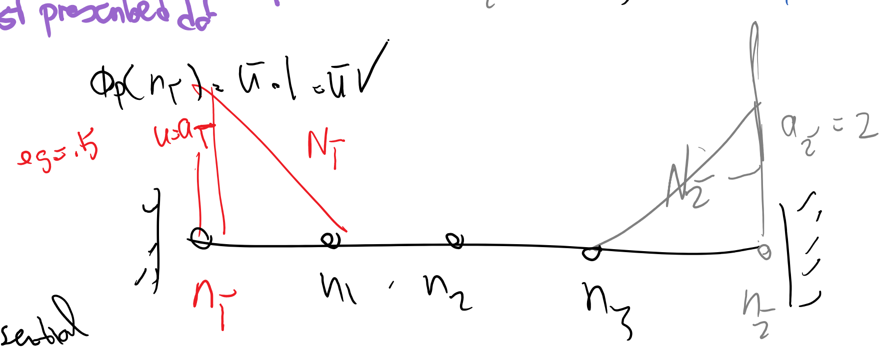
FEM provides a general means to form ϕ_p

$\phi_p = \bar{u} \, N_{\bar{1}}(x)$

$\bar{u} = a_{\bar{1}}$: value @ 1st prescribed dof



$N_{\bar{1}}(n_{\bar{1}}) = 1 \qquad \phi_p(n_{\bar{1}}) = \bar{u} \cdot 1 = \bar{u}$ ✓



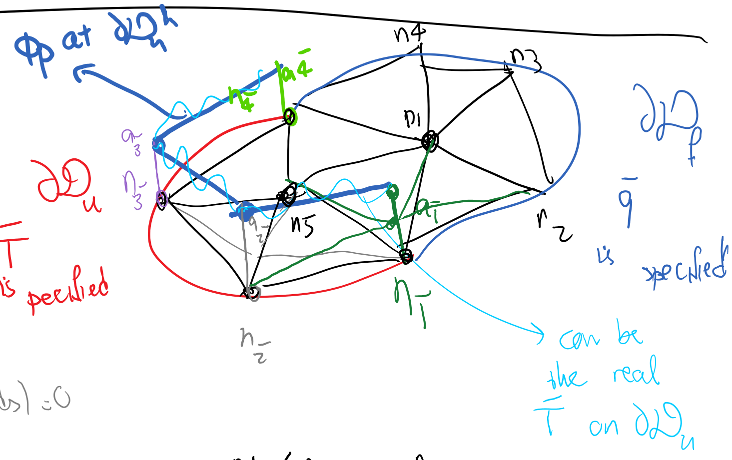
$\phi_p = a_{\bar{1}} N_{\bar{1}}(x) + a_{\bar{2}} N_{\bar{2}}(x)$

satisfies all essential BCs :)

Example from 2D ϕ_p at $\partial\Omega_u^h$

BCs :)

Example from 2D Heat conduction



$N_{\bar 1}(x)$ takes the value 1 @ node $\bar 1$ & zero at other nodes

$N_{\bar 2}(n_{\bar 2}) = 1$ & $N_{\bar 2}(\text{other nodes}) = 0$

$$\Phi_p(x) = a_{\bar 1} N_{\bar 1}(x) + a_{\bar 2} N_{\bar 2}(x) + a_{\bar 3} N_{\bar 3}(x) + a_{\bar 4} N_{\bar 4}(x)$$

$$\Phi_p(n_{\bar 1}) = a_{\bar 1} \underbrace{N_{\bar 1}(n_{\bar 1})}_{1} + a_{\bar 2} \underbrace{N_{\bar 2}(n_{\bar 1})}_{0} + a_{\bar 3} \underbrace{N_{\bar 3}(n_{\bar 1})}_{0} + a_{\bar 4} \underbrace{N_{\bar 4}(n_{\bar 1})}_{0}$$

$$= a_{\bar 1} \checkmark$$

The particular solution we form this way **satisfies the essential BC on ALL prescribed dofs** (node here). However, it may not match the exact Tbar between the nodes if Tbar is not linear between the nodes.

Is this a problem?

Discretization (infinite unknown -> finite unknowns)

$$u^h = \Phi_p + \sum_{i=1}^{n_f} N_i(x) a_i$$

results in error

$$\| u^h - u^{exact} \| \propto h^{p+1}$$ (p → order of element)

Any other source of error in calculating K and F in Ka = F that DOES not dominate discretization error is perfectly fine, because the error still behaves as

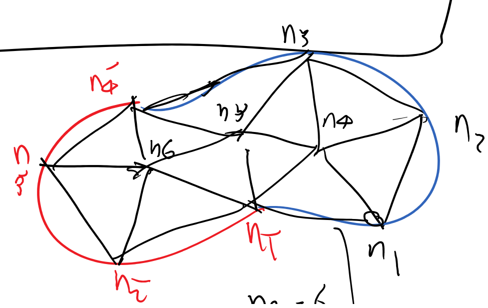
$$C h^{p+1}$$

Some examples:

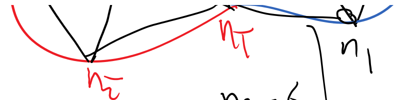
- Φ_p not satisfied essential BC between nodes
- Approximate sources term
- Numerical integration (quadrature)
- …

Summary:

$$u^h(x) = \Phi_p(x) + \sum_{i=1}^{n_f} a_i N_i(x)$$



$$u(x) = \phi_p(x) + \sum_{i=1} a_i N_i(x)$$



$$= \sum_{\bar{i}=1}^{n_p} a_{\bar{i}} N_{\bar{i}}(x) \quad + [N_1 \cdots N_{n_f}] \begin{bmatrix} a_1 \\ \vdots \\ a_{n_f} \end{bmatrix}$$

$n_p = 4$ $\quad n_f = 6$

$$= \underbrace{[N_{\bar{i}}(x) \cdots N_{\bar{n}_p}(x)]}\begin{bmatrix} a_{\bar{i}} \\ \vdots \\ a_{\bar{n}_p} \end{bmatrix} + [N_1 \cdots N_{n_f}] \begin{bmatrix} a_1 \\ \vdots \\ a_{n_f} \end{bmatrix}$$

$\nabla T = B_p a_p + B_f a_f$

$B_p = L_m(N_p)$

$= \nabla N_p$

$B_f = L_m(N_f) = \nabla N_f$

$$\boxed{u^h = \underbrace{\bar{N}\bar{a}}_{\phi_p} + Na}$$

$$\varepsilon(x) = \bar{N}'\bar{a} + N'a$$

$$= \bar{B}\bar{a} + Ba$$

$\bar{B}\bar{a}$: displacement to strain map for prescribed dof's

Ba: displacement to strain map for free dof's

also noted as

$$B_p a_p \quad + B_f a_f$$

Still don't have the formula for F_D

last time

$$F_D = \mathcal{A}(\underbrace{N^t}_{\phi^t}, \phi_p) = \int_{\mathcal{D}} \underbrace{L_m \begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix}} D \; L_m(\phi_p)\, dV$$

bar $L_m = (\;)'$

$B = N'$

$$F_D = \int_{\mathcal{D}} B_f^t D \, L_m\Big(\underbrace{\sum_{\bar{i}=1}^{n_p} a_{\bar{i}} N_{\bar{i}}}_{\phi_p}\Big) dV$$

$$= \int_{\mathcal{D}} B_f^t D \sum_{\bar{i}=1}^{n_p} a_{\bar{i}} \, L_m(N_{\bar{i}}) \, dV$$

because L_m is linear

$$= \int_{\mathcal{D}} B_f^t D \underbrace{[L_m(N_{\bar{i}}) \cdots L_m(N_{\bar{n}_p})]}_{B_p} \underbrace{\begin{bmatrix} a_{\bar{i}} \\ \vdots \\ a_{\bar{n}_p} \end{bmatrix}}_{a_p} dV$$

$\longrightarrow$ (F) ...

$$\left(F_D\right)_{n_f\times 1} = \left(K_{fp}\right)_{n_f\times n_p}\left(a_p\right)_{n_p\times 1}, \quad \text{where } K_{fp} = \int_{\mathcal{D}} \begin{bmatrix} B_1 \\ \vdots \\ B_{n_f} \end{bmatrix} D\left[B_1 \cdots B_{n_p}\right] dV = \int_{\mathcal{D}} B_f^t D B_p \, dV$$

Compare this with

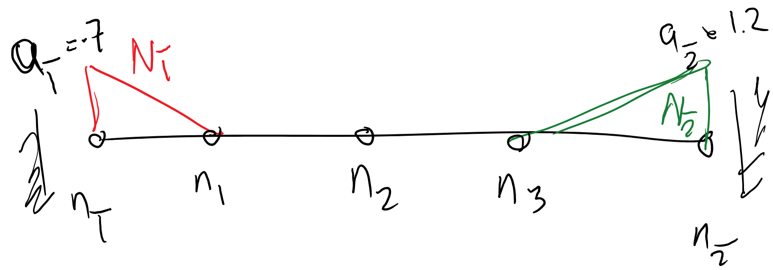
$$K = K_{ff} = \int_{\mathcal{D}} \begin{bmatrix} B_1 \\ \vdots \\ B_{n_f} \end{bmatrix} D\left[B_1 - - B_{n_f}\right] dV = \int_{\mathcal{D}} B_f^t D B_f \, dV$$

(2)

Calculation of FD is similar to K (= Kff) with the difference that we calculate Kfp and next multiply it by ap

Bar problem example

$$\left(K_{fp}\right)_{3\times 2} = \int_0^L \underbrace{\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}'}_{B_f^t} \underbrace{EA}_{D} \underbrace{\left[N_{\bar{1}}\ N_{\bar{2}}\right]'}_{B_p} dx$$



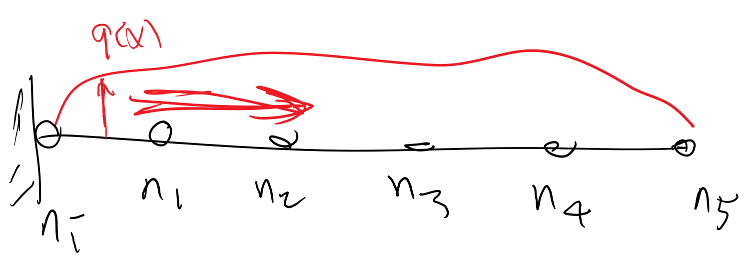
$$\left(F_D\right)_{3\times 1} = \left(K_{fp}\right)_{3\times 2} \underbrace{\begin{bmatrix} .7 \\ 1.2 \end{bmatrix}_{2\times 1}}_{a_p}$$

$$Ka = F, \qquad F = F_r + F_N - F_D$$

C. Source term force

(3)

$$F_r = \int_{\mathcal{D}} \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_{n_f} \end{bmatrix} \underbrace{r(x)}_{\text{source term}} \, dx$$



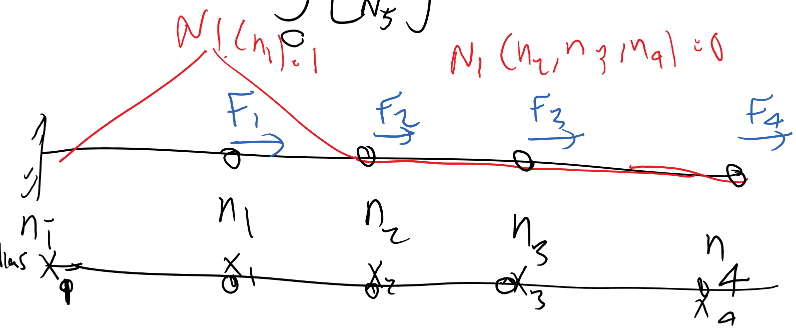
1D bar example

$$F_r = \int_0^L \begin{bmatrix} N_1 \\ \vdots \end{bmatrix} q(x)\, dx$$

D Source term

$$F_r = \int^L \begin{bmatrix} N_1 \\ \vdots \\ N_5 \end{bmatrix} q(x)\, dx$$

D. Point forces



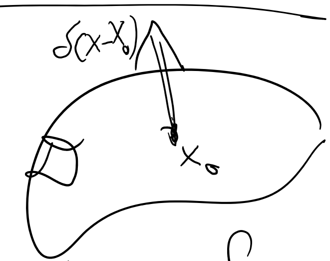
$$q(x) = F_1\, \delta(x - x_1) + F_2\, \delta(x - x_2) + F_3\, \delta(x - x_3) + F_4\, \delta(x - x_4)$$

Delta-Dirac function (we discussed this collocation method)

From eq (3) $\quad (F_r)_I = \int_0^L N_I(x)\, q(x)\, dx =$

$$\int_0^L N_I(x)\left(F_1\, \delta(x - x_1) + \cdots + F_{nf}\, \delta(x - x_{nf})\right) dx$$

$$= \sum_{J=1}^{nf} \int_0^L F_J \underbrace{N_I(x)}_{f}\, \delta(x - \underbrace{x_J}_{x_0})\, dx$$



$$\int_D f(x)\, \delta(x - x_0) = f(x_0)$$

$$= \sum_{J=1}^{nf} F_J \underbrace{N_I(x_J)}_{= 1 \text{ if } I = J,\ 0 \text{ otherwise}}$$

$$= F_I$$

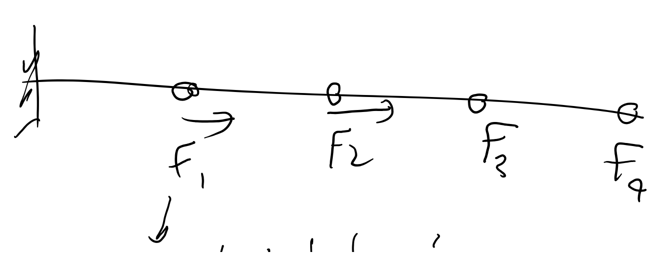
$$\boxed{(F_r)_I = F_I}$$

from concentrated force

we denote the force from concentrated force by F_n

& the formula is

$$F_n = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ \vdots \end{bmatrix} \qquad (4)$$



Γ_Ω $\quad f_{n_f}$

$t_1 \quad t_2 \quad t_3 \quad t_4$

concentrated q's

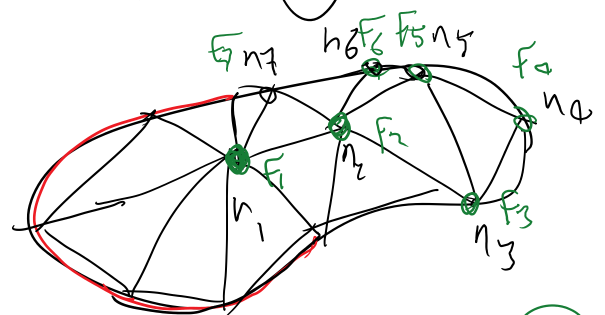
$(EA\,u')' + q = 0$

Example 2

heat conduction

$$\nabla \cdot q + Q = 0$$

source term = heat source

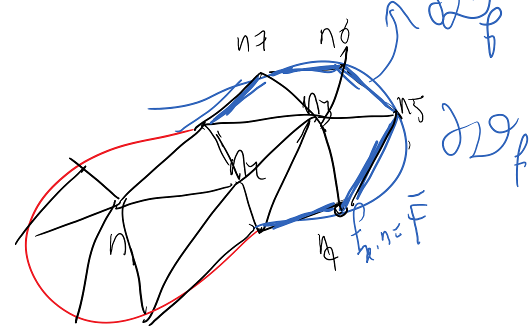


$f_1 \cdots f_{n_f}$ are concentrated Q and are scalar heat sources

— Example idealizing heat source from a larger

D. FN = the force from natural boundary condition

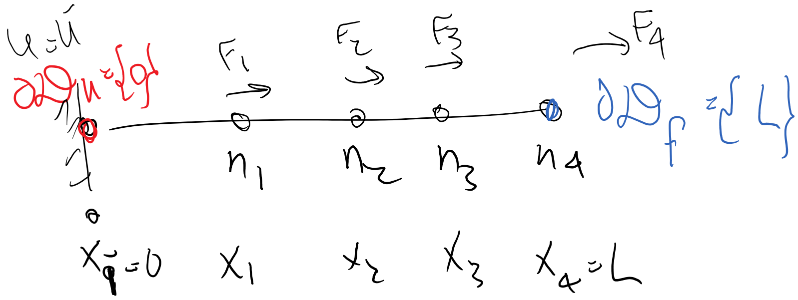
$$\boxed{F_N = \int_{\partial\Omega_f^h} \begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix} \bar{F}\, ds} \quad (5)$$



for 2D heat equation $\bar{F} = \bar{q}$ = net heat flux

$$F_N = \int_{\partial\Omega_f} \begin{bmatrix} N_1 \\ \vdots \\ N_7 \end{bmatrix} \bar{q}\, ds$$

Example 2: 1D bar



$$F_N = \int_{\partial\Omega_f} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \bar{F}\, ds$$

$\partial\Omega_f = \{x_4\}$ $\quad F = \begin{bmatrix} N_1(x_4) \\ N_2(x_4) \end{bmatrix} \bar{F}$ $\quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \vdots \end{bmatrix}$

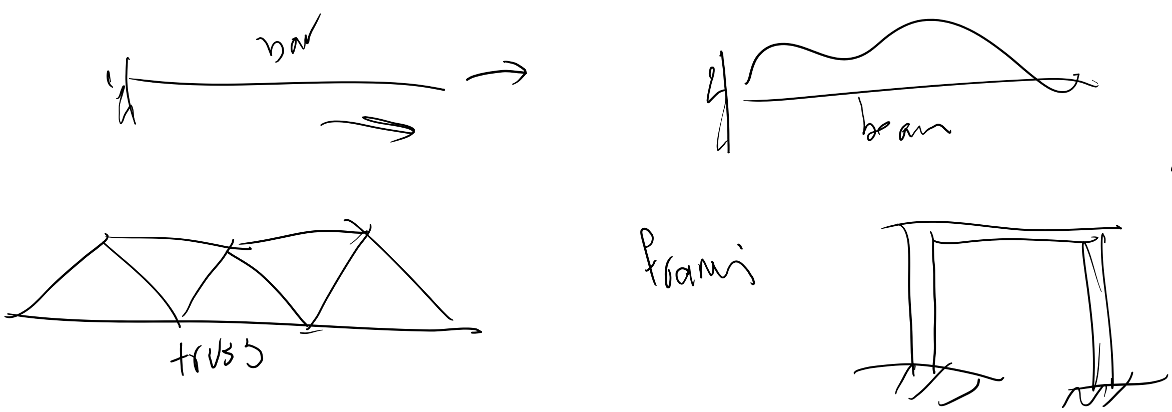
∂D_f

$$\partial D_f = \{x_4\} \qquad \underline{F_N} = \begin{bmatrix} N_1(x_4) \\ N_2(x_4) \\ N_3(x_4) \\ N_4(x_4) \end{bmatrix} F_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F_4 \end{bmatrix}$$

Also $\quad F_n = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \qquad Ka = F = F_r + F_n + F_N - F_D$

F_4 is double counted, we need to consider F_4 either as a nodal force or natural boundary condition

When dealing with 1D elements, we never calculate FN

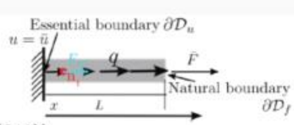


no ∂D_f

Summary: Force vectors

- Force vector is given by:

$$\boxed{\mathbf{F} = \mathbf{F}_r + \mathbf{F}_N + \mathbf{F}_n - \mathbf{F}_D} \qquad (311)$$



- $\mathbf{F}_r$, $\mathbf{F}_N$, $\mathbf{F}_n$ and $\mathbf{F}_D$ are given by (*cf.* (301) and (310))

$$\mathbf{F}_r = \left(\mathbf{N}^{\mathrm{T}}, q\right)_r = \int_{\mathcal{D}} \mathbf{N}^{\mathrm{T}} q \,\mathrm{d}v = \int_0^L \begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix} q \,\mathrm{d}x \qquad (312a)$$

$$\mathbf{F}_N = \left(\mathbf{N}^{\mathrm{T}}, \bar{F}\right)_N = \int_{\partial \mathcal{D}_f} \mathbf{N}^{\mathrm{T}} \bar{\mathbf{F}}.\mathbf{N} \,\mathrm{d}s = \left(\begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix} \bar{F} \right)_{x=L} \qquad (312b)$$

$$\mathbf{F}_D = \mathcal{A}\left(\mathbf{N}^{\mathrm{T}}, \phi_p\right) = \int_{\mathcal{D}} \frac{\mathrm{d}}{\mathrm{d}x} \mathbf{N}^{\mathrm{T}} EA \frac{\mathrm{d}}{\mathrm{d}x} \phi_p \,\mathrm{d}v \qquad (312c)$$

$$= \left\{ \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} EA \bar{\mathbf{B}} \,\mathrm{d}v \right\} \bar{\mathbf{a}} = \left\{ \int_0^L \begin{bmatrix} B_1 \\ \vdots \\ B_{n_f} \end{bmatrix} EA \left[\bar{B}_{\bar{1}} \quad \cdots \quad \bar{B}_{\bar{n}_p} \right] \mathrm{d}x \right\} \begin{bmatrix} \bar{a}_{\bar{1}} \\ \vdots \\ \bar{a}_{\bar{n}_p} \end{bmatrix} = \mathbf{K}_{fp} \bar{\mathbf{a}}$$

$$\mathbf{F}_n = \begin{bmatrix} F_{n1} \\ \vdots \\ F_{n n_f} \end{bmatrix} \qquad (312d)$$

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For next time:

Bar Example: Overview