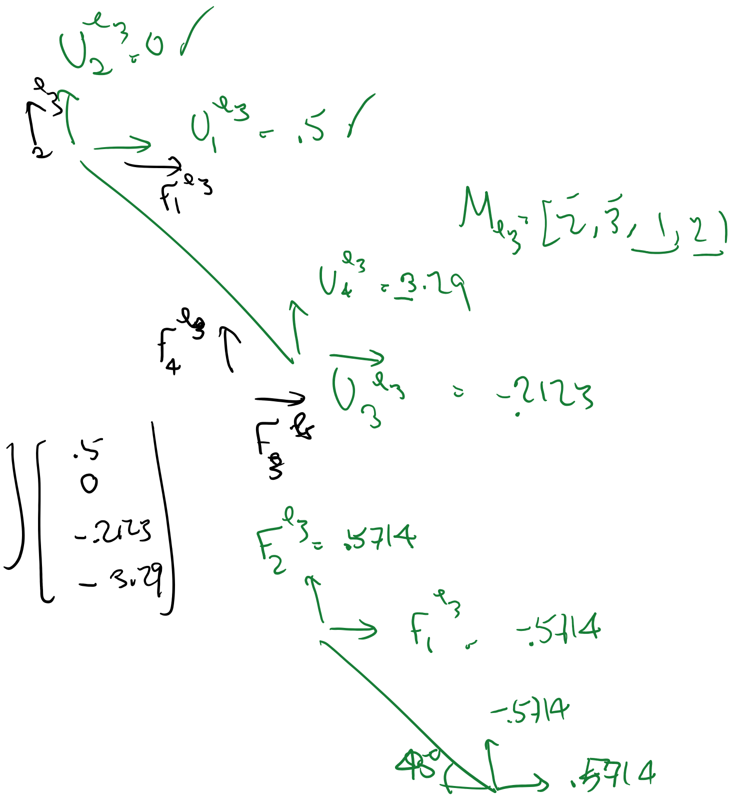
2022/11/01

Tuesday, November 1, 2022 11:21 AM

Calculating Support (prescribed dof) forces

from last time element # 3

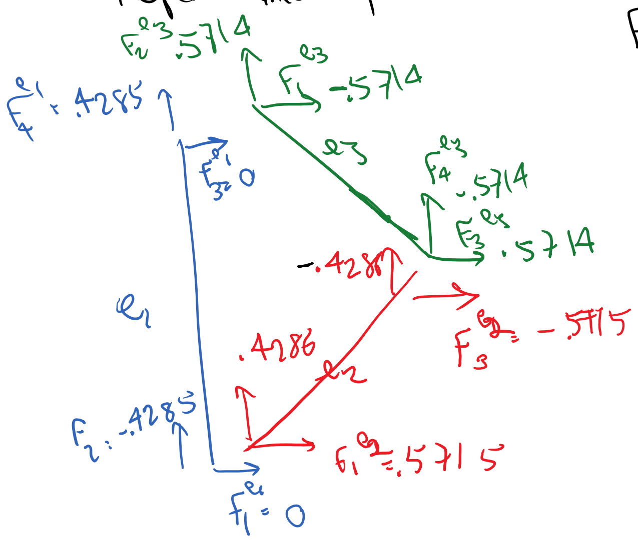
$$U = \begin{bmatrix} -.2123 \\ -3.29 \\ -1.2 \end{bmatrix}$$



$$F^{e_3} = k^{e_3} \underbrace{a^{e_3}}_{f_D^e} = \begin{bmatrix} .221 & & \\ & \ddots & \\ & & \ddots \end{bmatrix} \begin{bmatrix} .5 \\ 0 \\ -.2123 \\ -3.29 \end{bmatrix}$$

$$= \begin{bmatrix} -.5714 \\ .5714 \\ .5714 \\ -.5714 \end{bmatrix}$$

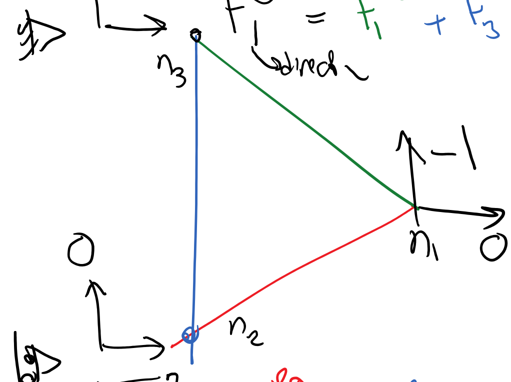
Repeat this process for all the elements



$$F_2^3 = F_2^{e_3} + F_4^{e_1} = 1$$

node # x dired

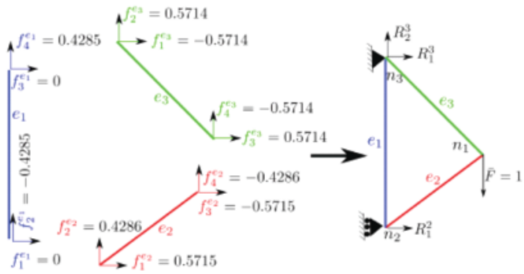
$$F_1^{(3)} = F_1^{e_3} + F_3^{e_1} = -.5714$$



node 2

$$F_1^2 = F_1^{e_2} + F_1^{e_1} = .5715$$

Truss Example: Reaction Forces



- First, we compute reaction forces by adding up forces from individual elements that contribute to reaction forces:

$$R_1^2 = f_1^{e_1} + f_1^{e_2} = 0 + 0.5715 = 0.5715 \quad (397a)$$
$$R_1^3 = f_3^{e_1} + f_1^{e_3} = 0 + -0.5714 = -0.5714 \quad (397b)$$
$$R_2^3 = f_4^{e_1} + f_2^{e_3} = 0.4285 + 0.5714 = 0.9999 \quad (397c)$$

Truss Example: Observations

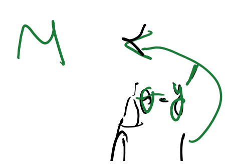
- The fact that our free dof forces in (398) are the values enforced is a necessary (but not sufficient) check on the correctness of our FEM solutions.
- We observe that exact (direct solutions in (399)) match our FEM solution in the last table and (397).
- The reason FEM and exact solutions match is that FEM shape functions (linear displacement within each bar) can capture the exact solution. In general, whenever, FEM approximate solution space can capture exact solution, FEM recovers the exact solution.
- Small error between FEM and direct method or about 0.0001 errors in some reaction and sum of forces in FEM method are finite numerical precision error which are different from discretization errors caused by approximating an infinite solution space by FEM shape functions. The former is caused by working with finite number of digits in our calculations.

Review: Euler Bernoulli Beam, Strong and WR statements

1 higher order DE $\Longrightarrow$

multiple dofs per node even in 1D

Sample Boundary value problems: Euler Bernoulli beam

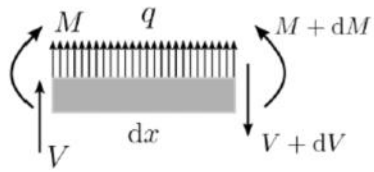


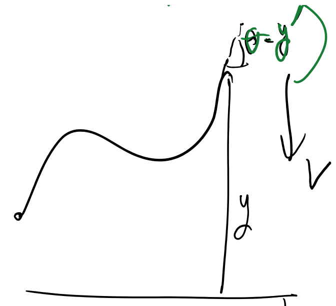
$$
\begin{aligned}
\nabla.\mathbf{F} - \mathbf{r} &= 0 && \text{Balance law} \\
\mathbf{F} &= \begin{bmatrix} M \\ V \end{bmatrix} && \text{Flux} \\
\mathbf{r} &= \begin{bmatrix} V \\ q \end{bmatrix} && \text{Source term} \\
M &= EI\kappa && \text{Constitutive equation} \\
\kappa &= \frac{d^2 y}{dx^2} && \text{Kinematic compatibility}
\end{aligned}
\Rightarrow
\begin{array}{l}
\boxed{\begin{aligned} \frac{dM}{dx} - V &= 0 \\ \frac{dV}{dx} - q &= 0 \end{aligned}} \\
\boxed{\frac{d^2 EI}{dx^2}\left(\frac{d^2 y}{dx^2}\right) - q = 0}
\end{array}
\qquad (28)
$$

Essential BC

where

M = Momentum
V = Shear force
q = Distributed load
E = Elastic modulus
I = Second moment of area
κ = Curvature
y = Vertical displacement





order

Kinematic $\left\{ \begin{array}{l} y \\ \theta : y' \end{array} \right.$ 0 1

Force-like $\left\{ \begin{array}{l} M = EIy'' \\ V = \frac{dM}{dx} = (EIy'')' \end{array} \right.$ 2 3

go with natural BC

DE $\quad (EIy'')'' + q = 0$

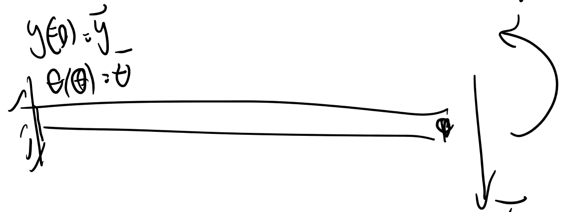
$M = 4$

Weighted residual statement

$R_i = (EIy'')'' + q$

∂D_f

∂D_u



$R_f = \bar{V} - V$

& $(\bar{M} - M)$

See slide 44 and after

Dropping the WR term on Essential Boundary $\partial\mathcal{D}_u$.

We noticed that in the original statement $w \in C^3(\mathcal{D})$ because of the weighted residual term on the essential boundary $\partial\mathcal{D}_u$. Suppose we could modify the weighted residual to (dropping the integral on $\partial\mathcal{D}_u$).

$$\forall \mathbf{w} \in \mathcal{W}^{\mathrm{WRS}} \; : \int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i \, dv + \int_{\partial\mathcal{D}_f} \mathbf{w}.\mathcal{R}_f \, ds = 0 \qquad (42)$$

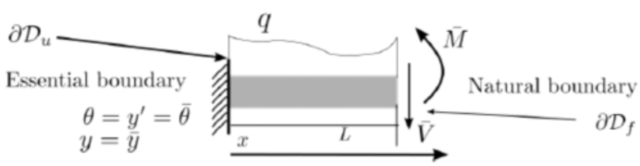
We will shortly discuss what the space $\mathcal{W}^{\mathrm{WRS}}$ would be. The weight and residual functions are:

Term	Domain	Weight: function	Weight: order	Residual: function	Residual: order
Interior	$\partial\mathcal{D}$	w	0	$\mathcal{R}_i = \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - q$	4
Natural Boundary	$\partial\mathcal{D}_f$	$\mathbf{w}_f = \begin{bmatrix} -w' \\ w \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\mathcal{R}_f = \begin{bmatrix} \bar{M} - M(y) \\ \bar{V} - V(y) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- What is the maximum derivative order for trial functions? 4 for the $\mathcal{R}_i$.
- What is the function space for trial functions? $C^4(\mathcal{D})$.
- What is the maximum derivative order for weight functions? 1.
- What would be the function space for the weight functions? $C^1(\mathcal{D})$.
- Trial and weight function spaces are $y \in C^4(\mathcal{D})$ and $w \in C^1(\mathcal{D})$.

- What is the maximum derivative order for weight functions? 1.
- What would be the function space for the weight functions? $C^1(\mathcal{D})$.
- Trial and weight function spaces are $y \in C^4(\mathcal{D})$ and $w \in C^1(\mathcal{D})$.

Weighted residual statement to Weak statement



Since we strongly enforce the essential boundary condition, the weighted residual for this problem simplifies to:

4 derivatives

cancel out after IBP

0 dr

$$
\begin{aligned}
0 &= \int_{\mathcal{D}} w\mathcal{R}_i(y)\,\mathrm{d}v + \int_{\partial\mathcal{D}_f} \mathrm{w}_f \mathcal{R}_f(y)\,\mathrm{d}s \\
&= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - q\right)\mathrm{d}x + \begin{bmatrix} -\frac{\mathrm{d}w}{\mathrm{d}x} \\ w \end{bmatrix} \cdot \begin{bmatrix} \cancel{\bar{M} - M} \\ \cancel{\bar{V} - V} \end{bmatrix}\Big|_{x=L} \qquad (55)\\
&= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - q\right)\mathrm{d}x - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y))|_{x=L} + w(\bar{V} - V(y))|_{x=L}
\end{aligned}
$$

1 IBP

Next, we transfer derivatives from y to w (trial function to weight function). We note that

1 3

$$
\begin{aligned}
\int_0^L w\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)\mathrm{d}x &= \int_0^L \left[-\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}EI\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)\right]\mathrm{d}x + \left[w\frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)\right]\Big|_{x=0}^{x=L} \qquad (56)\\
&= \underline{\int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right]\mathrm{d}x} + [wV(y)]\Big|_{x=0}^{x=L} - \left[\frac{\mathrm{d}w}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)\right]\Big|_{x=0}^{x=L}
\end{aligned}
$$

2 IBP

Plugging (55) in (56) yields,

$$
\begin{aligned}
0 &= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - q\right)\mathrm{d}x - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y))|_{x=L} + w(\bar{V} - V(y))|_{x=L} \\
&= \left\{\int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq\right]\mathrm{d}x + \left[wV(y)] - \frac{\mathrm{d}w}{\mathrm{d}x}M(y)\right]\Big|_{x=0}^{x=L}\right\} \\
&- \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y))|_{x=L} + w(\bar{V} - V(y))|_{x=L} \qquad (57)\\
&= \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq\right]\mathrm{d}x \\
&+ \left\{wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x}M(y) - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y)) + w(\bar{V} - V(y))\right\}_{x=L} \\
&- \left\{wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x}M(y)\right\}_{x=0}
\end{aligned}
$$

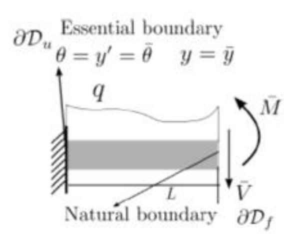
Weighted residual statement to Weak statement

This equation simplifies to

$$
0 = \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq\right]\mathrm{d}x + \left\{-\frac{\mathrm{d}w}{\mathrm{d}x}\bar{M} + w\bar{V}\right\}_{x=L} \qquad (58a)
$$

$$
+\left\{w\left(V(y) - V(y)\right) - \frac{\mathrm{d}w}{\mathrm{d}x}\left(M(y) - M(y)\right)\right\}_{x=L} \qquad (58b)
$$

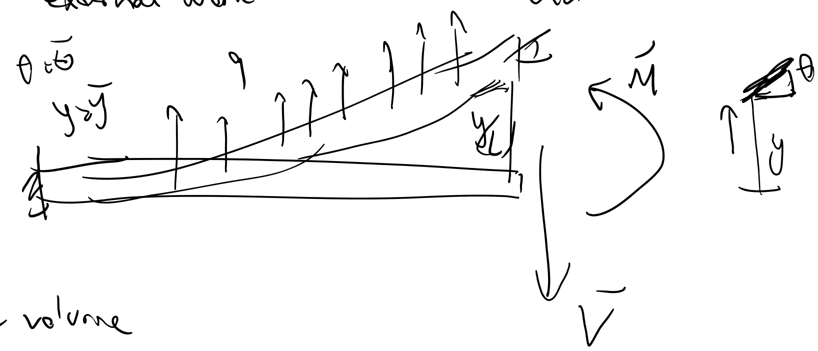
$$
-\left\{wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x}M(y)\right\}_{x=0} \qquad (58c)
$$



Above was the process of forming residuals, multiplying them by appropriate weight functions, IBP twice -> weak statement

Energy approach

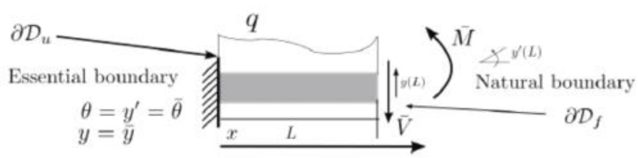
$$\Pi = \underbrace{V}_{\text{internal energy}} - \underbrace{W}_{\text{external work}}$$



$$W = -y(L)\bar{V} + \underbrace{\theta(L)}_{y'(L)}\bar{M} + \int_0^L y(x)\,q(x)\,dx$$

$$V = \int_{\mathcal{D}} \underbrace{e(\varepsilon)}_{\text{strain energy per volume}} dV$$

Example: Euler Bernoulli beam



We determined the internal energy of the beam to be (*cf.* (85c)),

$$V = \int_{\mathcal{D}} \frac{1}{2}\epsilon\sigma\,\mathrm{d}v = \int_0^L \left(\int_A \frac{1}{2}\epsilon^2 E\,\mathrm{d}A\right)\mathrm{d}x = \int_0^L \left(\int_A \frac{1}{2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}z\right)^2 E\,\mathrm{d}A\right)\mathrm{d}x$$

$$= \int_0^L \frac{1}{2}E\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)^2\underbrace{\left(\int_A z^2\,\mathrm{d}A\right)}_{I}\mathrm{d}x \Rightarrow$$

$EI\,y''^2$

$$\boxed{V = \int_0^L \frac{1}{2}\boxed{EI\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)^2}\mathrm{d}x} \qquad (100)$$

$$\delta\left(\frac{1}{2}EI\,Y^2\right) = \frac{\partial\,\frac{1}{2}EI\,Y^2}{\partial Y}\,\delta Y$$

for use $Y = y''$

$$\Pi(y) = V - W = \int_0^L \frac{1}{2}EI\,y''^2\,dx - \int_0^L y\,q\,dx - y'(L)\bar{M} + y(L)\bar{V}$$

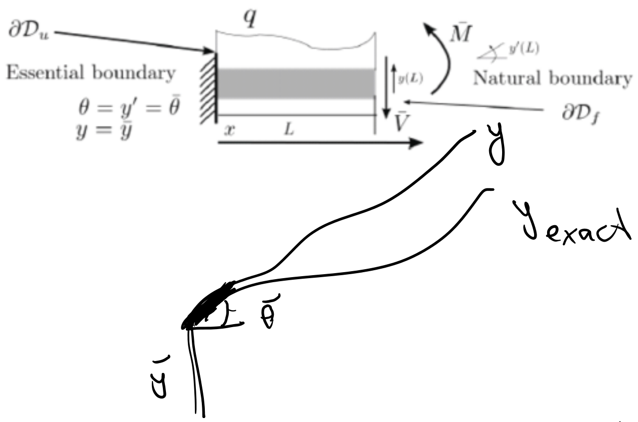
Minimize the energy

$$\delta\Pi = \int \underbrace{\frac{\partial\,\frac{1}{2}EI\,y''^2}{\partial y''}}_{EI\,y''}\delta y''\,dx - \int \delta y\,q\,dx - \delta y'(L)\bar{M} + \delta y(L)\bar{V}$$

Find $y \in \mathcal{V} = \{ f \in \cancel{C^2}\ C^1 \mid f(0) = \bar{y} \quad f'(0) = \bar{\theta} \}$ C^1

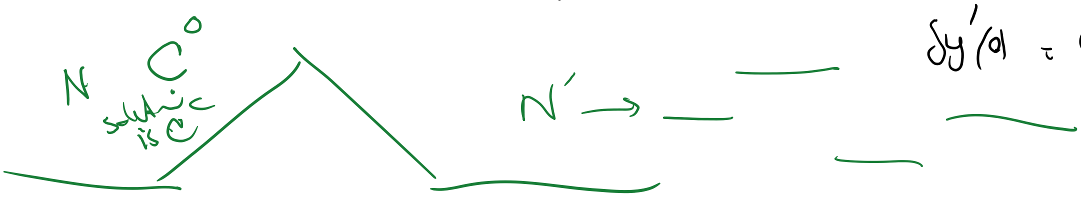
P such that $\forall\, \delta y \in W = \{ f \in \cancel{C^2} \mid f(0) = 0,\ f'(0) = 0 \}$

$$\int_0^L \overset{w''}{(\delta y)''}\, EI y''\, dx = \int_0^L \overset{w}{\delta y}\, q\, dx + \overset{w(L)}{(\delta y)'}(L)\bar{M} - \overset{w(L)}{\delta y(L)}\bar{V}$$



$\delta y(0) = y(0) - y_{exact}(0) = \bar{y} - \bar{y} = 0$

$\delta y'(0) = y'(0) - y'_{exact}(0) = \bar{\theta} - \bar{\theta} = 0$



bar problem

$$\int w' EA u'$$

$u \in V = \{ f \in C^0 \mid - \quad - \}$

Conventional (continuus) finite element methods:

Strong Form order $M = 2m$ $\Rightarrow$ Trial functions are C^{m-1}

Takeaways

for beams

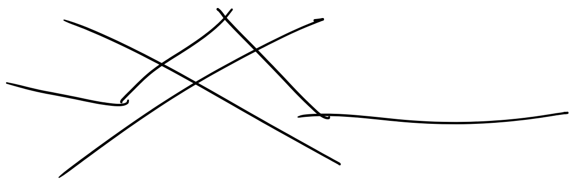
$$\int_0^L w'' \underbrace{EI}_{D.EI} y''\, dx = \int_0^L w\, q(x)\, dx + \ldots$$

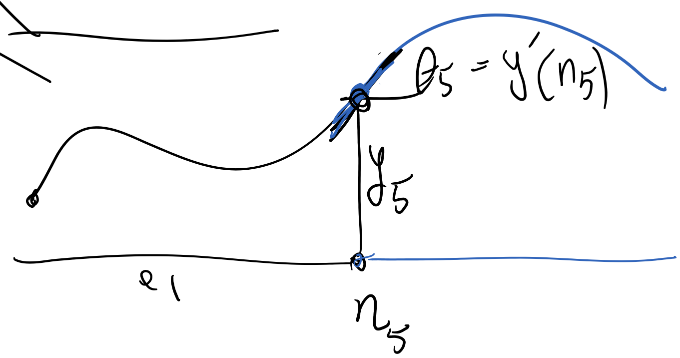
for beams $\int_0 w^* \underbrace{EI}_{D=EI} y\,dx = \int_0 w f(x)\,dx + \ldots$

$L_m(y) = (\;)''$

$M=4 \longrightarrow m=2$

C^{m-1} continuity is needed $\longrightarrow$

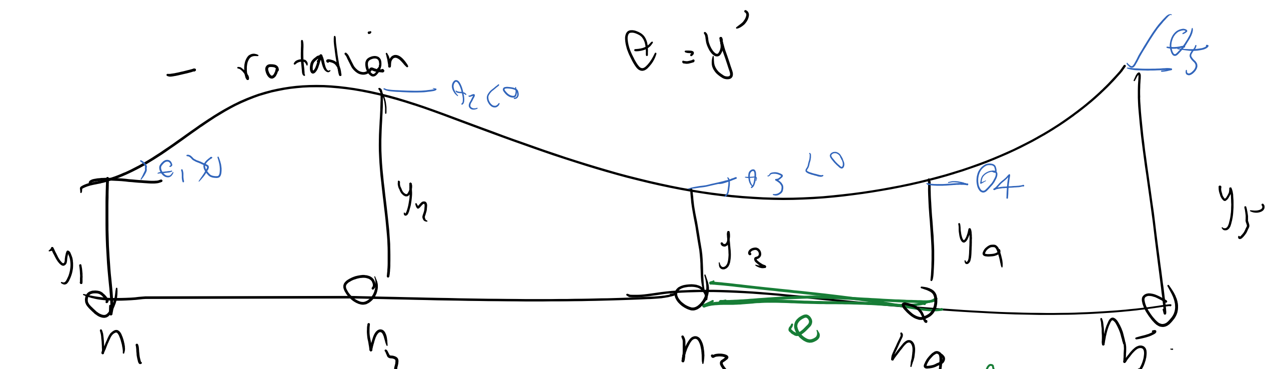




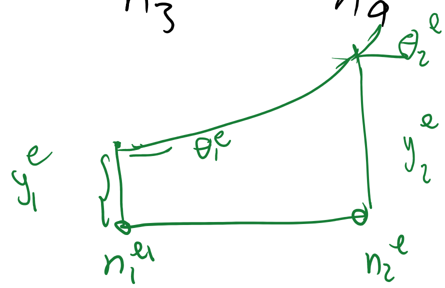
y & y' should be continuous across elements

$\Longrightarrow$ each node needs to have 2 global dofs to ensure C^1 continuity

- vertical displacement y
- rotation $\theta = y'$



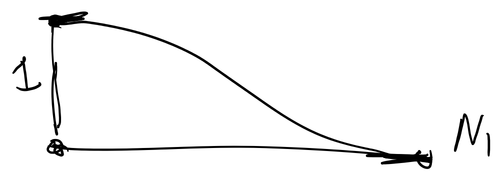
$$a^e = \begin{bmatrix} y_1^e \\ \theta_1^e \\ \hline y_2^e \\ \theta_2^e \end{bmatrix}$$



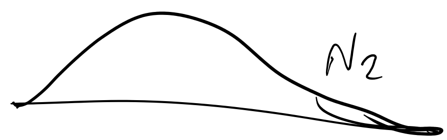
we need 4 shape functions

N_1 ...

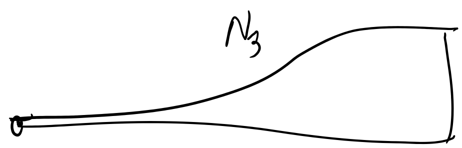
N_1 $y_1 = 1$, $\theta_1 = 0$, $y_2 = 0$, $\theta_2 = 0$



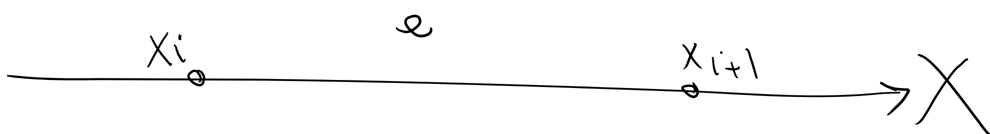
N_2 $y_1 = 0$, $\theta_1 = 1$, $y_2 = 0$, $\theta_2 = 0$



N_3 $y_2 = 1$



N_4



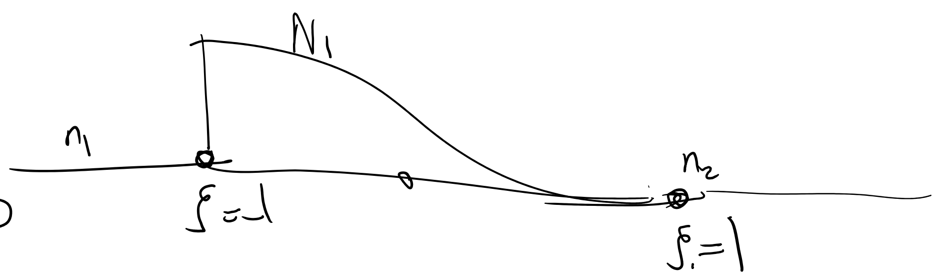
N_1
4 eqns

$$N_1(\xi_1) = N_1(\xi = -1) = 1$$

$$\frac{dN_1}{dx}(\xi_1) = \frac{dN_1}{dx}(\xi = -1) = 0$$

$$N_1(\xi_2) = N_1(\xi = 1) = 0$$

$$\frac{dN_1}{dx}(\xi_2) = \frac{dN_1}{dx}(\xi = 1) = 0$$



$$N_1(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3$$

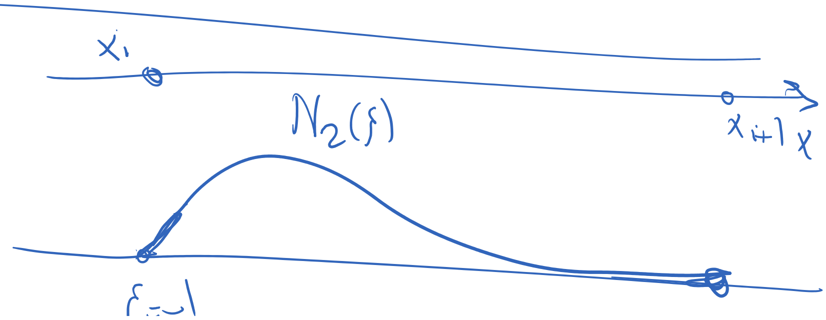
we need 3rd order shape functions (4 coefficient → unknowns)

We obtain these using the four equations above

Calculations for N_2

$$N_2(-1) = 0$$

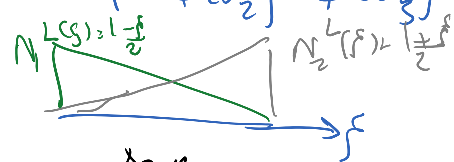
$$\frac{d}{dx} N_2(-1) = 1$$



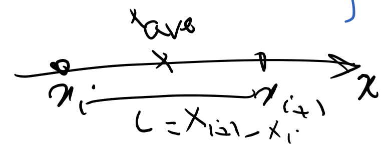
① $\begin{cases} \frac{d}{dx} N_2(-1) = 1 \\ N_2(1) = 0 \\ \frac{d}{dx} N_2(1) = 0 \end{cases}$

② $\begin{cases} N_2(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3 \\ \frac{dN_2}{d\xi}(\xi) = \alpha_1 + 2\alpha_2 \xi + 3\alpha_3 \xi^2 \end{cases}$

we need a relation between x & ξ



$x(\xi) = x_i N_1^L(\xi) + x_{i+1} N_2^L(\xi)$



$$\boxed{x(\xi) = x_{ave} + \frac{L}{2}\xi} \quad ③$$

② $\frac{dN_2}{dx}(\xi) = \frac{dN_2}{d\xi} \cdot \frac{d\xi}{dx} = \frac{dN_2}{d\xi} \cdot \frac{1}{\frac{dx}{d\xi}} = \frac{dN_2}{d\xi}\left(\frac{1}{L/2}\right)$

eq3 $\rightarrow \frac{dx}{d\xi} = \frac{L}{2}$

Noting $\frac{dN_2}{dx} = \frac{2}{L}\frac{dN_2}{d\xi}$ in eqn 1

④ $\begin{cases} N_2(-1) = 0 \\ \frac{2}{L}\frac{dN_2}{d\xi}(-1) = 1 \\ N_2(1) = 0 \\ \frac{2}{L}\frac{dN_2}{d\xi}(1) = 0 \end{cases}$

$N_2(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3$

$\frac{dN_2}{d\xi} = \alpha_1 + 2\alpha_2 \xi + 3\alpha_3 \xi^2$

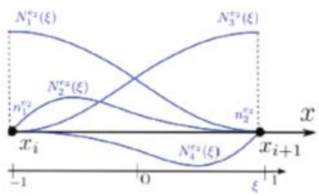
④ $\begin{cases} \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 = 0 \\ \alpha_1 - 2\alpha_2 + 3\alpha_3 = \frac{L}{2} \\ \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = 0 \end{cases}$

4 unknowns
4 eqns

$$\boxed{N_2(\xi) = \frac{L}{2}\left(1 - \xi + \xi^2 - \xi^3\right)}$$

Do the same process for all the other shape functions

FEM formulation of beam elements: Shape functions



- From (417) and (418) we get:

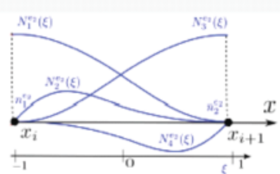
$$
\begin{array}{llllllll}
N_1(\xi=-1) & =1 & \frac{\mathrm{d}N_1}{\mathrm{d}\xi}(\xi=-1) & =0 & N_1(\xi=1) & =0 & \frac{\mathrm{d}N_1}{\mathrm{d}\xi}(\xi=1) & =0 \\
N_2(\xi=-1) & =0 & \frac{\mathrm{d}N_2}{\mathrm{d}\xi}(\xi=-1) & =\frac{L^e}{2} & N_2(\xi=1) & =0 & \frac{\mathrm{d}N_2}{\mathrm{d}\xi}(\xi=1) & =0 \\
N_3(\xi=-1) & =0 & \frac{\mathrm{d}N_3}{\mathrm{d}\xi}(\xi=-1) & =0 & N_3(\xi=1) & =1 & \frac{\mathrm{d}N_3}{\mathrm{d}\xi}(\xi=1) & =0 \\
N_4(\xi=-1) & =0 & \frac{\mathrm{d}N_4}{\mathrm{d}\xi}(\xi=-1) & =0 & N_4(\xi=1) & =0 & \frac{\mathrm{d}N_4}{\mathrm{d}\xi}(\xi=1) & =\frac{L^e}{2}
\end{array}
\tag{419}
$$

- Since each N_i has four conditions, we interpolate them with cubic polynomials:

$$N_i = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 \quad \Rightarrow \quad \frac{\mathrm{d}N_i}{\mathrm{d}\xi} = \alpha_1 + 2\alpha_2\xi + 3\alpha_3\xi^2 \tag{420}$$

α_j are determined from the conditions in (419).

FEM formulation of beam elements: Shape functions



- For example, to determine N_1 from (419) and (420) we observe:

$$
\left.\begin{array}{ll}
N_1(\xi=-1) & =1 \\
\frac{\mathrm{d}N_1}{\mathrm{d}\xi}(\xi=-1) & =0 \\
N_1(\xi=1) & =0 \\
\frac{\mathrm{d}N_1}{\mathrm{d}\xi}(\xi=1) & =0
\end{array}\right\}
\Rightarrow
\begin{array}{r}
\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 \\
\alpha_1 - 2\alpha_2 + 3\alpha_3 \\
\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \\
\alpha_1 + 2\alpha_2 + 3\alpha_3
\end{array}
\left.\begin{array}{l}
=1 \\ =0 \\ =0 \\ =0
\end{array}\right\}
\Rightarrow
\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}
=
\begin{bmatrix}\frac{1}{2}\\ -\frac{3}{4}\\ 0\\ \frac{1}{4}\end{bmatrix}
$$

$$\boxed{N_1(\xi) = \frac{1}{4}(2 - 3\xi + \xi^3)}$$

same

→ changes

- Similarly for N_2:

$$
\left.\begin{array}{ll}
N_2(\xi=-1) & =0 \\
\frac{\mathrm{d}N_2}{\mathrm{d}\xi}(\xi=-1) & =\frac{L^e}{2} \\
N_2(\xi=1) & =0 \\
\frac{\mathrm{d}N_2}{\mathrm{d}\xi}(\xi=1) & =0
\end{array}\right\}
\Rightarrow
\begin{array}{r}
\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 \\
\alpha_1 - 2\alpha_2 + 3\alpha_3 \\
\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \\
\alpha_1 + 2\alpha_2 + 3\alpha_3
\end{array}
\left.\begin{array}{l}
=0 \\ =\frac{L^e}{2} \\ =0 \\ =0
\end{array}\right\}
\Rightarrow
\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}
=
\frac{L^e}{8}\begin{bmatrix}1\\ -1\\ -1\\ 1\end{bmatrix}
$$

$$\boxed{N_2(\xi) = \frac{L^e}{8}(1 - \xi - \xi^2 + \xi^3)}$$

FEM formulation of beam elements: Shape functions

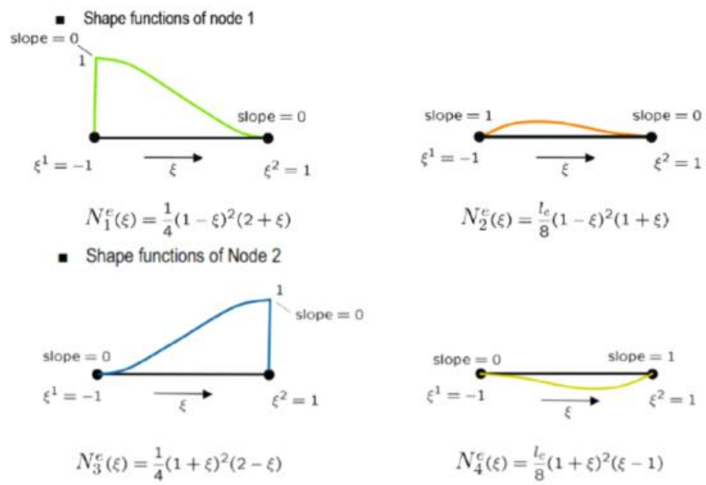


figure from F. Cirak

$$\int w'' \underbrace{EI}_{D} y'' \, dx$$

$$L_m(y) = y''$$

$$K = \int B^T D B \, dV$$

$$B = L_m(N) = N''$$