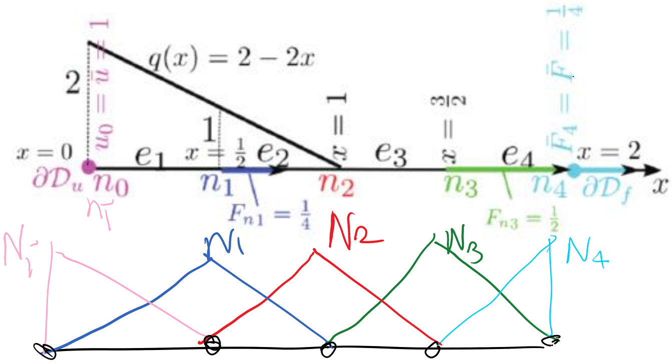
2022/10/18

Tuesday, October 18, 2022 11:23 AM

Bar Example: Overview



$n_f = 4$

$n_p = 1$

$K_{4\times4}\, a = F_{4\times1}$

$F = F_n + \cancel{F_N} + F_r - F_D$

$$\bar{F}_n = \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix}$$

nodal forces

Next $K\ (K_{ff})$ & $F_D = K_{fp}\, a_p$

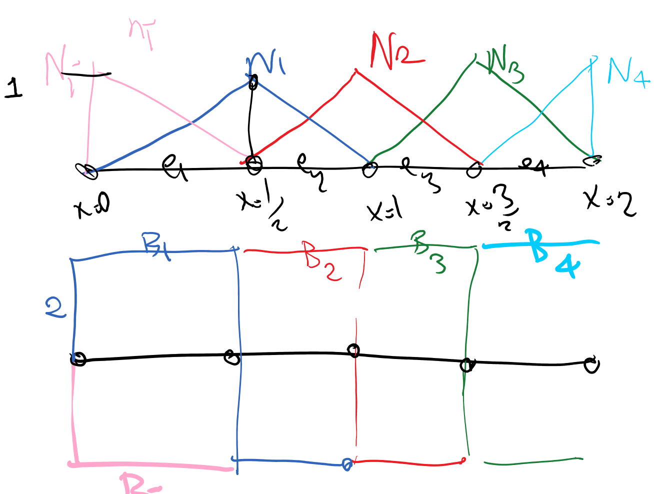
color coding is based on dof (no/node here)

$K_{ff} = \int_{\Omega} B_f^T D B_f \, dV$

$K_{fp} = \int_{\Omega} B_f^T D B_p \, dV$

$D = EA = 1$ for this problem

$B = \frac{d}{dx} N$ we need to calculate B's



N

$B = \frac{dN}{dx}$

$K_{11} = \int_0^2 B_1 \overset{1}{\widetilde{EA}} B_1 \, dx = \int_{e_1 \to [0,.5]} B_1 (1) B_1 \, dx + \int_{e_2\,[.5,1]} B_1 B_1 \, dx + \underbrace{0 + 0}_{e_3 \,\&\, e_4}$ contributions $= \int_0^{.5} 2\times2 \, dx + \int_{.5}^{1} (-2)(-2) \, dx = 4\times.5 + 4\times.5 = 4$

$$+\int_{e_2\,[.5,1]} B_1 B_1\,dx + \underbrace{0 + 0}_{e_3\,\&\,e_4 \text{ contributions}} = \int_0^{.5} 2\times 2\,dx + \int_{.5}^{1} (-2)(-2)\,dx = 4\times.5 + 4\times.5 = 4$$

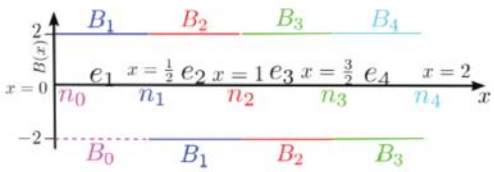
$$K_{12} = \int_0^2 B_1 \overset{1}{EA} B_2\,dx = \int_{.5}^{1} (-2)(2)\,dx = \underline{-2}$$

element 2

Doing this for the rest of the components we get:

$$K = \begin{bmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 4 & 2 \\ & & -2 & 2 \end{bmatrix}$$

Bar Example: Step 1: Stiffness matrix



$$\mathbf{K} = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} dx = \begin{bmatrix} \int_0^2 B_1B_1\,dx & \int_0^2 B_1B_2\,dx & \int_0^2 B_1B_3\,dx & \int_0^2 B_1B_4\,dx \\ & \int_0^2 B_2B_2\,dx & \int_0^2 B_2B_3\,dx & \int_0^2 B_2B_4\,dx \\ & & \int_0^2 B_3B_3\,dx & \int_0^2 B_3B_4\,dx \\ \text{sym.} & & & \int_0^2 B_4B_4\,dx \end{bmatrix}$$

$$= \begin{bmatrix} \int_{e_1} B_1B_1\,dx + \int_{e_2} B_1B_1\,dx & \int_{e_2} B_1B_2\,dx & 0 & 0 \\ & \int_{e_2} B_2B_2\,dx + \int_{e_3} B_2B_2\,dx & \int_{e_3} B_2B_3\,dx & 0 \\ & & \int_{e_3} B_3B_3\,dx + \int_{e_4} B_3B_3\,dx & \int_{e_4} B_3B_4\,dx \\ \text{sym.} & & & \int_{e_4} B_4B_4\,dx \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\cdot(2)\cdot(2) + \frac{1}{2}\cdot(-2)\cdot(-2) & \frac{1}{2}\cdot(-2)\cdot(2) & 0 & 0 \\ & \frac{1}{2}\cdot(2)\cdot(2) + \frac{1}{2}\cdot(-2)\cdot(-2) & \frac{1}{2}\cdot(-2)\cdot(2) & 0 \\ & & \frac{1}{2}\cdot(2)\cdot(2) + \frac{1}{2}\cdot(-2)\cdot(-2) & \frac{1}{2}\cdot(-2)\cdot(2) \\ \text{sym.} & & & \frac{1}{2}\cdot(2)\cdot(2) \end{bmatrix} \Rightarrow$$

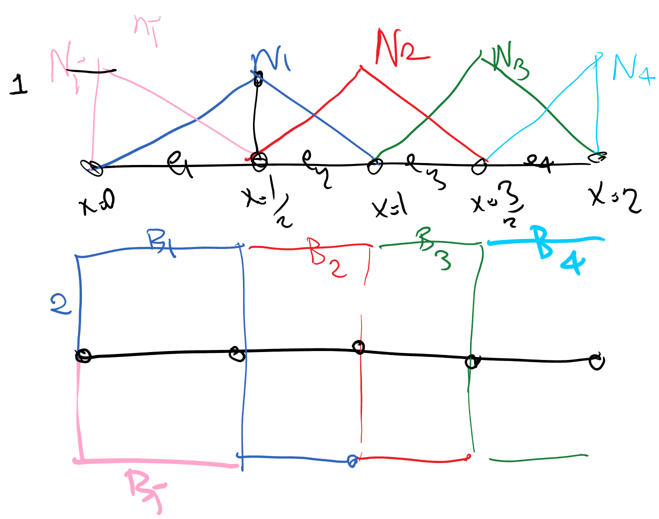
$$\mathbf{K} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ & & 4 & -2 \\ \text{sym.} & & & 2 \end{bmatrix} \qquad (316)$$

You are screening sharing 00:26:20 Stop Share

254 / 456

$$F_D = K_{fp}\, a_p$$

$$K_{fp} = \int_0^2 B_f^t \overset{EA=1}{D} B_p\,dx$$



$$B_f^t = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} \qquad B_p = B_T$$

$$K_{fp} = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} B_T\,dx$$

$$(K_{fp})_{11} = \int_{e_1} B_1 B_T\,dx = \int_0^{.5} 2(-2)\,dx = -2$$

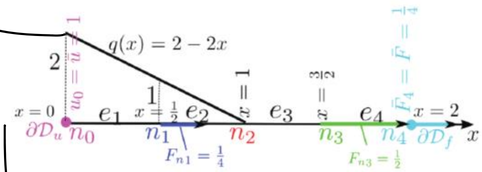
$$K_{fp} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Bar Example: Overview

e_1 u_0

$$K_{fp} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad q_p = 1$$

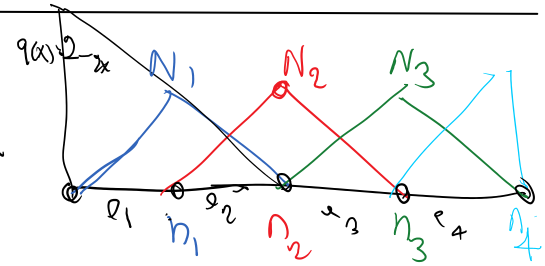
Bar Example: Overview



$$F_D = K_{fp}\, q_p = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}(1) \rightarrow \boxed{F_D = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$F_\gamma = ?$

$$F_\gamma = \int N^t q(x)\,dx = \int_0^2 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} q(x)\,dx$$



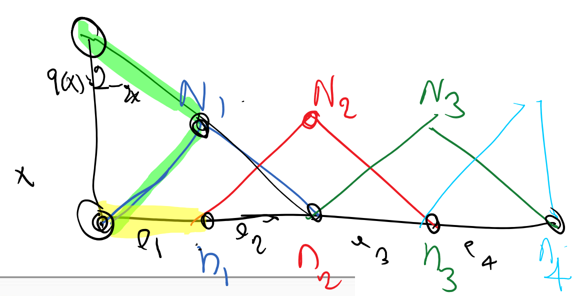
$$= \int_0^2 \begin{bmatrix} N_1 q(x) \\ N_2 q(x) \\ 0 \\ 0 \end{bmatrix} dx = \begin{bmatrix} \int_{e_1} N_1 q(x)\,dx + \int_{e_2} N_1 q(x)\,dx \\ \int_{e_2} N_2 q(x)\,dx \\ 0 \\ 0 \end{bmatrix}$$

$$\int_a^b f(x) g(x) = \frac{(b-a)}{6}\left(2 f_a g_a + 2 f_b g_b + f_b g_a + f_a g_b\right)$$

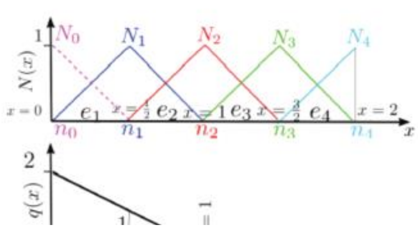
calculate

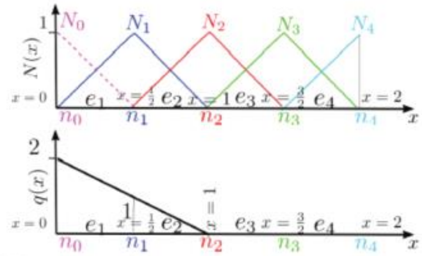
$$\int_{e_1} N_1(x)\, q(x)\,dx = \frac{\frac{1}{2}}{6}\left(2 \times 2 \times 0 + 2 \times 1 \times 1 + 2 \times 1 + 1 \times 0\right)$$

$$= \frac{4}{12} = \frac{1}{3}$$



Bar Example: Step 2.1: Source term force





From (312a),

$$\mathbf{F}_r = \int_0^L \begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix} q \, \mathrm{d}x = \begin{bmatrix} \int_0^2 N_1(x)q(x)\,\mathrm{d}x \\ \int_0^2 N_2(x)q(x)\,\mathrm{d}x \\ \int_0^2 N_3(x)q(x)\,\mathrm{d}x \\ \int_0^2 N_4(x)q(x)\,\mathrm{d}x \end{bmatrix} = \begin{bmatrix} \int_{e_1} N_1(x)q(x)\,\mathrm{d}x + \int_{e_2} N_1(x)q(x)\,\mathrm{d}x \\ \int_{e_2} N_2(x)q(x)\,\mathrm{d}x \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1/2}{6}((2).(0).(2)+(2).(1).(1)+(0).(1)+(1).(2)) + \frac{1/2}{6}((2).(1).(1)+(2).(0).(0)+(1).(0)+(0).(1)) \\ \frac{1/2}{6}((2).(0).(1)+(2).(1).(0)+(0).(0)+(1).(1)) \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$

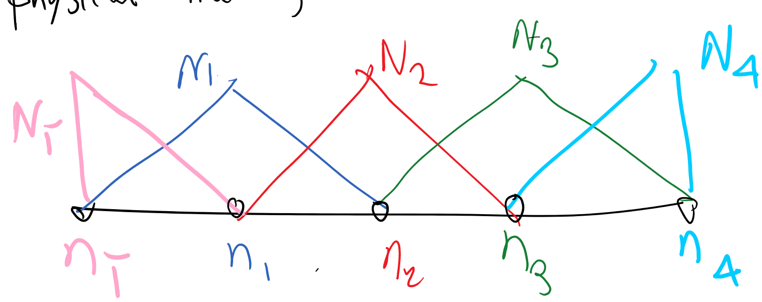
$$\mathbf{F}_r = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{12} \\ 0 \\ 0 \end{bmatrix} \tag{317}$$

$$Ka = F = F_r + F_n - F_D + \cancel{F_N} = \begin{bmatrix} 11/4 \\ 1/12 \\ 1/2 \\ 1/4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 4 & -2 \\ & & -2 & 2 \end{bmatrix} \qquad F_n = \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix} \qquad F_D = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a = \begin{bmatrix} 43/24 \\ 53/24 \\ 31/12 \\ 65/24 \end{bmatrix}$$

physical meaning of a's



$$u^h(x) = \phi_p + \sum N_i(x) a_i$$

$$= \sum_{i=1}^{np} N_i(x) a_i + \text{"}$$

$$= (N_T(x) a_T) + (a_1 N_1(x) + a_2 N_2(x) + a_3 N_3(x) + a_4 N_4(x))$$

$a_4 \to 65/24$

$$u^h(\underset{x=1}{n_2}) = \cancel{N_T(n_2) a_1} + a_1 \underbrace{N_1(n_2)}_{0} + a_2 \underbrace{N_2(n_2)}_{1} + a_3 \underbrace{N_3(n_2)}_{0} + a_4 \underbrace{N_4(n_2)}_{0}$$

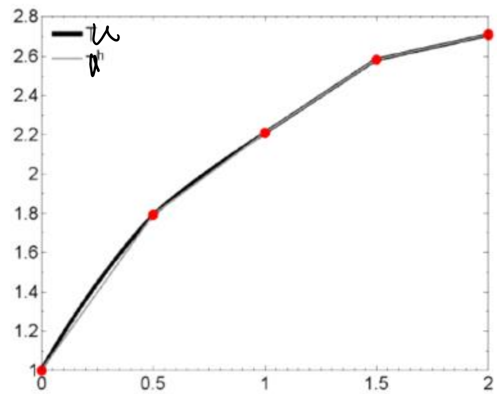
$$= a_2$$

$$u^h(n_i) = a_i$$

meaning of FEM dofs

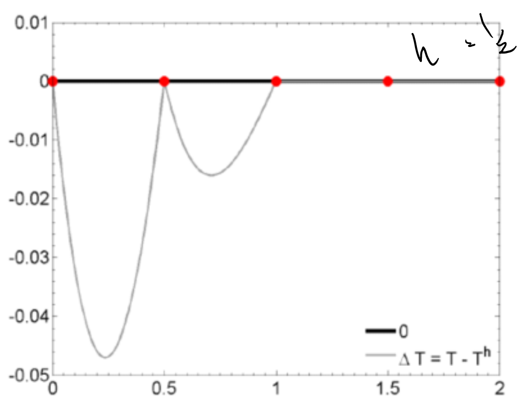
$$\boxed{\begin{aligned} u^h(n_i) &= a_i \\ u^h(n_T) &= a_T,\ u^h(n_2) = a_2 \cdots \end{aligned}}$$

Bar Example: solution values



- u^h and u match at all nodes n_0, n_1, n_2, n_3, and n_4. This holds for 1D solid elements with uniform AE and **does not hold** in general.

$u^h - u$



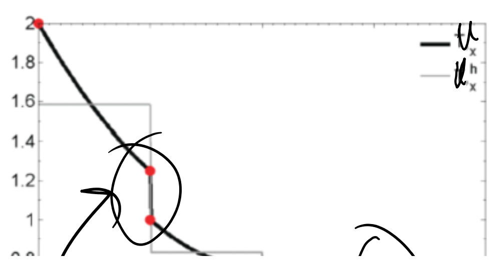
error $= C h^{\alpha}$

$= C h^{p+1}$

element order

- As mentioned before, the solution error at all nodes n_0, n_1, n_2, n_3, and n_4 is zero. This does not hold in general for FEM method.

Bar Example: solution derivatives ($\propto$ axial force)

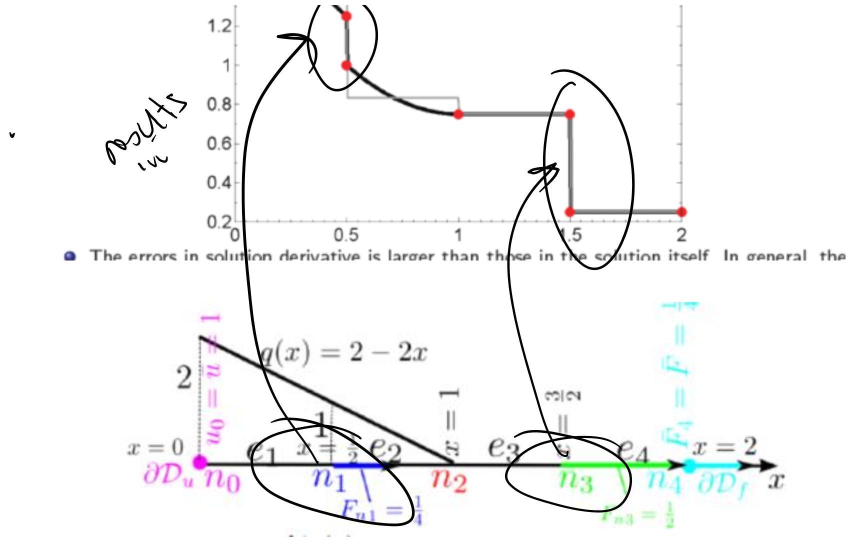


$= \varepsilon$

$= \varepsilon^h$

$F = A\sigma$

$= (AE)\,\varepsilon$



$(AE)\,\varepsilon$

$\|u^h_{,x} - u^{exact}_{,x}\| = Ch^1$

$u \rightarrow Ch^2$

1 derivative

$u_{,x} \rightarrow Ch^{2-1} = Ch^1$

- The errors in solution derivative is larger than those in the solution itself. In general, the accuracy of FE solution decreases for solution derivatives (*e.g.*, strains, stresses, *etc.*).
- Approximate solution u^h exhibits jumps in $\frac{du^h}{dx}$ at all interior nodes. This is because the solution is piece-wise constant in $H^1([0\ 2])$.
- Even the exact solution exhibits jumps in $\frac{du}{dx}$ at n_1 and n_3 from the concentrated forces.
- The $H^1([0\ 2])$, rather than $C^1([0\ 2])$, is the right solution space for u and u^h as none of them belong to the latter space.

264 / 456

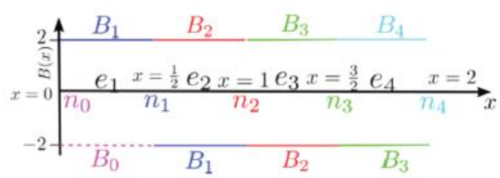
FEM local (element-centered view)

Much simpler tan global (node-centered view) which we just covered.

Motivation: With global view (which is the direct consequence of using hat functions in the weak statement), we anyways had to break the integrals to elements

Bar Example: Step 1: Stiffness matrix

Global view



$$\mathbf{K} = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} dx = \begin{bmatrix} \int_0^2 B_1 B_1\, dx & \int_0^2 B_1 B_2\, dx & \int_0^2 B_1 B_3\, dx & \int_0^2 B_1 B_4\, dx \\ & \int_0^2 B_2 B_2\, dx & \int_0^2 B_2 B_3\, dx & \int_0^2 B_2 B_4\, dx \\ & & \int_0^2 B_3 B_3\, dx & \int_0^2 B_3 B_4\, dx \\ \text{sym.} & & & \int_0^2 B_4 B_4\, dx \end{bmatrix}$$

$$= \begin{bmatrix} \int_{e_1} B_1 B_1\, dx + \int_{e_2} B_1 B_1\, dx & \int_{e_2} B_1 B_2\, dx & 0 & 0 \\ & \int_{e_2} B_2 B_2\, dx + \int_{e_3} B_2 B_2\, dx & \int_{e_3} B_2 B_3\, dx & 0 \\ & & \int_{e_3} B_3 B_3\, dx + \int_{e_4} B_3 B_3\, dx & \int_{e_4} B_3 B_4\, dx \\ \text{sym.} & & & \int_{e_4} B_4 B_4\, dx \end{bmatrix}$$

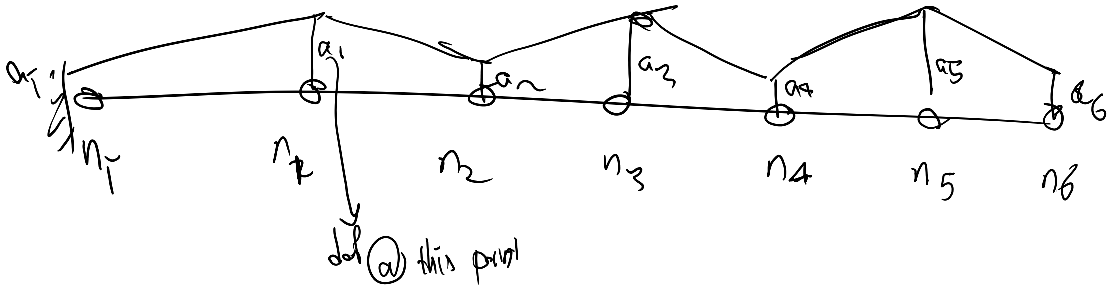
$$= \begin{bmatrix} \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 & 0 \\ & \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 \\ & & \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) \\ \text{sym.} & & & \frac{1}{2}.(2).(2) \end{bmatrix} \Rightarrow$$

$$\mathbf{K} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ \text{sym.} & & 4 & -2 \\ & & & 2 \end{bmatrix} \tag{316}$$

254 / 456

Because of this (breaking the integrals to elements), we would take care of the contributions of element at a time -> element centered approach

Now, it's a good time to distinguish nodes and dofs

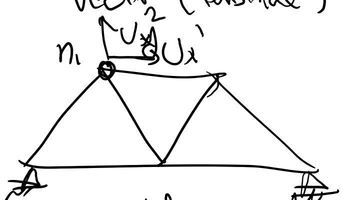


a_i's are solution coefficients = displacement @ nodes

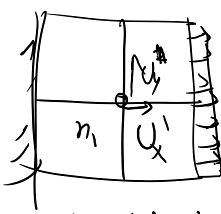
These are the degrees of freedom (dof)

Examples of more than 1 dof per node

vector (tensorial) problems



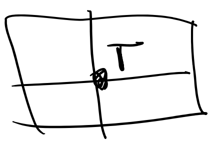
2 dof node



$$U = \begin{bmatrix} U_x \\ U_y \end{bmatrix}$$

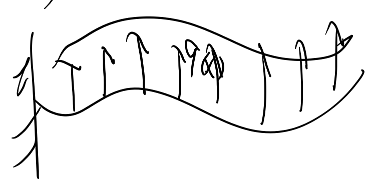
in contrast problem with scalar unknown

heat conduction



High order DE's

$$(EIy'')'' + q = 0$$



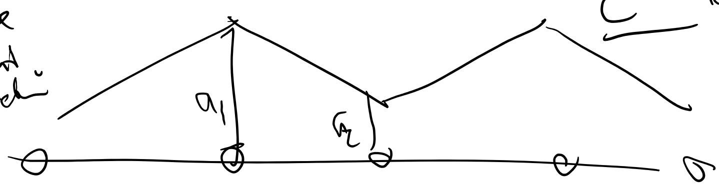
$M = 4$

$m = \frac{M}{2} = 2$

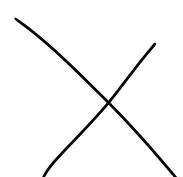
C^{m-1} continuity is needed

C^1 functions

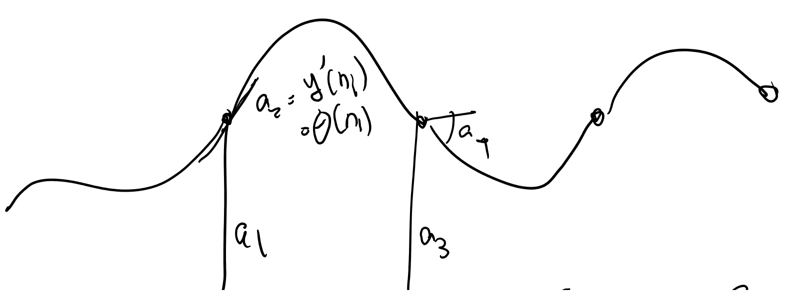
use linear function

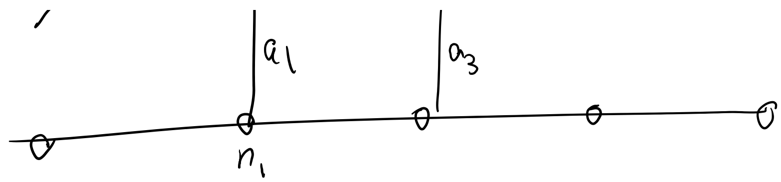


for beam problem



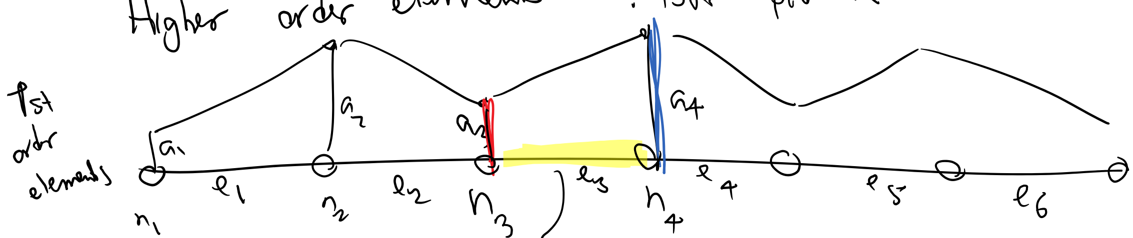
for beam

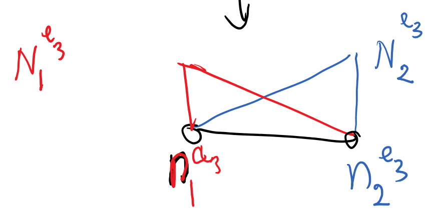




we have $m = \frac{M}{2}$ dofs per node for beam

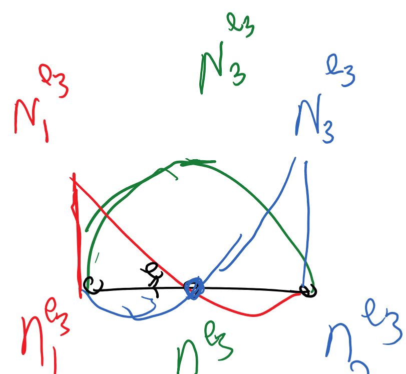
Higher order elements : Bar problem





$a_1^{e_3} = a_3$ $\quad a_2^{e_3} = a_4$ $\quad$ of the global problem

$$u^h_{e_3} = a_1^{e_3} N_1^{e_3} + a_2^{e_3} N_2^{e_3} = a_3 N_3 + a_4 N_4$$



we added interior nodes for higher order elements