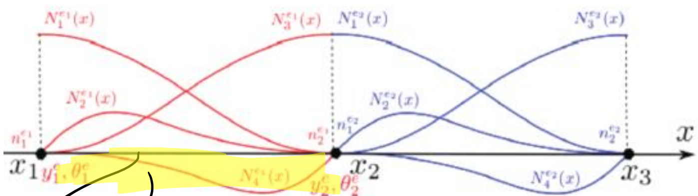
2022/10/20

Thursday, October 20, 2022 11:23 AM

Shape functions take the value of 1 at one of the element dofs (NOT NODE) and zero elsewhere



$$\text{dofs} = \begin{pmatrix} y_1^e \\ \theta_1^e \\ y_2^e \\ \theta_2^e \end{pmatrix}$$

$y_1^e = y_2^e = \theta_1^e = 0$ but $\theta_2^e = 1$ for this shape function

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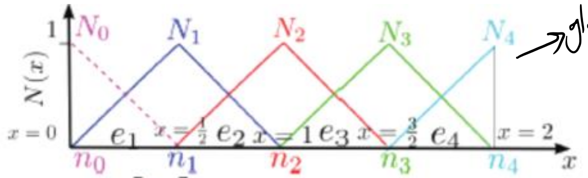
Solving the same problem but now with element-centered approach

This is how we did it before



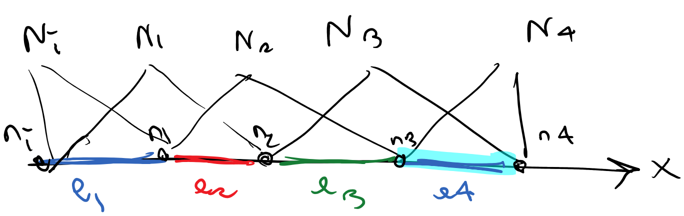
global/nodal approach

$$K = \int_\Omega B^t D B \, dv$$

$$K_{12} = \int_0^L B_1 EA B_2 \, dx = \int_{e_1} B_1 EA B_2 \, dx + \int_{e_2} B_1 EA B_2 \, dx \ldots$$

$$K_{11} = \int B_1 EA B_1 \, dx \quad \underbrace{\int_{e_1} B_1 EA B_1 \, dx}_{\neq 0} + \underbrace{\int_{e_2} B_1 EA B_1 \, dx}_{\neq 0} + \int_3 + \int_4$$

In the "local" of "element-centered" approach, we take care of all things pertained to one element, and then move to the next one.

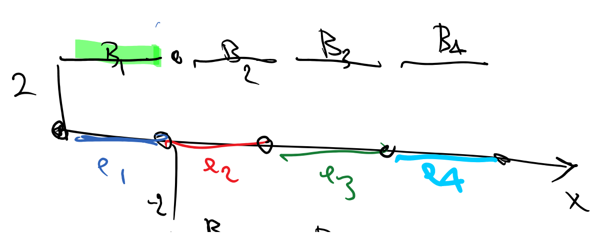


$$K = \int_\Omega B^t D B \, dv = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} dx$$

$$= K_{e_1} + K_{e_2} + K_{e_3} + K_{e_4}$$

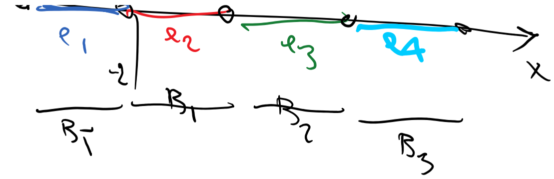
$$K_{e_i} = \int_{e_i} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} dx$$

$$K_{e_1} = \int \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} (1) \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} dx = \int^{.5} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix} dx$$



$K_{e_1} = \int \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} [B_1\ B_2\ B_3\ B_4] \, dx = \int_0 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} [2\ 0\ 0\ 0] \, dx$

$e_1 = [0, .5]$



$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

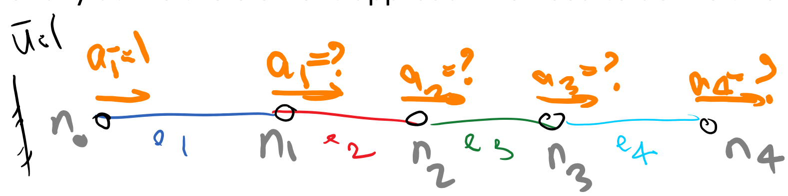
$$K_{e_2} = \int_{e_2} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} [B_1\ B_2\ B_3\ B_4] \, dx = \int_{.5}^{1} \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} [-2\ 2\ 0\ 0] \, dx = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_3 = \int_{e_3} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} [B_1\ B_2\ B_3\ B_4] \, dx = \int_{1}^{1.5} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix} [0\ -2\ 2\ 0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_{e_4} = \int_{e_4} [B^T B] \, dx = \int \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2 \end{bmatrix} [0\ 0\ -2\ 2] \, dx = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

$$K = \sum K_{e_i} = \begin{bmatrix} 2+2 & -2 & & \\ -2 & 2+2 & -2 & \\ & -2 & 2+2 & -2 \\ & & -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 4 & -2 \\ & & -2 & 2 \end{bmatrix}$$

To fully utilize the element-approach we need to define two maps:



• dofs

• nodes

$n_e = 4 \quad n_p = 1$

$LEM_{e_1} = [0,1]$ $LEM_{e_2} = [1,2]$ $LEM_{e_3} = [2,3]$ $LEM_{e_4} = [3,4]$

LEM (1) ← Maps

Local element map (nodal map)

All geometry:

what nodes form the element

$M_{e_1} = [\bar 1, 1]$ $M_{e_2} = [1,2]$ $M_{e_3} = [2,3]$ $M_{e_4} = [3,4]$

M (2)

element dof map

what are the dofs of element (in order)

These two maps are not the same:

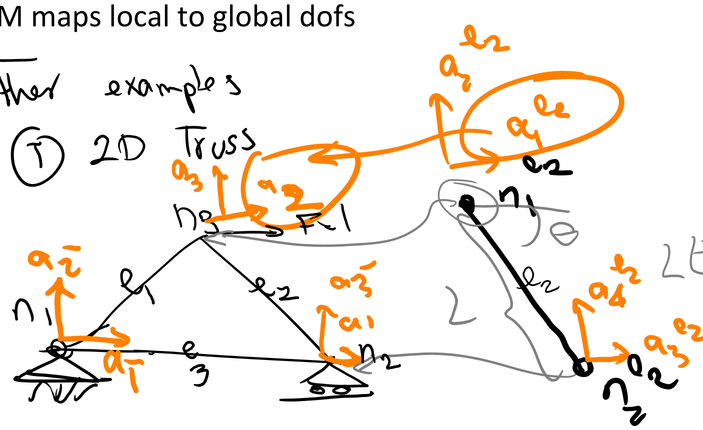
- LEM provides the element geometry
- M maps local to global dofs

e_2

- LEM provides the element geometry
- M maps local to global dofs

Other examples

(1) 2D Truss



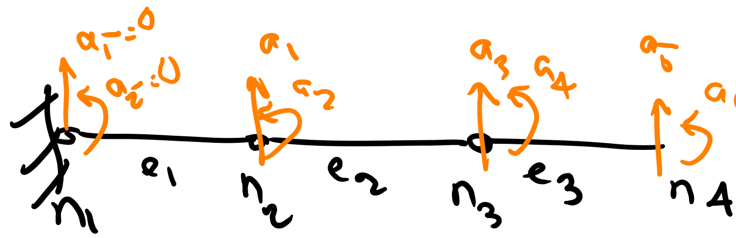
$LEM_2 = [3, 2]$ used for geometry of element (θ, L)

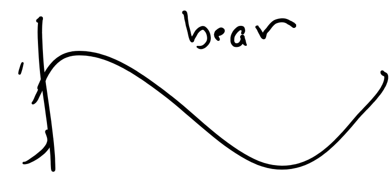
$M_2 = [2, 3, 1, \bar{3}]$

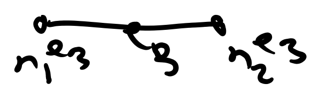
local to global K & F assembly

— input

— dofs we need to number these



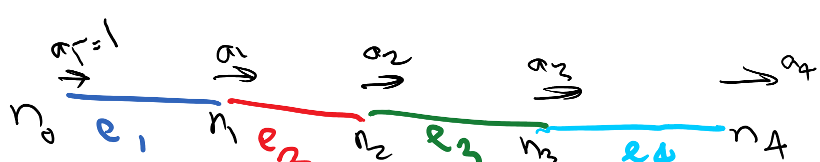




$LEM_3 = [3, 4]$

$M_3 = [3, 4, 5, 6]$

Let's go back to our bar problem and use the M map



$M_1 = [\bar{1}, 1]$ $M_2 = [1, 2]$ $M_3 = [2, 3]$ $M_4 = [3, 4]$

I'll show that element level stiffness is

$$k_e = \left(\frac{AE}{L}\right)_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For this problem for all elements $A=1, E=1$

$$L_e = \frac{L}{n} = \frac{2}{4} = \frac{1}{2} \rightarrow k_{e_1} = k_{e_2} = k_{e_3} = k_{e_4} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$k_{e_1} = \begin{array}{c|cc} & \bar{1} & 1 \\ \hline \bar{1} & 2 & -2 \\ 1 & -2 & 2 \end{array} \quad k_{e_2} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 2 & -2 \\ 2 & -2 & 2 \end{array} \quad k_{e_3} = \begin{array}{c|cc} & 2 & 3 \\ \hline 2 & 2 & -2 \\ 3 & -2 & 2 \end{array} \quad k_{e_4} = \begin{array}{c|cc} & 3 & 4 \\ \hline 3 & 2 & -2 \\ 4 & -2 & 2 \end{array}$$

K =

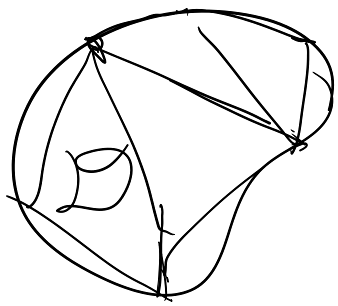
	1	2	3	4
1	2+2	-2		
2	-2	2+2	-2	
3		-2	2+2	-2

$$
\begin{array}{c|cccc}
3 & & -2 & 2+2 & -2 \\
4 & & & -2 & 2
\end{array}
$$

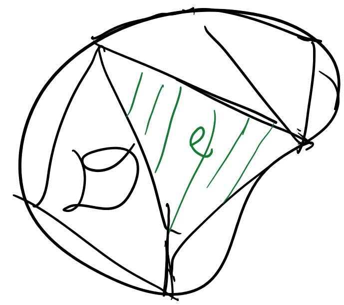
How do local and global K's and F's are related?

Global Approach (node-based)	Local Approach (element-based)

$K = \int_{\Omega} B^t D B \, dV$



$K_e = \int_{e} B_e^t D B_e \, dV$



Forces

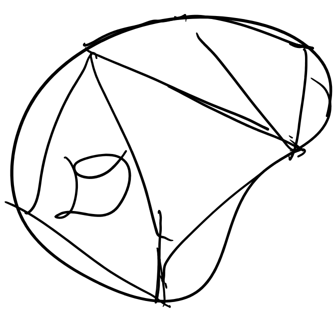
$F = \underbrace{(F_r + F_N - F_D)}_{F_e \; + \; F_{eq}} + F_n$ → nodal

element forces

F_e is assembled from element f_e's

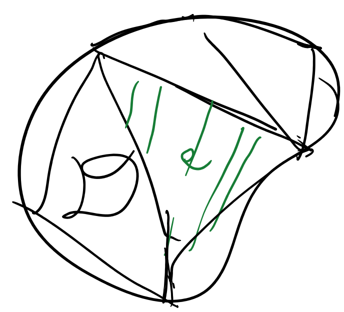
ⓐ F_r

$F_r = \int_{\Omega} N^t r \, dV$

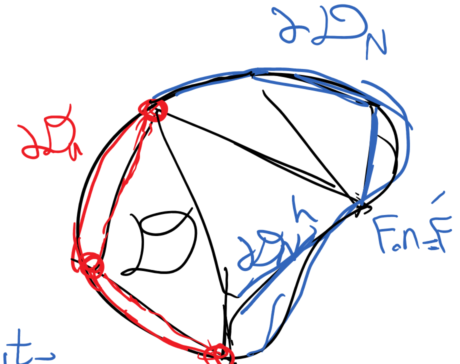


f_r^e

$f_r^e = \int_{e} N_e^t r \, dV$



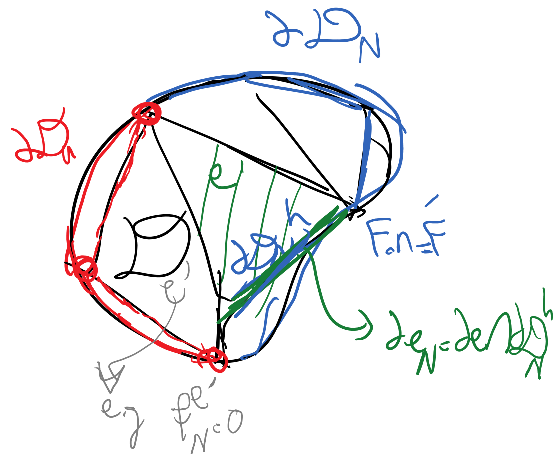
ⓑ F_N → Neumann



$F_N = \int N^t \bar{F} \, ds$

f_N^e

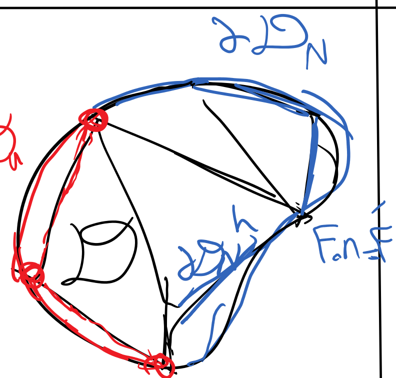
$f_N^e = \int_{\partial e_N} N_e^t \bar{F} \, ds$



$f_N = \int_{\partial\Omega_N^h} N^t \bar{t}\, ds$

e.g. $f_N^{e} = 0$

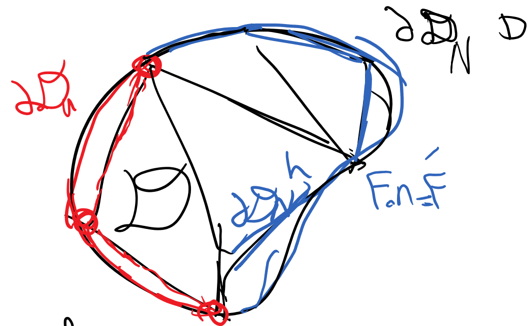
ⓒ F_D



$F_D = K_{fp} a_p$

$$K_{fp} = \int_{\Omega} B_f^t D B_p \, dV$$

f_D^e



$f_D^e = k^e a^e$

for free dofs (?) put zero

for prescribed ones put the correct value

No need to calculate K_{fp}!

So, we learned how to assemble local element stiffness to global element stiffness

$$f^e = f_r^e + f_N^e - f_D^e$$

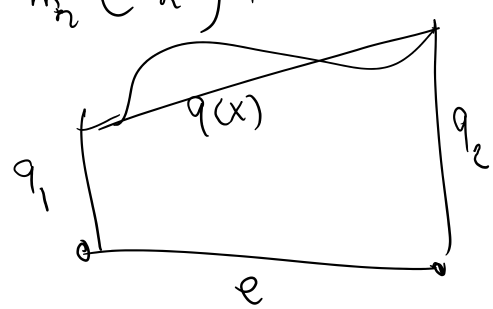
I'll motivate why this is true next time

next time

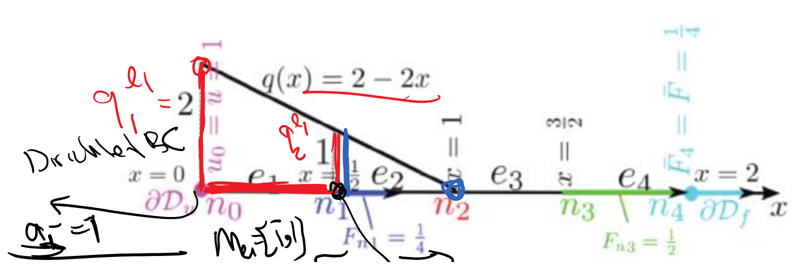
Map_e: $m_1, m_2, \ldots, m_n$ $\begin{bmatrix} f_1^e \\ \vdots \\ f_n^e \end{bmatrix}$ → added to row m_1

$$k^e = \left(\frac{AE}{L}\right)^e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

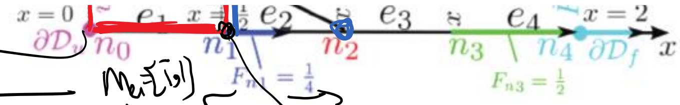
$$f_r^e = r^e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad r^e = \frac{L^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$



Let's do our problem with the element-based approach



We obtained K



$a_{\bar 1} = 1$ $\quad M_{e_1} = [\bar 1, 1]$ $\quad a_1 = ?$

Calculating $F_{4\times 1}$

$f^{e_1} = f_r^{e_1} + \cancel{f_N^{e_1}} - f_D^{e_1}$

Determined

$f_r^{e_1} = r^{e_1} q^{e_1}, \quad q^{e_1} = \begin{bmatrix}2\\1\end{bmatrix}$

$r^{e_1} = \frac{L^{e_1}}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix} = \frac{1}{12}\begin{bmatrix}2&1\\1&2\end{bmatrix}$

$f_r^{e_1} = \frac{1}{12}\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{12}\begin{bmatrix}5\\4\end{bmatrix}$

$f_D^{e_1} = k^{e_1} a^{e_1}, \quad k^{e_1} = \begin{bmatrix}2&-2\\-2&2\end{bmatrix} \quad a^{e_1} = \begin{bmatrix}a_{\bar 1}\\a_1\end{bmatrix} = \begin{bmatrix}1\\\cancel{?}\end{bmatrix}$

$= \begin{bmatrix}1\\0\end{bmatrix} \quad f_D^{e_1} = \begin{bmatrix}2&-2\\-2&2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix}$

$f^{e_1} = f_r^{e_1} + \cancel{f_N^{e_1}} - f_D^{e_1}$

$= \frac{1}{12}\begin{bmatrix}5\\4\end{bmatrix} - \begin{bmatrix}2\\-2\end{bmatrix} = \begin{array}{c} \bar 1 \\ 1 \end{array}\begin{bmatrix}-19/12\\28/12\end{bmatrix}$

$M^{e_1} = [\bar 1, 1]$

$f^{e_2} = f_r^{e_2} + \cancel{f_N^{e_2}} - f_D^{e_2}$

$f_r^{e_2} = r^{e_2} q^{e_2} = \frac{1}{12}\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} =$

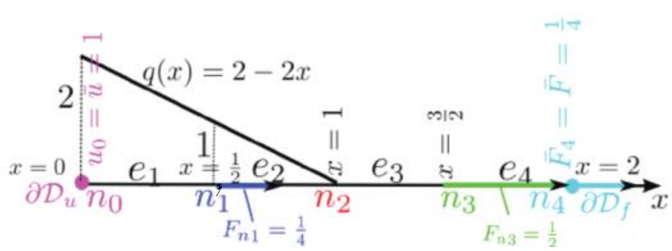
$= \begin{bmatrix}2/12\\1/12\end{bmatrix}$

$f^{e_2} = \begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix}2/12\\1/12\end{bmatrix}$

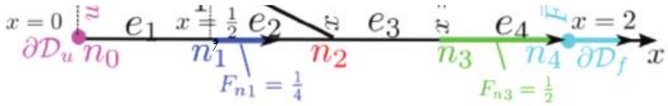
$M_{e^2} = [1, 2]$

$F = F_e + F_n \qquad F_e = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix}28/12 + 2/12\\1/12\\ \\ \end{bmatrix}$

$F_n = \begin{bmatrix}1/4\\0\\1/2\\1/4\end{bmatrix} \quad \rightarrow \quad F = F_n + F_e = \begin{bmatrix}11/4\\1/12\\1/2\\1/4\end{bmatrix}$



$a = K^{-1}F = \begin{bmatrix}43/24\\53/24\\31/12\\ \end{bmatrix}$



$$a = K\, f = \begin{bmatrix} 53/24 \\ 31/12 \\ 65/24 \end{bmatrix}$$