2022/11/03

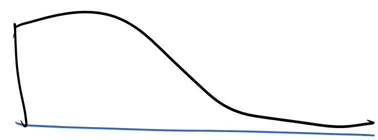
Thursday, November 3, 2022 11:20 AM

Calculating the stiffness matrix for the beam element

$$\int_{\Omega} \underbrace{w''}_{L_m(w)=w''} \, EI \, \underbrace{y''}_{L_m(y)=y''} \, dx$$

LHS from weak statement

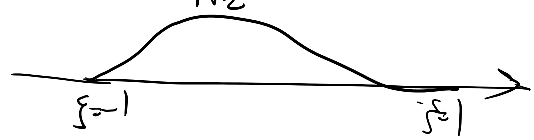
N1



$$k^e = \int_e B_e^t D B^e \, dv$$

$$B_e = L_m(N) = \left[\underset{B_1}{\frac{d^2 N_1}{dx^2}} \quad \cdots \quad \underset{B_4}{\frac{d^2 N_4}{dx^2}} \right]$$

N2



$$B_2 = ? \quad , \quad N_2(\xi) = \frac{L_e}{8}(1 - \xi - \xi^2 + \xi^3)$$

$$\frac{dN_2}{dx} = \frac{1}{\frac{dx}{d\xi}} \cdot \frac{dN_2}{d\xi} = \frac{1}{\frac{L_e}{2}} \cdot \frac{L_e}{8}(-1 - 2\xi + 3\xi^2) = \frac{1}{4}(-1 - 2\xi + 3\xi^2)$$

from last time $\frac{L_e}{2}$

$$\frac{d^2 N_2}{dx^2} = \frac{1}{\frac{dx}{d\xi}} \frac{d\left(\frac{dN_2}{d\xi}\right)}{d\xi} = \frac{1}{\frac{L_e}{2}} \cdot \frac{1}{4}(-2 + 6\xi) = \boxed{\frac{-1+3\xi}{L_e}}$$

Calculating other ones

$$\boxed{B = \frac{d^2 N}{dx^2} = \frac{1}{J^2} \underset{B_\xi}{\frac{d^2 N}{d\xi^2}} = \frac{4}{L^2} B_\xi = \left[\frac{6\xi}{L_e^2} \quad \frac{-1+3\xi}{L_e} \quad \frac{-6\xi}{L_e^2} \quad \frac{1+3\xi}{L_e} \right]}$$

$J = \frac{L_e}{2}$

$$k^e = \int_e B^{e^t} D B^e \, (dx \rightarrow J d\xi) \Rightarrow$$

$$\boxed{k^e = \int_{-1}^{1} \begin{bmatrix} \frac{6\xi}{L^2} \\ \frac{-1+3\xi}{L} \\ \frac{-6\xi}{L^2} \\ \frac{1+3\xi}{L} \end{bmatrix} EI(\xi) \left[\frac{6\xi}{L^2} \quad \frac{-1+3\xi}{L} \quad \frac{-6\xi}{L^2} \quad \frac{1+3\xi}{L} \right] \frac{L}{2} d\xi}$$

FEM

K

for EI constant $\rightarrow$

- If E and I are constant, we can take those out of the equation and have:

$$\mathrm{k}^e = \frac{EI}{L^{e3}}\begin{bmatrix} 12 & 6L^e & -12 & 6L^e \\ & 4L^{e2} & -6L^e & 2L^{e2} \\ & & 12 & -6L^e \\ \text{sym.} & & & 4L^{e2}\end{bmatrix} \quad \text{for constant } E \text{ and } I \qquad (427)$$

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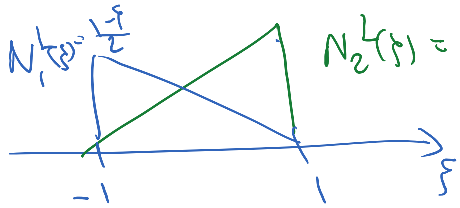
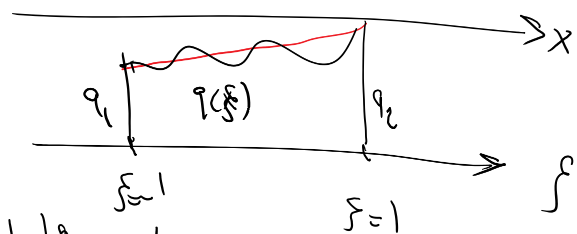
Beam elements: Forces: A. Source term forces

$q(\xi) = q_1 N_1^L(\xi) + q_2 N_2^L(\xi)$

$N_1^L(\xi) = \frac{1-\xi}{2}$ $\quad N_2^L(\xi) = \frac{1+\xi}{2}$





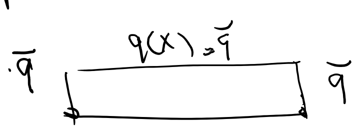
$$f_r^e = \int_e N^t q\, dV = \int_{-1}^{1}\begin{bmatrix}N_1\\N_2\\N_3\\N_4\end{bmatrix} q(\xi)\, \underbrace{dx}_{\frac{L}{2}d\xi} = \int_{-1}^{1}\begin{bmatrix}N_1(\xi)\\N_2(\xi)\\N_3(\xi)\\N_4(\xi)\end{bmatrix}\left[N_1^L(\xi)\ N_2^L(\xi)\right]\begin{bmatrix}q_1\\q_2\end{bmatrix}\frac{L_e}{2}d\xi$$

take it out

$$f_r^e = (r_e)_{4\times 2}\begin{bmatrix}q_1\\q_2\end{bmatrix} \qquad r_e = \int_{-1}^{1}\begin{bmatrix}N_1\\ \vdots \\N_4\end{bmatrix}\left[N_1^L\ N_2^L\right]\frac{L_e}{2}d\xi$$

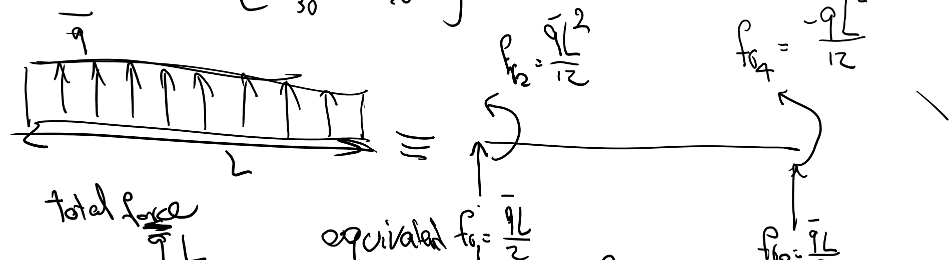
$$\mathbf{f}_r^e \approx \mathbf{r}^e\begin{bmatrix}q_1\\q_2\end{bmatrix} \quad \text{where } \mathbf{r}^e = L^e\begin{bmatrix}\frac{7}{20} & \frac{3}{20}\\ \frac{1}{20}L^e & \frac{1}{30}L^e \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{30}L^e & -\frac{1}{20}L^e\end{bmatrix} \quad \text{exact for linear } q \qquad (433)$$

Example



$$f_r = L\begin{bmatrix}\frac{7}{20} & \frac{3}{20}\\ \frac{1}{20}L^e & \frac{1}{30}L^e \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{30}L & -\frac{1}{20}L^e\end{bmatrix}\begin{bmatrix}\bar q\\ \bar q\end{bmatrix}$$

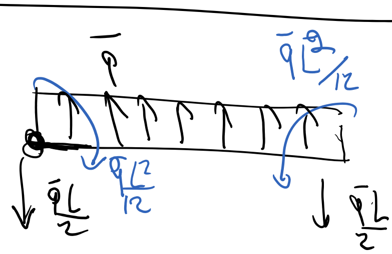
$$f_r = L\bar q\begin{bmatrix}\frac{1}{2}\\ \frac{L}{12}\\ \frac{1}{2}\\ -\frac{L}{12}\end{bmatrix}$$



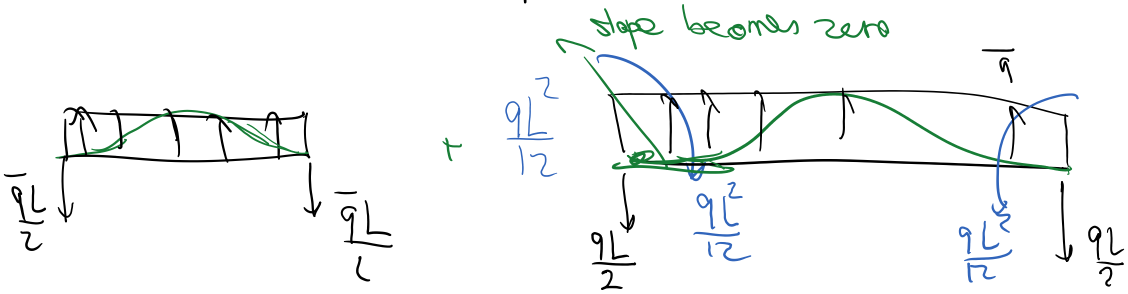
$\left[\frac{-L}{12}\right]$ total force $\bar{q}L$ equivalent $f_{r_1} = \frac{\bar{q}L}{2}$ $f_{r_3} = \frac{\bar{q}L}{2}$

f_r

$-f_r$ & distributed forces



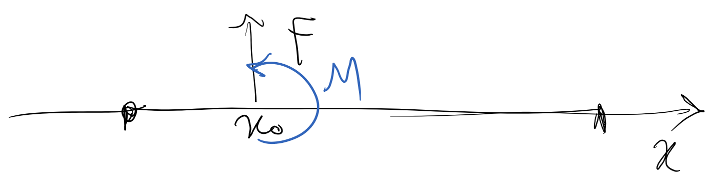
After FEM solution we need to put $-f_r$ forces at the nodes to have equilibrium with source term q.

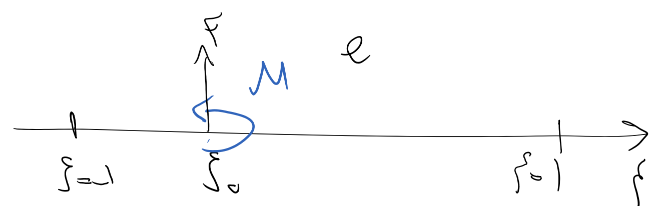


$$F_r = \int \begin{bmatrix} N_1 \\ \vdots \\ N_4 \end{bmatrix} \overbrace{q(x)}^{F\delta(x - x_0)}\, dx$$

$$= \begin{bmatrix} N_1 \\ \vdots \\ N_4 \end{bmatrix}(x_0)\, F$$

$$= \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}(\xi_0)\, F$$

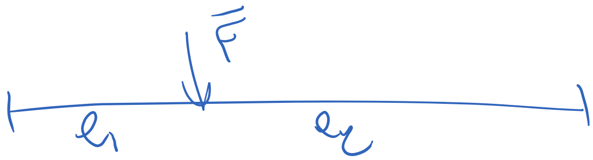


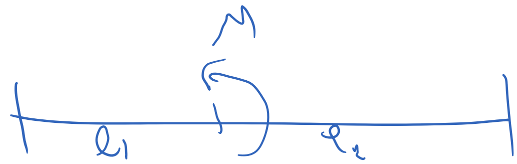


$$F_r^M = \frac{d}{dx}\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}(\xi_0)\, M$$

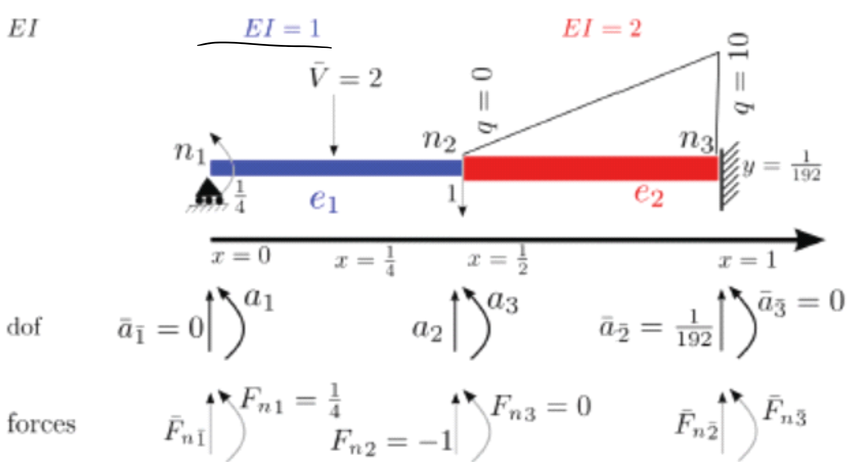
$$F_r^M = \frac{1}{\underbrace{L/2}_{\frac{dx}{d\xi}}} \cdot \frac{d}{d\xi}\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}(\xi_0)\, M$$

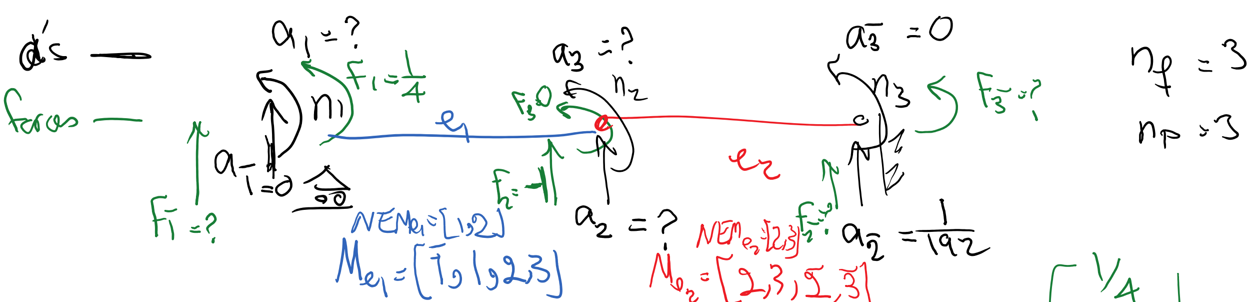
In practice we'll break the domain there





$e_1 \quad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad e_2 \quad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$





$$Ka = F \quad , \quad F = \underset{\text{nodes}}{F_n} + \underset{\text{element}}{F_e} \qquad F_n = \begin{bmatrix} 1/4 \\ -1 \\ 0 \end{bmatrix}$$

both elements are prismatic

$$\mathrm{k}^e = \frac{EI}{L^{e3}} \begin{bmatrix} 12 & 6L^e & -12 & 6L^e \\ & 4L^{e2} & -6L^e & 2L^{e2} \\ & & 12 & -6L^e \\ \text{sym.} & & & 4L^{e2} \end{bmatrix} \quad \text{for constant } E \text{ and } I \qquad (427)$$

e1

$l_{e_1} = \frac{1}{2} \quad , \quad EI_{e_1} = 1$

e2

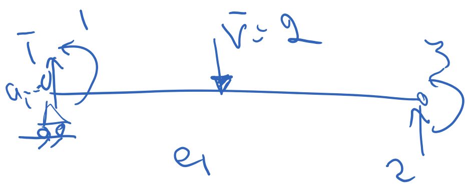
$l_2 = \frac{1}{2} \quad EI_{e_2} = 2$

e	e_1	e_2
$\mathbf{k}^e$	$\mathbf{k}^{e_1} = \frac{1}{(\frac{1}{2})^3}\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$ $= \begin{array}{c\|cccc} & \bar{1} & 1 & 2 & 3 \\ \hline \bar{1} & 96 & 24 & -96 & 24 \\ 1 & 24 & 8 & -24 & 4 \\ 2 & -96 & -24 & 96 & -24 \\ 3 & 24 & 4 & -24 & 8 \end{array}$	$\mathbf{k}^{e_2} = \frac{2}{(\frac{1}{2})^3}\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$ $= \begin{array}{c\|cccc} & 2 & 3 & \bar{2} & \bar{3} \\ \hline 2 & 192 & 48 & -192 & 48 \\ 3 & 48 & 16 & -48 & 8 \\ \bar{2} & -192 & -48 & 192 & -48 \\ \bar{3} & 48 & 8 & -48 & 16 \end{array}$

$M_{e_1} = [\bar{1}, 1, 2, 3]$ $\qquad$ $M_{e_2} = [2 \;\; 3 \;\; \bar{2} \;\; \bar{3}]$

$$\mathbf{K} = \begin{bmatrix} 8 & -24 & 4 \\ & 96+192 & -24+48 \\ \text{sym.} & & 8+16 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -24 & 4 \\ & 288 & 24 \\ \text{sym.} & & 24 \end{bmatrix}$$



$f^{e_1} = f_r^{e_1} + \cancel{f_N^{e_1}} - f_D^{e_1} \to 0$

1D element

$M_{e_1} = [\bar{1}, 1, 2, 3]$

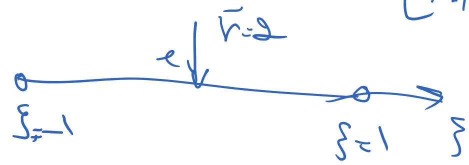
$f_D^{e_1} = k^{e_1} a^{e_1} = k^{e_1}\begin{bmatrix} a_{\bar{1}} \to 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

put zero for free dofs

$f_r^{e_1} = (-2)\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}(\xi = 0) = \begin{bmatrix} -1 \\ -1/8 \\ -1 \\ 1/8 \end{bmatrix}$

force in the middle

center



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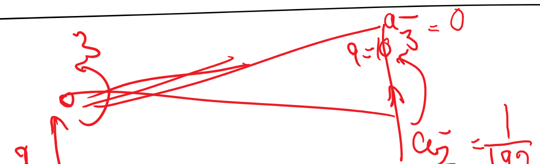
element 2



$f^{e_2} = f_r^{e_2} + \cancel{f_N^{e_2}} - f_D^{e_2}$

1D

$M_{e_2} = [2, 3, \bar{2}, \bar{3}]$

$f_r^{e_2} = (r_{e_2})_{4\times2}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = ((r_e)_2)_{4\times2}\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/12 \\ 1/4 \\ -1/8 \end{bmatrix}$

$f_D^{e_2} = K^{e_2}_{4\times4}\begin{bmatrix} a_2 \to 0 \\ a_3 \to 0 \\ a_{\bar{2}} \end{bmatrix} = k^{e_2}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

for free ones

$$f_D^{e_2} = K^{e_2}_{4\times4}\begin{bmatrix} a_2 \\ a_3 \\ \bar{a}_2 \\ \bar{a}_3 \end{bmatrix} = k^{e_2}\begin{bmatrix} 0 \\ 0 \\ 1/192 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1/4 \\ 1 \\ -3/4 \end{bmatrix}$$

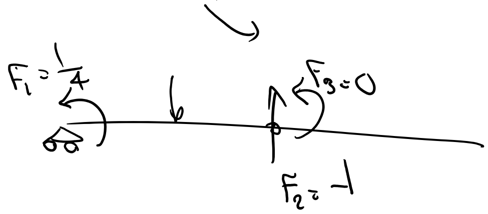
$\rightarrow 0$ ones

a_{e_2}

$\mathbf{f}_r^e$	(440) (1st eqn) ($\xi = 0$) $\bar{V}\begin{bmatrix} N_1^{e_1}(\xi_0) \\ N_2^{e_1}(\xi_0) \\ N_3^{e_1}(\xi_0) \\ N_4^{e_1}(\xi_0) \end{bmatrix} = -2\begin{bmatrix} \frac{1}{2} \\ \frac{1}{8} \\ \frac{1}{2} \\ -\frac{1}{8} \end{bmatrix} = \begin{array}{c} \bar{1} \\ 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} -1 \\ -\frac{1}{8} \\ -1 \\ \frac{1}{8} \end{bmatrix}$	equation (433); $\mathbf{r}^e\,[q_1 \quad q_2]^T$ $\frac{1}{2}\begin{bmatrix} \frac{7}{20} & \frac{3}{20} \\ \frac{1}{40} & \frac{1}{60} \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{60} & -\frac{1}{40} \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{array}{c} 2 \\ 3 \\ \bar{2} \\ \bar{3} \end{array}\begin{bmatrix} \frac{3}{4} \\ \frac{1}{12} \\ \frac{7}{4} \\ -\frac{1}{8} \end{bmatrix}$
$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e =$ $\begin{bmatrix} 96 & 24 & -96 & 24 \\ 24 & 8 & -24 & 4 \\ -96 & -24 & 96 & -24 \\ 24 & 4 & -24 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{array}{c} 1 \\ \bar{1} \\ 2 \\ 3 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e =$ $\begin{bmatrix} 192 & 48 & -192 & 48 \\ 48 & 16 & -48 & 8 \\ -192 & -48 & 192 & -48 \\ 48 & 8 & -48 & 16 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \frac{1}{192} \\ 0 \end{bmatrix} = \begin{array}{c} 2 \\ 3 \\ \bar{2} \\ \bar{3} \end{array}\begin{bmatrix} -1 \\ -\frac{1}{4} \\ 1 \\ -\frac{1}{4} \end{bmatrix}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \mathbf{f}_r^{e_1} + \mathbf{f}_N^{e_1} - \mathbf{f}_D^{e_1} = \begin{array}{c} \bar{1} \\ 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} -1 \\ -\frac{1}{8} \\ -1 \\ \frac{1}{8} \end{bmatrix}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \mathbf{f}_N^{e_2} - \mathbf{f}_D^{e_2} = \begin{array}{c} 2 \\ 3 \\ \bar{2} \\ \bar{3} \end{array}\begin{bmatrix} \frac{7}{4} \\ \frac{1}{3} \\ \frac{3}{4} \\ \frac{1}{8} \end{bmatrix}$

$$F_e = \begin{bmatrix} -1/8 \\ -1 + 7/4 \\ 1/8 + 1/3 \end{bmatrix}$$

$$F = F_n + F_e = \begin{bmatrix} 1/4 \\ -1 \\ 0 \end{bmatrix} + F_e = \begin{bmatrix} -1/8 \\ -3/4 \\ 11/24 \end{bmatrix}$$



$$\mathbf{F} = \mathbf{F}n + \mathbf{F}e$$

$$= \begin{bmatrix} \frac{1}{4} \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ -1+\frac{7}{4} \\ \frac{1}{8}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} \\ -\frac{3}{4} \\ \frac{11}{24} \end{bmatrix} \Rightarrow$$

$$\boxed{\begin{aligned} \mathbf{U} &= \mathbf{K}^{-1}\mathbf{F} \\ &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{\ } \end{bmatrix} = \begin{bmatrix} -.00607 \\ -.00332 \\ .023434 \end{bmatrix} \end{aligned}}$$

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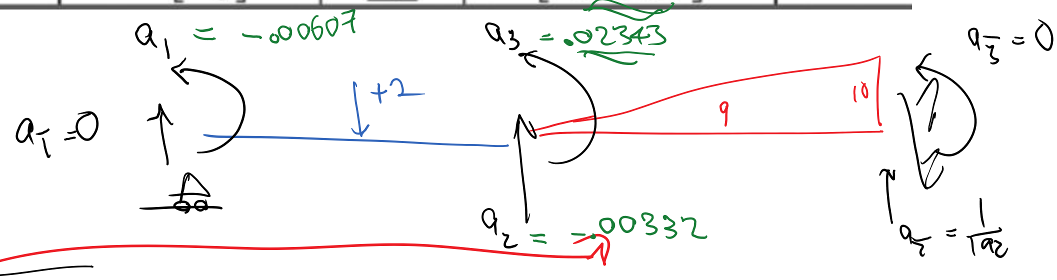
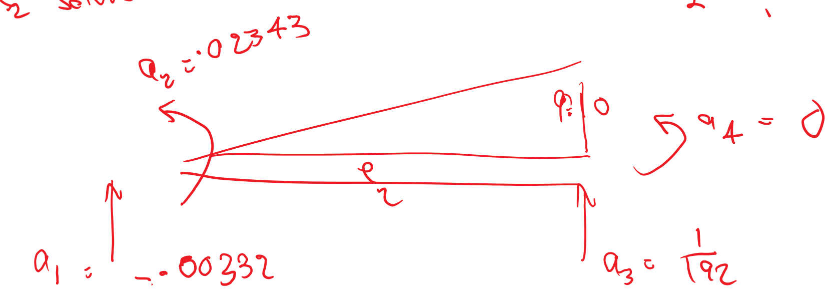
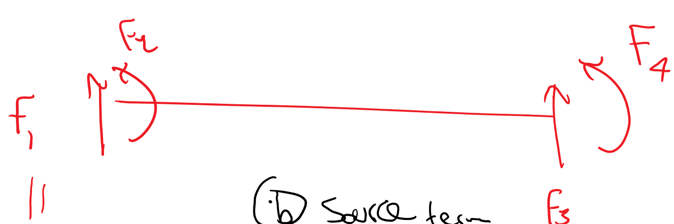


Remaining thing: element solution's & support forces

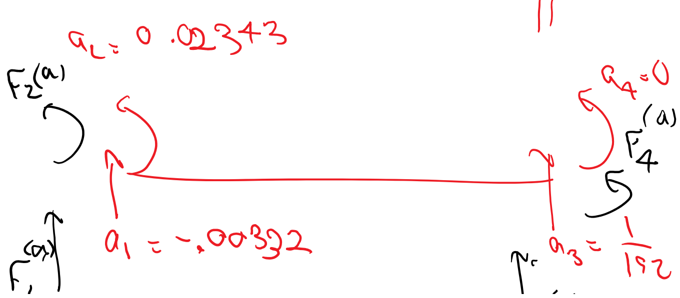
for $F_{\bar{2}}$ & $F_{\bar{3}}$ calculate e_2 solution

$M_{e_2} = [2\ 3\ \bar{2}\ \bar{3}]$

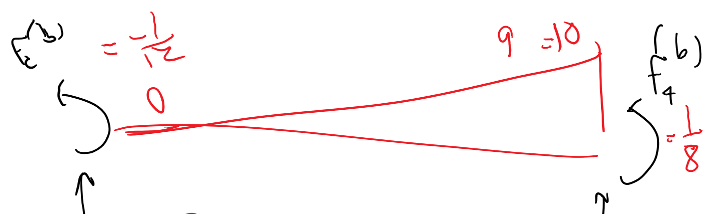




(a) nodal solutions



(b) Source term q:



$f_1^{(a)}$ $a_1 = -.00392$

$a_3 = \frac{1}{192}$ $f_3^{(a)}$

$f_1^{(b)} = -\frac{3}{4}$ $= \frac{1}{8}$ $f_3^{(b)} = -\frac{7}{4}$

$-f_r$

$f^{(a)} = k^{e_2} a^{e_2}$

$$k^{e_2} \begin{bmatrix} 192 & 48 & -192 & 48 \\ 48 & 16 & -48 & 8 \\ -192 & -48 & 192 & -48 \\ 48 & 8 & -48 & 16 \end{bmatrix} \quad a^{e_2} \begin{bmatrix} -\frac{23}{6912} \\ \frac{3}{128} \\ \frac{1}{192} \\ 0 \end{bmatrix}$$

equation (433); $\mathbf{r}^e \; [q_1 \quad q_2]^T$

$$\frac{1}{2}\begin{bmatrix} \frac{7}{20} & \frac{3}{20} \\ \frac{1}{40} & \frac{1}{60} \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{60} & -\frac{1}{40} \end{bmatrix} \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{array}{c} 2 \\ 3 \\ \bar{2} \\ \bar{3} \end{array} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{12} \\ \frac{7}{4} \\ -\frac{1}{8} \end{bmatrix} \Bigg\} f_r^{e_2}$$

$$f = f^{(a)} + f^{(b)} = \underbrace{ka}_{f_D} - f_r - f_N$$

In general FF are not f_N

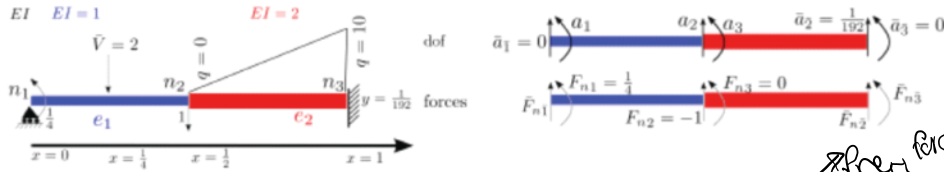
$$f^e = f_r + f_N - f_D$$

$$= f_D - f_r - f_N$$

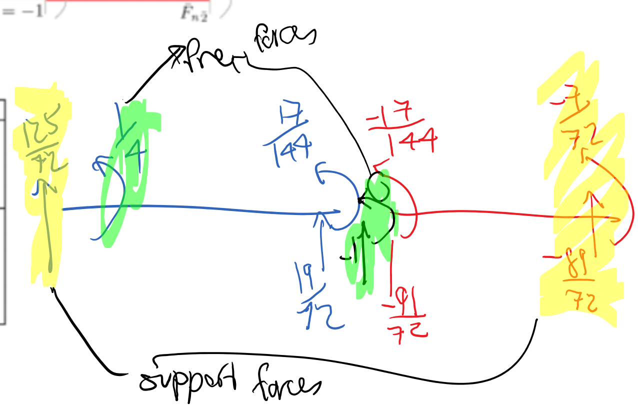
element forces

$$\boxed{f = ka - f_r - f_N = f_D - f_r - f_N = -f^e}$$

Beam Example: Resultant nodal forces



e	e_1	e_2
$\mathbf{u}^e$	$\begin{bmatrix} 0 \\ -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{128} \end{bmatrix}$	$\begin{bmatrix} -\frac{23}{6912} \\ \frac{3}{128} \\ \frac{1}{192} \\ 0 \end{bmatrix}$
$-f^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e - \mathbf{f}_T^{e_1} - \mathbf{f}_N^{e_1} =$ $\begin{bmatrix} 96 & 24 & -96 & 24 \\ 24 & 8 & -24 & 4 \\ -96 & -24 & 96 & -24 \\ 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{128} \end{bmatrix} - \begin{bmatrix} -1 \\ -\frac{1}{8} \\ -1 \\ \frac{1}{8} \end{bmatrix} = \begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} \frac{125}{72} \\ \frac{1}{4} \\ \frac{19}{72} \\ \frac{17}{144} \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e - \mathbf{f}_T^{e_2} - \mathbf{f}_N^{e_2} =$ $\begin{bmatrix} 192 & 48 & -192 & 48 \\ 48 & 16 & -48 & 8 \\ -192 & -48 & 192 & -48 \\ 48 & 8 & -48 & 16 \end{bmatrix} \begin{bmatrix} -\frac{23}{6912} \\ \frac{3}{128} \\ \frac{1}{192} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{3}{4} \\ \frac{1}{12} \\ \frac{7}{4} \\ -\frac{1}{8} \end{bmatrix} = \begin{array}{c} 2 \\ 3 \\ 2 \\ 3 \end{array} \begin{bmatrix} -\frac{91}{72} \\ -\frac{17}{144} \\ -\frac{89}{72} \\ -\frac{7}{72} \end{bmatrix}$



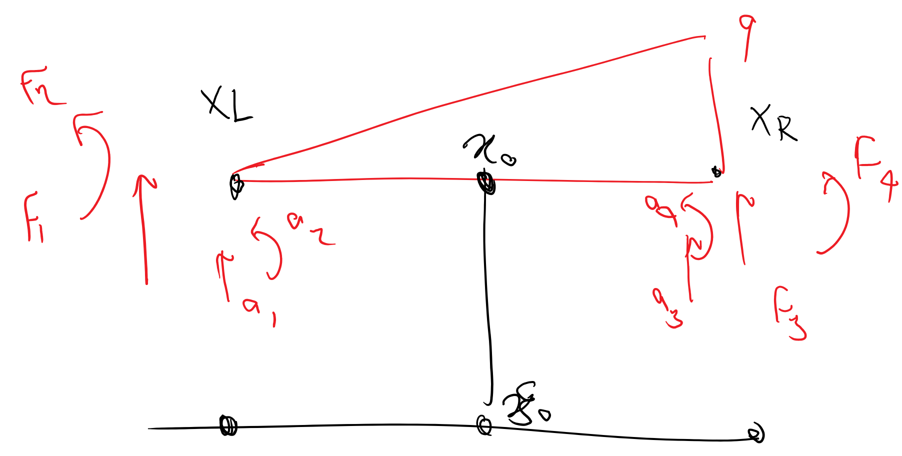
- **Prescribed forces $\mathbf{F}_p$ (Reaction forces):**

$$F_{n1} = f_1^{e1} = \frac{125}{72} = 1.736111 \qquad \text{vertical load at the left support} \qquad (442a)$$

$$F_{n2} = f_3^{e2} = -\frac{89}{72} = -1.236111 \qquad \text{vertical load at the right support} \qquad (442b)$$

$$F_{n3} = f_4^{e2} = -\frac{7}{72} = -0.0972222 \qquad \text{CCW moment at the right support} \qquad (442c)$$

last thing element solutions

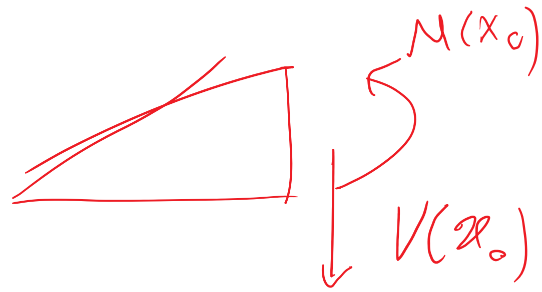


$$\xi_0 = \frac{x_0 - x_L}{x_R - x_L}(1) + \frac{x_R - x_0}{x_R - x_L}(-1)$$

$$y(\xi_0) = N_1(\xi_0)\,a_1 + N_2(\xi_0)\,a_2 + N_3(\xi_0)\,a_3 + N_4(\xi_0)\,a_4$$

$$\theta(\xi_0) = \frac{dy}{dx}(\xi) = \frac{dN_1}{dx}(\xi_0)\,a_1 \qquad + \frac{dN_4(\xi_0)}{dx}\,a_4$$

$$\frac{dN_i(\xi_0)}{dx} = \frac{1}{L/2}\frac{dN_i}{d\xi}$$



$$M = EI\,y'' = EI\,Ba$$

$$= EI\,\left(B_1(\xi_0)a_1 + B_2(\xi_0)a_2 + B_3(\xi_0)a_3 + B_4(\xi_0)a_4\right)$$

constant EI

$$V = \frac{d}{dx}M = EI\left(\frac{dB_1}{dx}(\xi)a_1 + \cdots + \frac{dB_4}{dx}(\xi)a_4\right)$$

$$\frac{dB_i}{dx}(\xi_0) = \frac{1}{\frac{L}{2}}\left.\frac{dB_i(\xi)}{d\xi}\right|_{\xi_0}$$

Beam Example: Calculation of y, θ, M, V within element

- After the solution of global free dofs, they are transferred to elements.
- Once element dofs are know, we have the Displacement in the entire elements:

$$y(\xi) = N_1^e(\xi)a_1^e + N_2^e(\xi)a_2^e + N_3^e(\xi)a_3^e + N_4^e(\xi)a_4^e.$$

element shape functions are given in (421).

- Rotation: Obtained by differentiating previous equation w.r.t. x & noting that $\frac{dx}{d\xi} = \frac{L^e}{2}$:

$$\theta(\xi) = \frac{dy}{dx}(\xi) = \frac{\frac{dy}{d\xi}(\xi)}{\frac{dx}{d\xi}(\xi)} = \frac{2}{L^e}\left\{\frac{dN_1^e}{d\xi}(\xi)a_1^e + \frac{dN_2^e}{d\xi}(\xi)a_2^e + \frac{dN_3^e}{d\xi}(\xi)a_3^e + \frac{dN_4^e}{d\xi}(\xi)a_4^e\right\}$$

- Moment is directly obtained by differentiating the above equation:

$$M(\xi) = E(\xi)I(\xi)\frac{d^2y}{dx^2}(\xi) = E(\xi)I(\xi)\mathbf{B}^e(\xi)$$
$$= E(\xi)I(\xi)\left\{B_1^e(\xi)a_1^e + B_2^e(\xi)a_2^e + B_3^e(\xi)a_3^e + B_4^e(\xi)a_4^e\right\} \quad \textit{cf.}\ (424)\ \text{for}\ \mathbf{B}^e$$

- Shear force is obtained by differentiating M w.r.t. x. It's a similar process to deriving θ from y with the difference that if EI are not constant we need to take it into account. For constant EI we have:

$$V(\xi) = \frac{dM}{dx}(\xi) = \frac{\frac{dM}{d\xi}(\xi)}{\frac{dx}{d\xi}(\xi)} = \frac{2EI}{L^e}\left\{\frac{dB_1^e}{d\xi}(\xi)a_1^e + \frac{dB_2^e}{d\xi}(\xi)a_2^e + \frac{dB_3^e}{d\xi}(\xi)a_3^e + \frac{dB_4^e}{d\xi}(\xi)a_4^e\right\}$$

- To obtain these fields for the entire beam we evaluate these equations for all elements.

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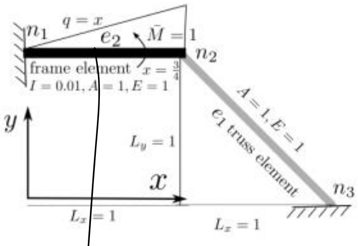
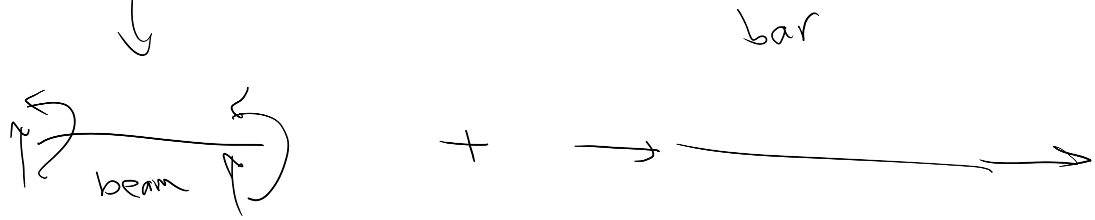


Figure 3: Frame and truss example.



(d) Obtain displacement (y), rotation ($\theta = \frac{\mathrm{d}y}{\mathrm{d}x}$), and moment ($M = EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$) for the frame element at $x = 0.5$. Note that $y(\xi) = \sum_{i=1}^{4} N_i^e(\xi)a_i^e$. Also, since $\mathbf{B}^e = \frac{\mathrm{d}^2\mathbf{N}^e}{\mathrm{d}x^2} \Rightarrow M = EI\sum_{i=1}^{4} B_i^e(\xi)a_i^e$. (**30 Points**)

threat  as beam and use formulas above