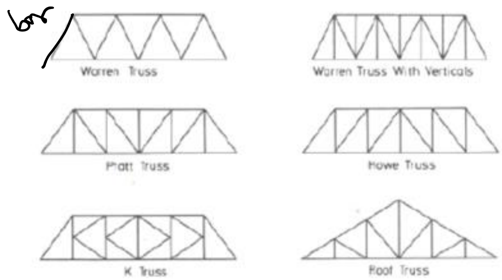
Trusses



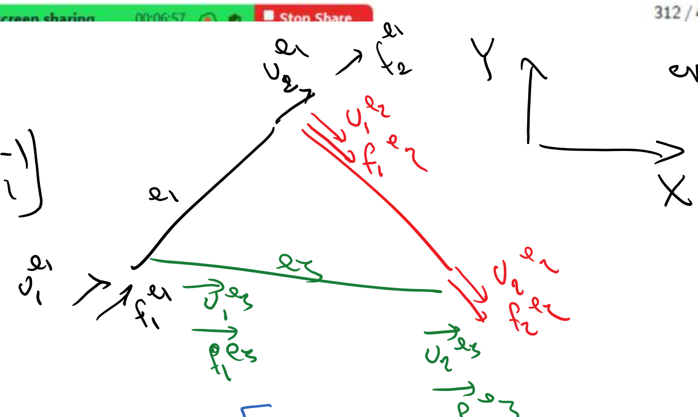
Types of simple Plane truss

- Trusses are 2D or 3D ensemble of bars.
- The main load transfer mechanism of these bars is axial force as the hinge connection at nodes prevent generation and transfer of moments.
- Although in bar elements we could have body force, in trusses we do not apply any type of load between the nodes (except the weight of bars themselves which may be neglected in many applications).
- Generally, top and bottom bars carry the moments and middle diagonal and vertical bars carry shear forces if we think of truss as a big bar.

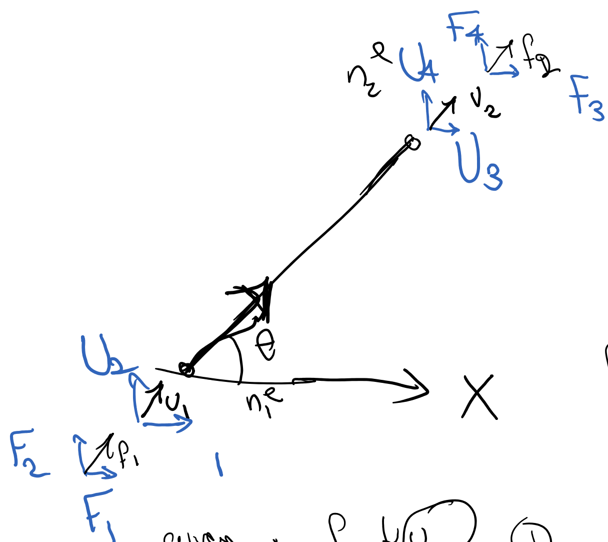
312 / 456

– key idea

$$\begin{bmatrix} f_1^{e_1} \\ f_2^{e_1} \end{bmatrix} = \left(\frac{AE}{L}\right)^{e_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



every element f, & displacement is in this system



$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

θ angle $\overrightarrow{n_1 n_2}$ relative to X axis

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = k^e_{2\times2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

given: $f = k u$ (1)

$u_{2\times1} = T_u U_{2\times4} U_{4\times1}$ (2)

Transfer matrix for $U_{4\times1}$ to $u_{2\times1}$

transfer matrix for $U_{4\times1}$ to $_{2\times1}$

Goal : $F = \underbrace{(\quad)}_{K \text{ in global coordinate system}} U$

① ② $\qquad f = (k T_{uU}) U$ ③

$$\boxed{\underset{4\times1}{F} = \underset{4\times2}{T_{Ff}}\ \underset{2\times1}{f}}$$ ④ $\qquad$ if we find T_{Ff}

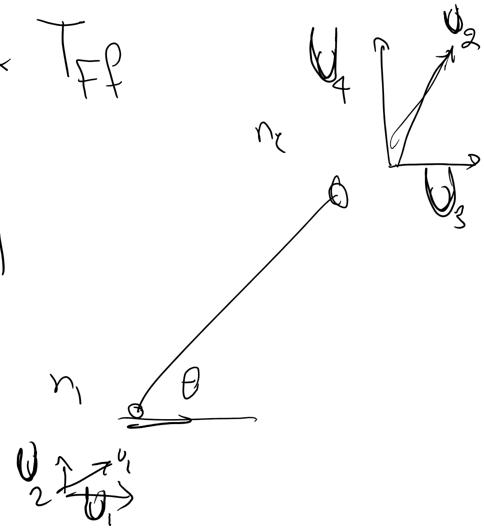
premultiply ③ by T_{Ff} :

$$F = T_{Ff} f = \underbrace{(T_{Ff}\, k\, T_{uU})}_{K} U$$

stiffness matrix in global coordinate ← K

$$\boxed{K_{4\times4} = {T_{Ff}}_{4\times2}\ k_{2\times2}\ {T_{uU}}_{2\times4}}$$ ⑤

All we need now is calculating T_{uU} & T_{Ff}



$c = \cos\theta$, $s = \sin\theta$

T_{uU}

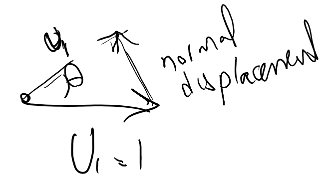
$$T_{uU} \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}$$

the result is column 1 of the matrix

Column 1 $\qquad U_1 = 1$, $U_2 = 0$, $U_3 = 0$ $U_4 = 0$

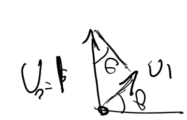
$u_1 = \cos\theta = c$

$u_2 = 0$

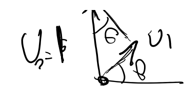


colum 2 $\qquad U_1 = 0$ $U_2 = 1$ $U_3 = 0$ $U_4 = 0$

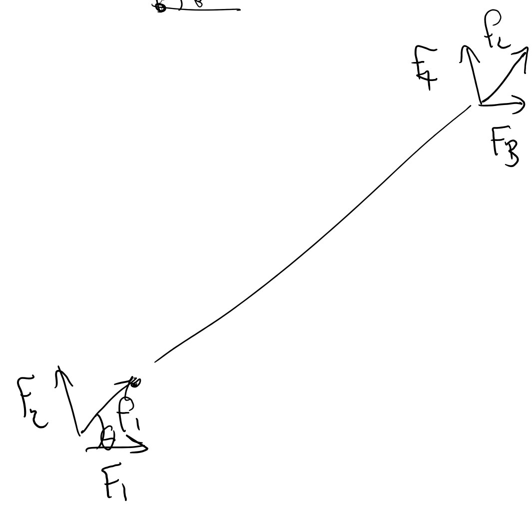
$u_1 = \sin\theta = s$ $\quad u_2 = 0$



$U_1 = \sin\theta = S \qquad U_2 = 0$



Calculating



$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \underbrace{\begin{bmatrix} C & 0 \\ S & 0 \\ 0 & C \\ 0 & S \end{bmatrix}}_{T^t} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$f_1 = 1 \qquad F_1 = \cos\theta$

$f_2 = 0 \qquad F_2 = \sin\theta$

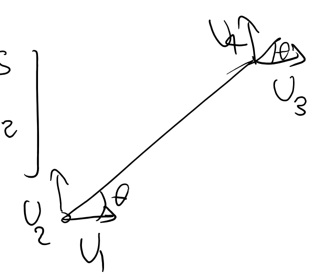
$F_3 = F_4 = 0$

recall $\quad T = T_0 U = \left[\begin{array}{cc|cc} c & s & 0 & 0 \\ 0 & 0 & c & s \end{array}\right]$

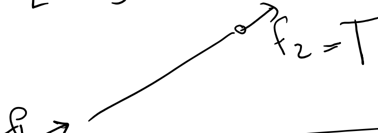
$$K = T^t k T = \left[\begin{array}{c|c} c & 0 \\ s & 0 \\ \hline 0 & c \\ 0 & s \end{array}\right] \left(\frac{AE}{L}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left[\begin{array}{cc|cc} c & s & 0 & 0 \\ 0 & 0 & c & s \end{array}\right]$$

I

$$\boxed{K = \frac{AE}{L} \left[\begin{array}{c|c} k_b & -k_b \\ \hline -k_b & k_b \end{array}\right], \quad k_b = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}}$$



$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = k T_0 U \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left[\begin{array}{cc|cc} c & s & 0 & 0 \\ 0 & 0 & c & s \end{array}\right] \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$



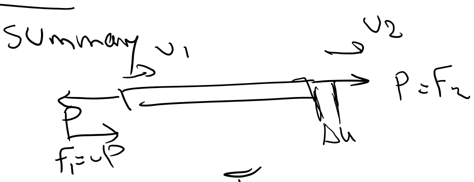
$T = f_2 = \frac{AE}{L}\left(c\,(\ldots\right.$

II

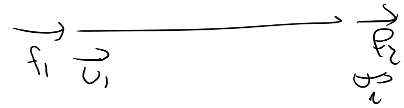
$$\boxed{T = f = \frac{AE}{L}\left(c\,(U_3 - U_1) + s\,(U_4 - U_2)\right)}$$

axial force

summary

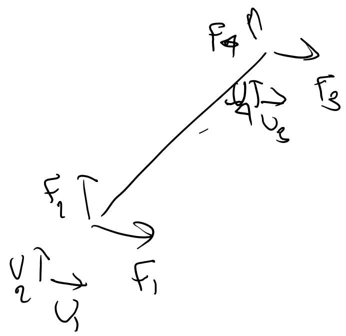


$P = \bar{k}\Delta u$



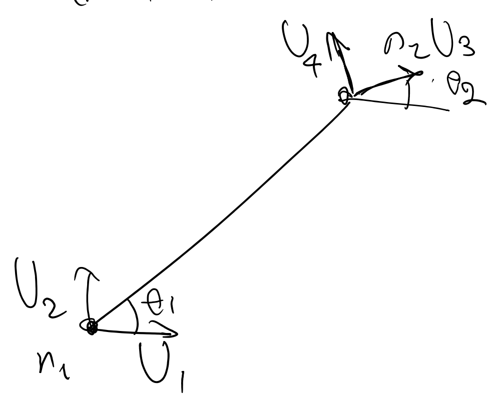
$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \bar{k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

needed this for 1D FEM

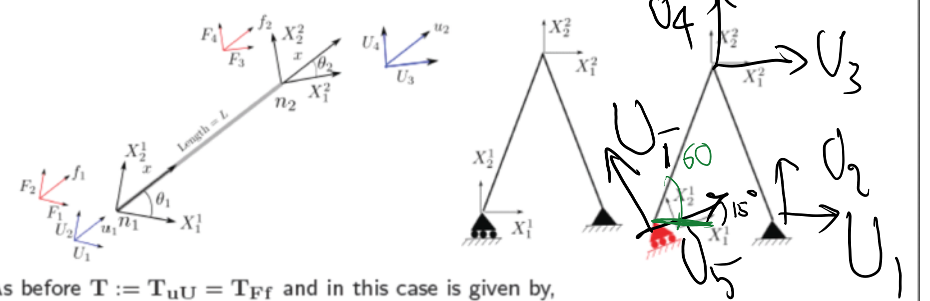


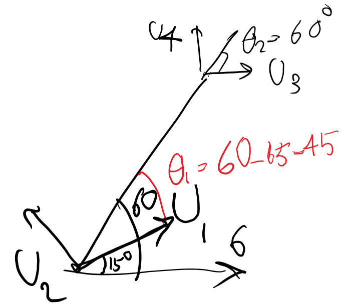
$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \bar{k}\left(\begin{array}{c|c} k_b & -k_b \\ \hline -k_b & k_b \end{array}\right)$$

There are cases that the two sides of the element don't have the same coordinate system



Truss element /two different coordinate systems





- As before $\mathbf{T} := \mathbf{T_{uU}} = \mathbf{T_{Ff}}$ and in this case is given by,

$$\mathbf{T} = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & c_2 & s_2 \end{bmatrix} \tag{393}$$

- Accordingly, from $\mathbf{K} = \mathbf{T}^{\mathrm{T}}\mathbf{k}\mathbf{T}$ we obtain,

$$\mathbf{K} = \frac{AE}{L}\begin{bmatrix} c_1^2 & c_1 s_1 & -c_1 c_2 & -c_1 s_2 \\ c_1 s_1 & s_1^2 & -c_2 s_1 & -s_1 s_2 \\ -c_1 c_2 & -c_2 s_1 & c_2^2 & c_2 s_2 \\ -c_1 s_2 & -s_1 s_2 & c_2 s_2 & s_2^2 \end{bmatrix} \tag{394}$$

- Finally the axial tensile force in the bar, which is the second line of $\mathbf{k}\mathbf{T_{uU}} = \mathbf{k}\mathbf{T}$ is (compare to one global coordinate in (387)):

$$\boxed{T = AE/L\,(-c_1 U_1 - s_1 U_2 + c_2 U_3 + s_2 U_4)} \tag{395}$$

319 / 456

$\rightarrow$ HW
Truss example
1 or 2 elements have different θ's on the two ends

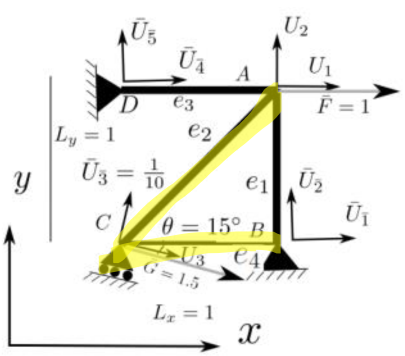
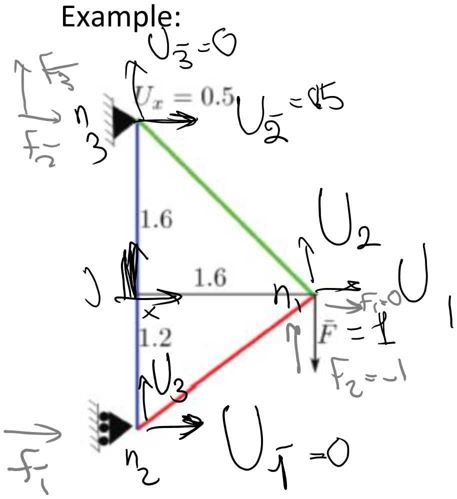


Figure 2: 3 dof truss with an angled support

Example:



$n = 3$ nodes $* \; 2 \frac{\text{dof}}{\text{node}} = 6$ total # of dofs

Start from node 1 to number dofs

$n_f = 3 \qquad n_p = 3$

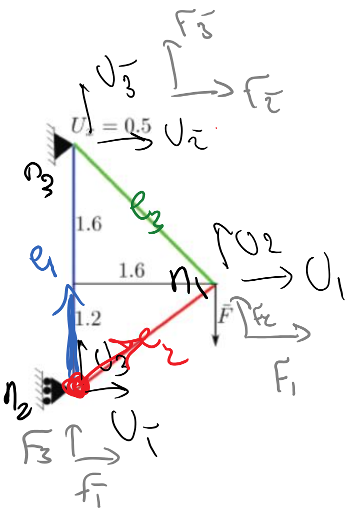
$$U_f = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = ? \qquad U_p = \begin{bmatrix} U_{\bar{1}} \\ U_{\bar{2}} \\ U_{\bar{3}} \end{bmatrix} = \begin{bmatrix} 0 \\ .5 \\ 0 \end{bmatrix}$$

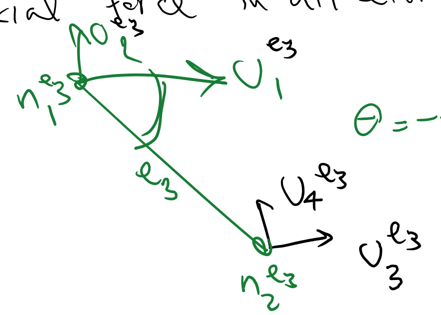
$$F_f = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad F_p = \begin{bmatrix} F_{\bar{1}} \\ F_{\bar{2}} \\ F_{\bar{3}} \end{bmatrix} = ?$$

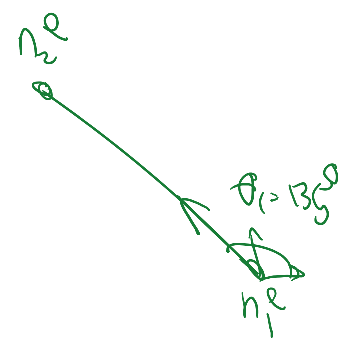
$n_1 : (1.6, 0)$
$n_2 : (0, -1.2)$
$n_3 : (0, 1.6)$

Reaction forces

we want to find
- U_f
- F_p
- axial force in all elements







$LEM_{e_3} = [3, 1]$

$LEM_{e_3} = [1, 3]$

this LEM is given in input files

My option

$M_{e_3} = [\bar{2}, \bar{3}, 1, 2]$

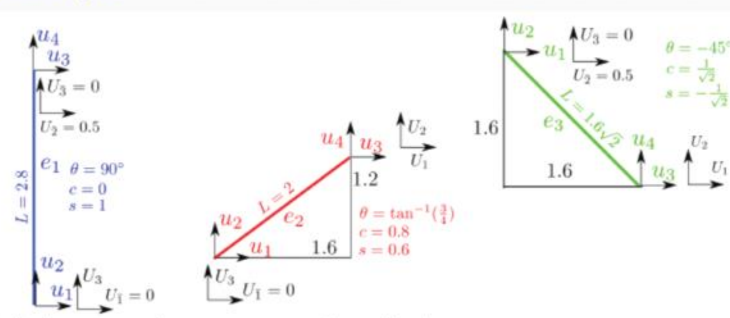
in FEM code this is formed internally

e_1: $\theta = 90°$ $L_{e_1} = 1.2 + 1.6 = 2.8$ $LEM_{e_1} = [2, 3]$ $M_{e_1} = [\bar{1}, 3, \bar{2}, \bar{3}]$

e_2: $\theta_1, \theta_2 = tg^{-1}\frac{3}{4}$ $L_{e_2} = \sqrt{(1.2)^2 + (1.6)^2} = 2$ $LEM_{e_2} = [2, 1]$ $M_{e_2} = [\bar{1}, 3, 1, 2]$

e_3 $\theta_1, \theta_2 = -45°$ $L_{e_3} = \sqrt{2}\,(1.6)$ $LEM = [3, 1]$ $M_{e_3} = [\bar{2}, \bar{3}, 1, 2]$

Truss Example



- Table below summarizes parameters for each element:

e	L^e	θ	c	s	$\mathbf{M}_t^e$
e_1	2.8	$90°$	0	1	$[\bar{1}\ \ 3\ \ \bar{2}\ \ \bar{3}]$
e_2	2	$\tan^{-1}(\frac{3}{4})$	0.8	0.6	$[\bar{1}\ \ 3\ \ 1\ \ 2]$
e_3	$1.6\sqrt{2}$	$-45°$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$[\bar{2}\ \ \bar{3}\ \ 1\ \ 2]$

Slide 322:

e_3	$1.6\sqrt{2}$	$-45°$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	[2 3 1 2]

Slide 322:

I'll do the calculation for e_3

$$K_{e_3} = \left(\frac{AE}{L}\right)_{e_3} \left[\begin{array}{c|c} K_b & -K_b \\ \hline -K_b & K_b \end{array}\right]$$

$AE = 1$

$L_{e_3} = 1.6\sqrt{2}$

$c = +\frac{\sqrt{2}}{2}$ $(\cos(-45°))$

$s = -\frac{\sqrt{2}}{2}$ $(\sin(-45°))$

$$K_b = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

$$K_{e_3} = \begin{array}{c|cccc} & \bar{2} & \bar{3} & 1 & 2 \\ \hline \bar{2} & .221 & -.221 & & \\ \bar{3} & -.221 & .221 & & \\ 1 & & & .221 & -.221 \\ 2 & & & -.221 & .221 \end{array}$$

$$K = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & +.221 & -.221 & \\ 2 & -.221 & .221 & \\ 3 & & & \end{array}$$

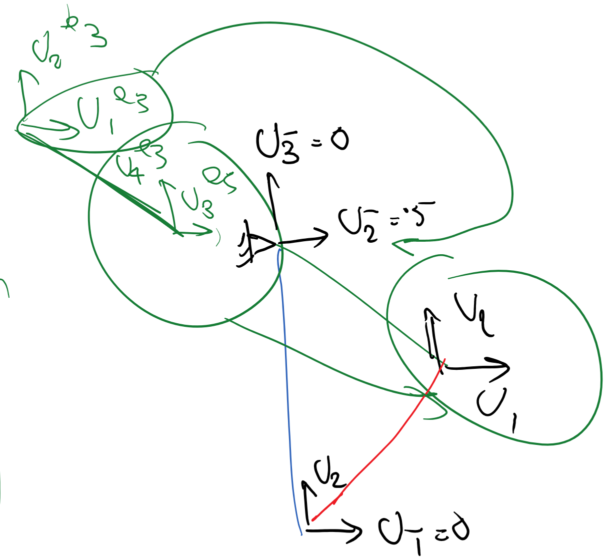
Forces of e_3:

$$f^{e_3} = f_r^{e_3} + \cancel{f_N^{e_3}} - f_D^{e_3}$$

$\rightarrow$ take care of this

$f_r^{e_3} \rightarrow 0$, ignoring the weight of the truss element

$f_N^{e_3} \rightarrow$ 1D element



$$f_D^{e_3} = K^{e_3} a^{e_3}$$

$$= \begin{bmatrix} .221 & -.221 & -.221 & +.221 \\ -.221 & .221 & +.221 & -.221 \\ -.221 & .221 & .221 & -.221 \\ .221 & -.221 & -.221 & .221 \end{bmatrix} \begin{bmatrix} .5 \\ 0 \\ \cancel{0} \\ \cancel{0} \end{bmatrix}$$

$$f_D^{e_3} = \begin{bmatrix} .1105 \\ -.1105 \\ -.1105 \\ .1105 \end{bmatrix}$$

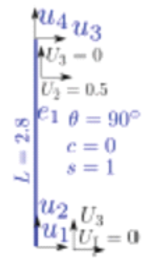
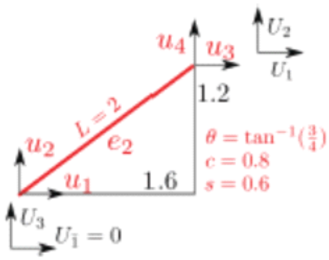
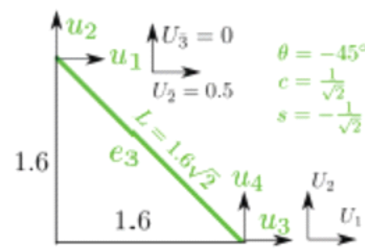
$$f^{e_3} = f_r^{e_3} + f_N^{e_3} - f_D^{e_3}$$

$$F_e = \begin{array}{c|c} 1 & +.1105 \\ 2 & -.1105 \\ \bar{3} & \end{array}$$

global element based force space of free dofs

$$= \begin{array}{c} \bar{2} \\ \bar{3} \\ 1 \\ 2 \end{array} \begin{bmatrix} \cancel{-.1105} \\ \cancel{.1105} \\ .1105 \\ -.1105 \end{bmatrix}$$

Truss example: Assembly of global system

e	e_1	e_2	e_3
			
$\mathbf{k}^e$	$\mathbf{k}^{e_1} = \frac{(1)(1)}{2.8}\begin{bmatrix} 0&0&0&0 \\ 0&1&0&-1 \\ 0&0&0&0 \\ 0&-1&0&1 \end{bmatrix}$ $= \begin{array}{c|c} & \bar 1 \;\; 3 \;\; \bar 2 \;\; \bar 3 \\ \begin{array}{c} \bar 1 \\ 3 \\ \bar 2 \\ \bar 3 \end{array} & \begin{bmatrix} 0&0&0&0 \\ 0&0.3571&0&-0.3571 \\ 0&0&0&0 \\ 0&-0.3571&0&0.3571 \end{bmatrix} \end{array}$	$\mathbf{k}^{e_2} = \frac{(1)(1)}{2}\begin{bmatrix} 0.64&0.48&-0.64&-0.48 \\ 0.48&0.36&-0.48&-0.36 \\ -0.64&-0.48&0.64&0.48 \\ -0.48&-0.36&0.48&0.36 \end{bmatrix}$ $= \begin{array}{c|c} & \bar 1 \;\; 3 \;\; 1 \;\; 2 \\ \begin{array}{c} \bar 1 \\ 3 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.32&0.24&-0.32&-0.24 \\ 0.24&0.18&-0.24&-0.18 \\ -0.32&-0.24&0.32&0.24 \\ -0.24&-0.18&0.24&0.18 \end{bmatrix} \end{array}$	$\mathbf{k}^{e_3} = \frac{(1)(1)}{1.6\sqrt{2}}\begin{bmatrix} 0.5&-0.5&-0.5&0.5 \\ -0.5&0.5&0.5&-0.5 \\ -0.5&0.5&0.5&-0.5 \\ 0.5&-0.5&-0.5&0.5 \end{bmatrix}$ $= \begin{array}{c|c} & \bar 2 \;\; \bar 3 \;\; 1 \;\; 2 \\ \begin{array}{c} \bar 2 \\ \bar 3 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.221&-0.221&-0.221&0.221 \\ -0.221&0.221&0.221&-0.221 \\ -0.221&0.221&0.221&-0.221 \\ 0.221&-0.221&-0.221&0.221 \end{bmatrix} \end{array}$
$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e = \begin{bmatrix} 0&0&0&0 \\ 0&0.3571&0&-0.3571 \\ 0&0&0&0 \\ 0&-0.3571&0&0.3571 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} = \begin{array}{c} \bar 1 \\ 3 \\ \bar 2 \\ \bar 3 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e = \begin{bmatrix} 0.32&0.24&-0.32&-0.24 \\ 0.24&0.18&-0.24&-0.18 \\ -0.32&-0.24&0.32&0.24 \\ -0.24&-0.18&0.24&0.18 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{array}{c} \bar 1 \\ 3 \\ 1 \\ 2 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_3}\mathbf{a}_3^e = \begin{bmatrix} 0.221&-0.221&-0.221&0.221 \\ -0.221&0.221&0.221&-0.221 \\ -0.221&0.221&0.221&-0.221 \\ 0.221&-0.221&-0.221&0.221 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{array}{c} \bar 2 \\ \bar 3 \\ 1 \\ 2 \end{array}\begin{bmatrix} 0.1106 \\ -0.1105 \\ -0.1105 \\ 0.1106 \end{bmatrix}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \mathbf{f}_r^{e_1} + \mathbf{f}_N^{e_1} - \mathbf{f}_D^{e_1} = \begin{array}{c} \bar 1 \\ 3 \\ \bar 2 \\ \bar 3 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \mathbf{f}_N^{e_2} - \mathbf{f}_D^{e_2} = \begin{array}{c} \bar 1 \\ 3 \\ 1 \\ 2 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{f}_e^{e_3} = \mathbf{f}_r^{e_3} + \mathbf{f}_N^{e_3} - \mathbf{f}_D^{e_3} = \begin{array}{c} \bar 2 \\ \bar 3 \\ 1 \\ 2 \end{array}\begin{bmatrix} -0.1105 \\ 0.1105 \\ 0.1105 \\ -0.1105 \end{bmatrix}$

$$\mathbf{K} = \begin{bmatrix} 0.32+0.221 & 0.24-0.221 & -0.24 \\ 0.24-0.221 & 0.18+0.221 & -0.18 \\ -0.24 & -0.18 & 0.3571+0.18 \end{bmatrix}$$

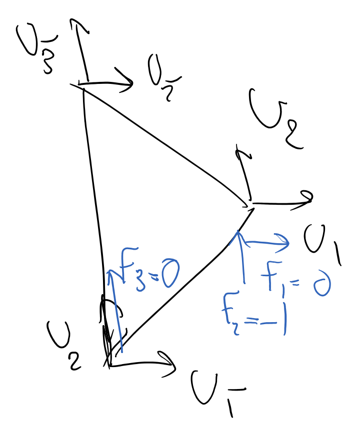
$$F_e = \begin{bmatrix} 0.1106 \\ -0.1105 \\ 0 \end{bmatrix}$$

the assembly of all element forces

$$F_E = F_N = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

free forces

nodal force



[.1105 [0 [.1105

free ... nodal force

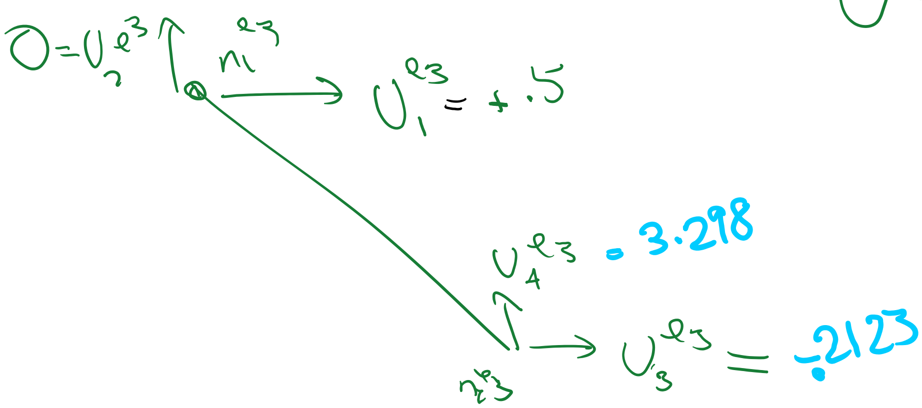
$$F = F_e + F_n = \begin{bmatrix} .1105 \\ -.1105 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} .1105 \\ -1.1105 \\ 0 \end{bmatrix}$$

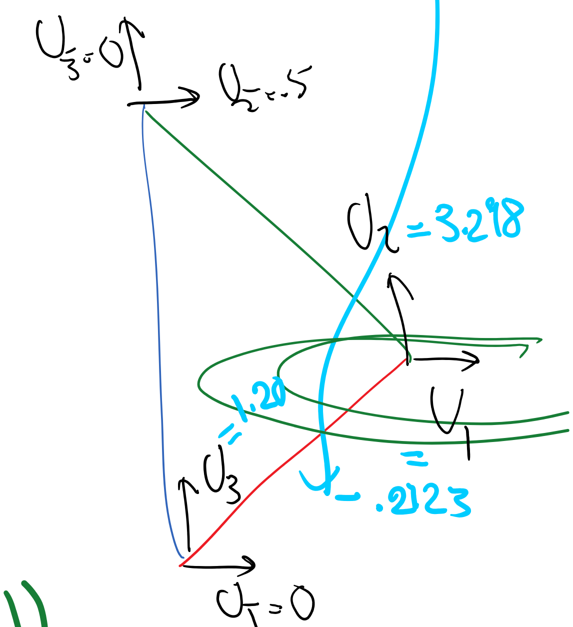
$$KU = F \longrightarrow \quad U = K^{-1}F = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} -.2123 \\ 3.298 \\ -1.20 \end{bmatrix}$$

— element axial forces

— Reaction forces

LEM = [2, 3, 1, 2]





$$T_{e3} = \left(\frac{AE}{L}\right)^{e_3}\left(c(\underbrace{U_3}_{-.2123} - \underbrace{U_1}_{.5}) + s(\underbrace{U_4}_{3.298} - \underbrace{U_2}_{0})\right)$$

$= 0.8064$