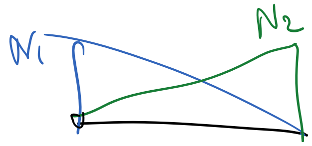
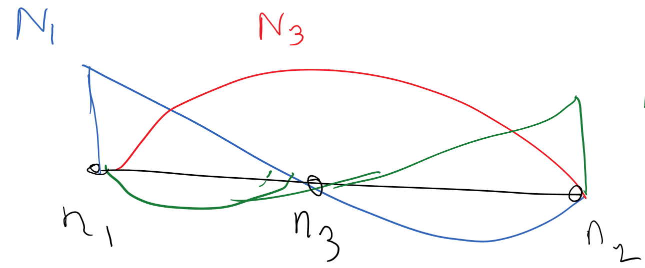
2022/11/15

Tuesday, November 15, 2022 11:14 AM

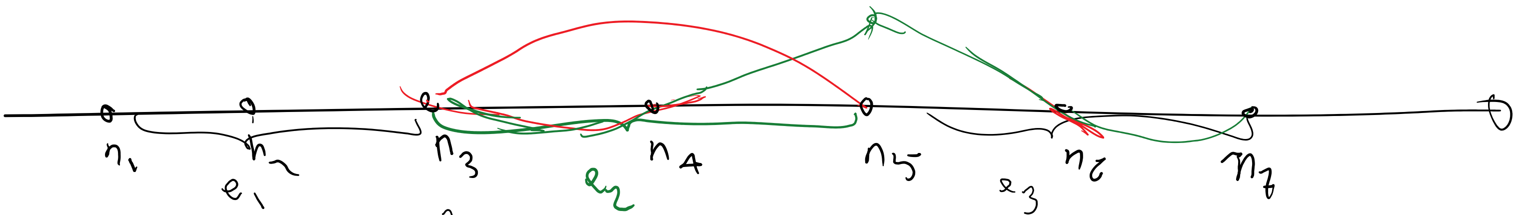
Higher order elements



error $= Ch^2$



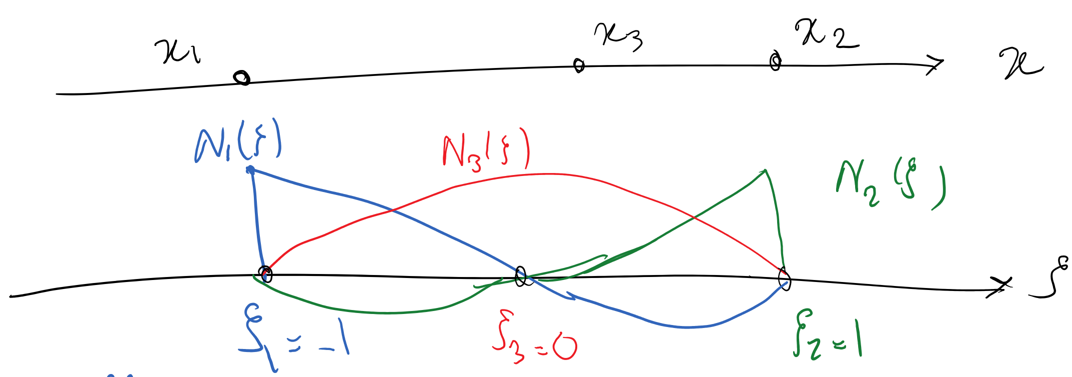
order $p = 2$



error $= Ch^3$

element order p $\longrightarrow$ error $\approx Ch^{p+1}$

Shape functions



$N_1(\xi) = ?$ Difficult way $N_1(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2$ $N_1(-1) = 1$, $N_1(1) = 0$ $N_1(0) = 0$

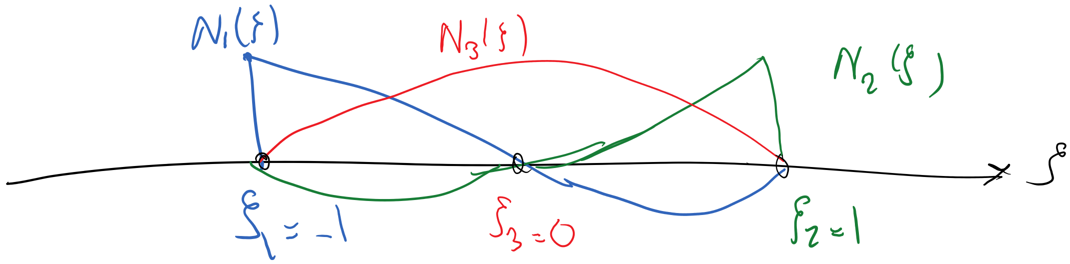
Easier way:

Lagrange Polynomials $N_1(\xi) = \underbrace{L_1(\xi)}_{\text{Lagrange}} = \dfrac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)}$

$N_1(\xi_1) = \dfrac{(\xi_1 - \xi_2)(\xi_1 - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = 1$ $\quad N_1(\xi_2) = \dfrac{\cancel{(\xi_2 - \xi_2)}(\xi_2 - \xi_3)}{\ldots} = 0$ $\quad$ similarly $N_1(\xi_3) = 0$

$$N_1(\xi_1) = \frac{(\xi_1-\xi_2)(\xi_1-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = 1 \qquad N_1(\xi_2) = \frac{\cancel{(\xi_2-\xi_2)}(\xi_2-\xi_3)}{\text{denom}} = 0 \qquad \text{similarly } N_1(\xi_3) = 0$$

$$N_1(\xi) = \frac{(\xi-1)(\xi-0)}{(-1-1)(-1-0)} = \frac{\xi(\xi-1)}{2}$$



$$N_2(\xi) = L_2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = \frac{(\xi-(-1))(\xi-0)}{(1-(-1))(1-0)} = \frac{\xi(\xi+1)}{2}$$

$$N_3(\xi) = L_3(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} = \frac{(\xi-(-1))(\xi-1)}{(0-(-1))(0-1)} = 1-\xi^2$$

$$\boxed{L_i(\xi) = \frac{\prod_{j=1}^{n,\, j\neq i}(\xi-\xi_j)}{\prod_{j=1}^{n,\, j\neq i}(\xi_i-\xi_j)}}$$

Calculating the stiffness

$$\boxed{N(\xi) = [N_1 \;\; N_2 \;\; N_3] = \left[\frac{\xi(\xi-1)}{2} \quad \frac{\xi(\xi+1)}{2} \quad 1-\xi^2\right]} \qquad (1)$$

WK: bar

$$\int_{\Omega} w' \, EA \, u' \, dx \qquad L_m = ()' \quad B$$

$$B = \frac{d}{dx}(N) = \frac{d}{dx} N$$

(2) $$\boxed{k = \int_e B^t D B \, dx}$$

$B = L_m(N) = \frac{d}{dx} N$

$D = EA$

$$B = \frac{d}{dx}\left[N_1(\xi) \quad N_2(\xi) \quad N_3(\xi)\right] = \frac{1}{\underbrace{\frac{dx}{d\xi}}_{J}} \underbrace{\frac{d}{d\xi}\left[N_1(\xi) \quad N_2(\xi) \quad N_3(\xi)\right]}_{B_\xi = \frac{d}{d\xi} N}$$

(3) $$\boxed{B = \frac{1}{J} B_\xi}, \quad B_\xi = \frac{d}{d\xi} N = \frac{d}{d\xi}\left[\frac{\xi(\xi-1)}{2} \quad \frac{\xi(\xi+1)}{2} \quad 1-\xi^2\right]$$

(4) $$\boxed{B_\xi = \left[\xi - \tfrac{1}{2}, \; \xi + \tfrac{1}{2}, \; -2\xi\right]}$$

(5) $$\boxed{dx = \frac{dx}{d\xi} d\xi = J \, d\xi}$$

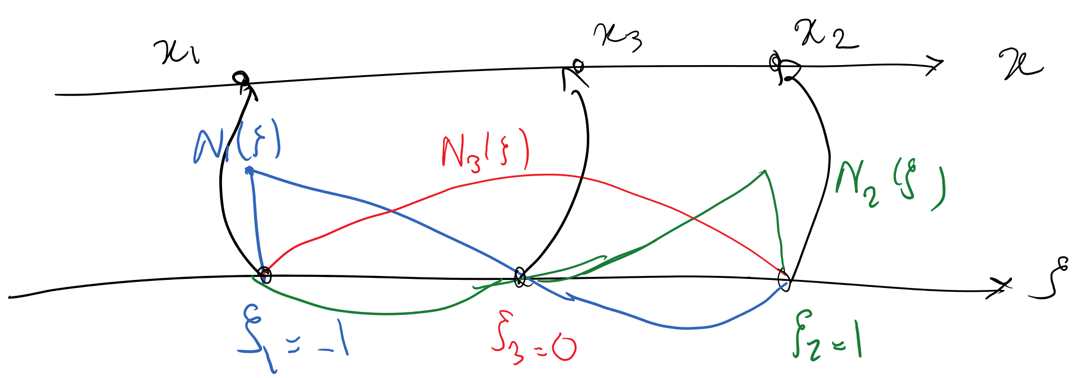
plug (4) → (3), (3) → (2)

(5) → (2)

$$k = \int_{\xi=-1}^{1} \left(\frac{B_\xi}{J}\right)^t EA \left(\frac{B_\xi}{J}\right) J \, d\xi$$

(6) $$\boxed{K = \int_{\xi=-1}^{1} \frac{1}{J} B_\xi^t \, EA(\xi) \, B_\xi \, d\xi}$$

$J = \frac{dx}{d\xi}$



$\xi_1 = -1 \longrightarrow x_1$

$\xi_2 = 1 \longrightarrow x_2$

$\xi_3 = 0 \longrightarrow x_3$

Difficult: $x(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2$

Note $u = N_1(\xi) u_1 + N_2(\xi) u_2 + N_3(\xi) u_3$

Easy $x = N_1(\xi) \, x_1 + N_2(\xi) x_2 + N_3(\xi) x_3$

$N \to$ solution

$N \to$ geometry

same N's for solution & geometry

isoparametric

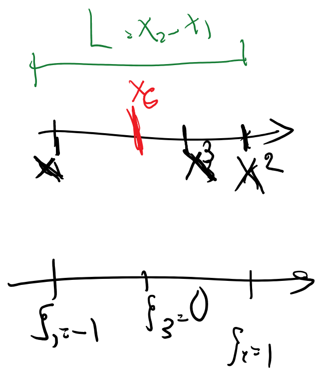
(7) $$x(\xi) = \frac{\xi(\xi-1)}{2} x_1 + \frac{\xi(\xi+1)}{2} x_2 + (1-\xi^2) x_3$$

(7) $$\boxed{x(\xi) = \frac{\xi(\xi-1)}{2}x_1 + \frac{\xi(\xi+1)}{2}x_2 + (1-\xi^2)x_3}$$

same N's & geometry
isoparametric

$$J = \frac{dx(\xi)}{d\xi} = \underbrace{\left(\xi - \frac{1}{2}\right)}_{B_{\xi_1}} x_1 + \underbrace{\left(\xi + \frac{1}{2}\right)}_{B_{\xi_2}} x_2 \underbrace{- 2\xi}_{B_{\xi_3}} x_3$$

$$J(\xi) = 2\left(\underbrace{\frac{x^1+x^2}{2}}_{x_c} - x^3\right) + \frac{1}{2}\underbrace{(x^2 - x^1)}_{L}$$



(8) $$\boxed{J(\xi) = 2(x_c - x^3) + \frac{1}{2}L}$$

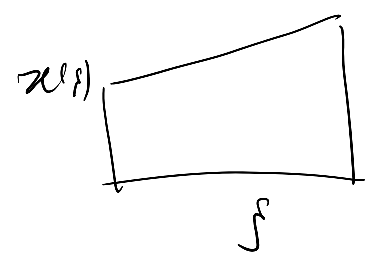
if $x_3 = x_c$ $\quad J(\xi) = \dfrac{L}{2}$

$\to x = \left(\frac{1-\xi}{2}\right)x_1 + \left(\frac{1+\xi}{2}\right)x_2$
$= x_c + \xi\frac{L}{2}$ $\quad$ for $x_3 = x_c$

for constant J

$$J = \frac{\text{real length}}{\xi - \text{length}}$$

$$J = \frac{dx}{d\xi}$$



(4), (6), (8) $\to$

(9) $\quad$ N: order p $\to B_\xi^{bar}$: order p-1

$$k^e = \int_{\xi=-1}^{\xi=1} \frac{AE(\xi)}{J(\xi)} \underbrace{\begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix}}_{B_\xi^t} \underbrace{\left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi\right]}_{B_\xi \quad \text{order} = 1} d\xi$$

$J(\xi) \to 2\xi(x_c - x^3) + \frac{L}{2}$

if $\begin{cases} AE(\xi) = AE \text{ constant : Homogeneous material, constant area section} \\ J(\xi) = \frac{L}{2} \text{ const} \longrightarrow \text{element is not skewed} \end{cases}$

under these conditions we deal with an integrand of order 2

(... polynom of order p $\to$ 2(p-1))

under these conditions we deal with an ...

(general: for bar problem of order $p \rightarrow 2(p-1)$)

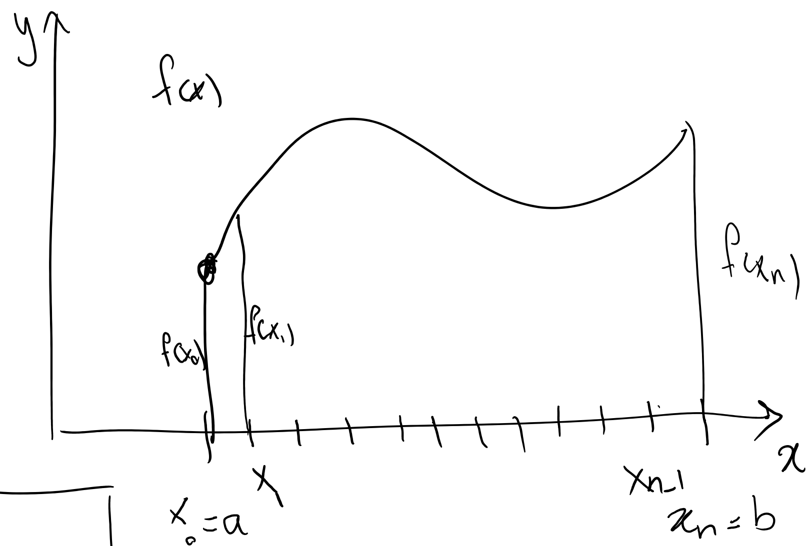
we can integrate it exactly to get

$$K = \frac{AE}{L}\begin{bmatrix} 7/3 & -8/3 & 1/3 \\ & 7/3 & -8/3 \\ \text{sym} & & 16/3 \end{bmatrix}$$

Quadrature: Numerical integration

Example: We dealt with a 2nd order polynomial (AE, J = const) and a non-polynomial otherwise. We want to learn to integrate this.

Newton-Cotes



n segments $\equiv (n+1)$ points $x_0 \ldots x_n$

$L = b - a$

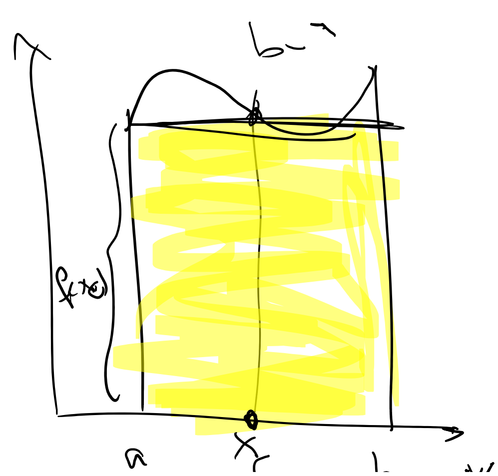
$x_i = x_0 + \frac{iL}{n}$

(10)
$$\boxed{\int_{x_0=a}^{x_n=b} f(x)\,dx = L\left(\sum_{i=0}^{n} \underbrace{w_i}_{\text{quadrature weights}} \underbrace{f(x_i)}_{\text{quadrature points}}\right) \qquad L = b - a}$$

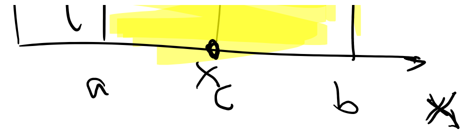
Examples $n = 1$

$$\int_a^b f(x)\,dx \approx (b-a)\, f(x_c)$$

rectangular rule

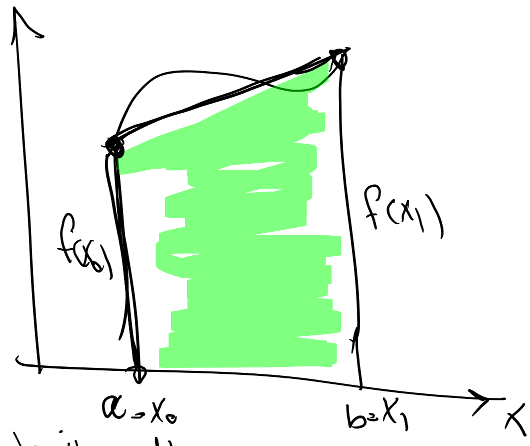


rectangular rule

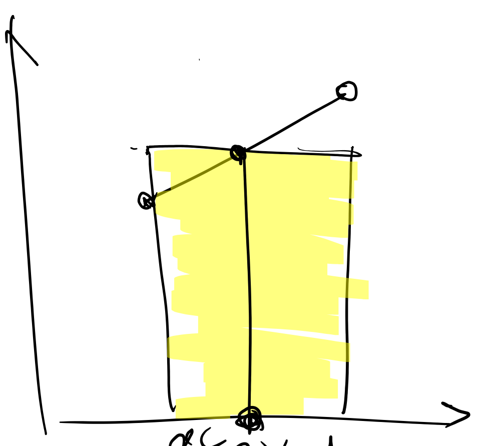


n = 2

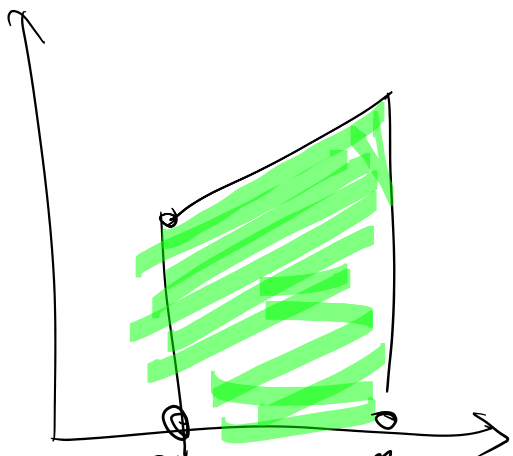
$$\int_a^b f(x)\,dx = (b-a)\Big(\underbrace{\tfrac{1}{2}}_{w_0} f(x_0) + \underbrace{\tfrac{1}{2}}_{w_1} f(x_1)\Big)$$



Linear function: which scheme(s) integrate it exactly



exact with 1 pt

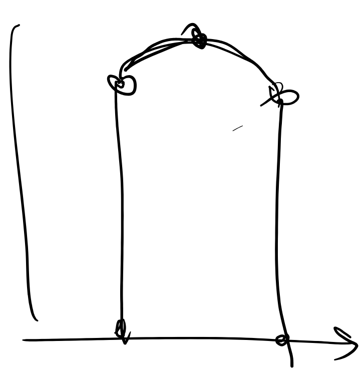


exact with 2 pts

this repeats: NC schemes with 2, 4, 6, — —
have the same order of accuracy as 1, 3, 5, — —

3 → Simpson's rule

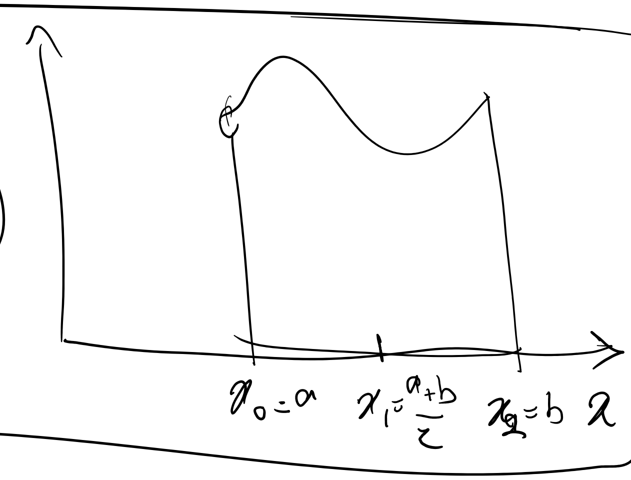
pts



Simpson's rule integrates a 2nd order (in fact 3 order) polynomial exactly

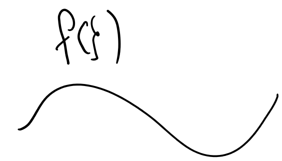
$n=3$

$$\int_a^b f(x)\,dx \approx \underbrace{(b-a)}_{2}\left(\underset{\omega_0\ \ x_0}{\frac{1}{6}f(a)} + \underset{\omega_1\ \ x_1}{\frac{4}{6}f\left(\frac{a+b}{2}\right)} + \underset{\omega_2\ \ x_2}{\frac{1}{6}f(b)}\right) \qquad (11)$$



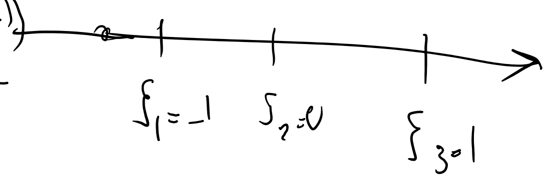
Why?

Let's use ξ coordinate



$$\int_{\xi=-1}^{1} f(\xi)\,d\xi = (1-(-1))\left(\omega_1 f(\xi_1) + \omega_2 f(\xi_2) + \omega_3 f(\xi_3)\right) \quad (*)$$

$$= 2\left(\omega_1 f(-1) + \omega_2 f(0) + \omega_3 f(1)\right)$$



ξ_1, ξ_2, ξ_3 known

we want to obtain $\omega_1, \omega_2, \omega_3$

$$f(\xi) = \underbrace{\alpha_0 + \alpha_1\xi + \alpha_2\xi^2}_{\text{we can integrate these exactly}} + \alpha_3\xi^3 + \alpha_4\xi^4 + \cdots$$

Find $\omega_1, \omega_2, \omega_3$ such that we can integrate $1, \xi, \xi^2$ exactly

$f(\xi)=1 \quad \int_{-1}^{1} f(\xi)d\xi = \int_{-1}^{1} 1\,d\xi = 2 \overset{(*)}{=} 2(\omega_1 f(-1) + \omega_2 f(0) + \omega_3 f(1)) = 2(\omega_1+\omega_2+\omega_3) \rightarrow \boxed{\omega_1+\omega_2+\omega_3 = 1}$ i

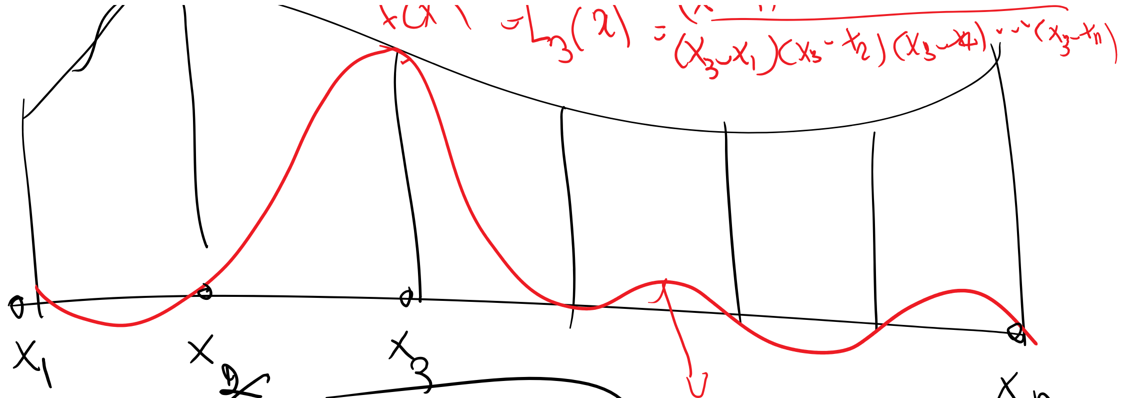
$f(\xi)=\xi \quad \int_{-1}^{1} f(\xi)d\xi = \int_{-1}^{1}\xi\,d\xi = \frac{\xi^2}{2}\Big|_{-1}^{1} = 0 \overset{(*)}{=} \quad // \quad = 2(\omega_1(-1)+\omega_2(0)+\omega_3(1)) \rightarrow \boxed{-\omega_1+\omega_3=0}$ ii

$f(\xi)=\xi^2 \quad \int_{-1}^{1} f(\xi)d\xi = \int_{-1}^{1}\xi^2 d\xi = \frac{\xi^3}{3}\Big|_{-1}^{1} = \frac{2}{3} \overset{(*)}{=} \quad // \quad = 2(\omega_1(-1)^2+\omega_2(0)^2+\omega_3(1)^2) \Rightarrow \boxed{\omega_1+\omega_3 = \frac{1}{3}}$ iii

ii, iii $\quad \omega_1 = \omega_3 = \frac{1}{6}$ $\quad$, i $\rightarrow$ $\omega_2 = \frac{4}{6}$ eqn (11)

Easier way:

$$f(x) \approx L_3(x) = \frac{(x-x_1)(x-x_2)(x-x_4)\cdots(x-x_n)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)\cdots(x_3-x_n)}$$



n pt
NC scheme

$$\int_{x_1}^{x_n} f(x)\,dx = (x_n - x_1)\sum_{i=1}^{n} \omega_i \, f(x_i) = (x_n - x_0)\,\omega_3 \underbrace{f(x_3)}_{1}$$

Goal find ω_i's $\rightarrow$ $\omega_3 = \dfrac{\int_{x_0}^{x_n} f(x)\,dx}{x_n - x_0}$

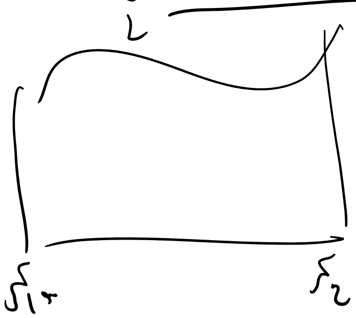
NC : $\omega_i = \frac{1}{b-a}\int_a^b L_i(x)\,dx$, $L_i(x) = \dfrac{(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$

easier way to get the weights for NCs

Gauss quadrature

2 point quadrature scheme

NC
$$\int_{-1}^{1} f(\xi)\,d\xi = \frac{2}{2}\left(\omega_1 f(\xi_1) + \omega_2 f(\xi_2)\right)$$



Gauss

Let's also find ξ_1, ξ_2

$$\int_{-1}^{1} f(\xi)\,d\xi = \cancel{2}\left(\omega_1 f(\xi_1) + \omega_2 f(\xi_2)\right)$$

Gauss quadrature is always done in the parent $\xi \in [-1,1]$ & the factor 2 is absorbed in ω's

Deriving $\underbrace{\omega_1, \omega_2, \xi_1, \xi_2}_{4\text{ unknowns}}$ for Gauss quadrature

... $1, \xi, \xi^2, \xi^3$ exactly

$\overrightarrow{\text{4 unknowns}}$

We should be able to integrate $\underbrace{1, \xi, \xi^2, \xi^3}_{4}$ exactly

$$\int_{-1}^{1} f(\xi)\,d\xi = \omega_1 f(\xi_1) + \omega_2 f(\xi_2)$$

$f(\xi) = 1$ $\quad \int_{-1}^{1} f(\xi)\,d\xi = \int_{-1}^{1} 1\,d\xi = 2 = \omega_1(1) + \omega_2(1) = 2$ (i)

$f(\xi) = \xi$ $\quad$ " $= \int_{-1}^{1} \xi\,d\xi = 0 = \omega_1 \xi_1 + \omega_2 \xi_2 = 0$ (ii) $\ast\, \xi_2^2$

$f(\xi) = \xi^2$ $\quad$ " $= \int_{-1}^{1} \xi^2\,d\xi = \frac{2}{3} = \omega_1 \xi_1^2 + \omega_2 \xi_2^2 = \frac{2}{3}$ (iii)

$f(\xi) = \xi^3$ $\quad$ " $= \int_{-1}^{1} \xi^3\,d\xi = 0 = \omega_1 \xi_1^3 + \omega_2 \xi_3^3 = 0$ (iv)

iv) $-$ ii) $\ast\, \xi_2^2 \;\rightarrow\; \omega_1 \xi_1^3 - \omega_1 \xi_1 \xi_2^2 = 0 \;\rightarrow\; \omega_1 \xi_1 (\xi_1 - \xi_2)(\xi_1 + \xi_2) = 0$

$\omega_1 = 0$, $\xi_1 = \xi_2$ $\rightarrow$ 1 pt :($\qquad \xi_1 = 0 \longrightarrow \xi_2 = 0 \;\ldots$

$\xi_1 + \xi_2 = 0 \;\rightarrow\; \boxed{\xi_2 = -\xi_1} \;\rightarrow$ (2) $\omega_1 - \omega_2 = 0$, i) $\omega_1 + \omega_2 = 2$ $\rightarrow \omega_1 = \omega_2 = 1$

iii $\;\rightarrow\; 2\xi_1^2 = \frac{2}{3} \;\rightarrow\; \boxed{\xi_1 = \mp \frac{1}{\sqrt{3}}}$

$$\boxed{\begin{aligned} \xi_1 &= \frac{-1}{\sqrt{3}}, & \omega_1 &= 1 \\ \xi_2 &= \frac{1}{\sqrt{3}}, & \omega_2 &= 1 \end{aligned}}$$

$-1 \qquad -\frac{1}{\sqrt{3}} \qquad \frac{1}{\sqrt{3}} \qquad -1$

integrate 3rd polynomial

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454

$\mp \frac{1}{\sqrt{3}}$

$\mp$

Intervals, i	No. of Points, n	C_0	C_1	C_2	C_3	C_4	C_5	C_6
1	2	1/2	1/2			(trapezoid rule)		
2	3	1/6	4/6	1/6		(Simpson s 1/3 rule)		
3	4	1/8	3/8	3/8	1/8	(Simpson s 3/8 rule)		

F

0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189

i	Points, n	C_0	C_1	C_2	C_3	C_4	C_5	C_6
1	2	1/2	1/2			(trapezoid rule)		
2	3	1/6	4/6	1/6		(Simpson s 1/3 rule)		
3	4	1/8	3/8	3/8	1/8	(Simpson s 3/8 rule)		
4	5	7/90	32/90	12/90	32/90	7/90		
5	6	19/288	75/288	50/288	50/288	75/288	19/288	
6	7	41/840	216/840	27/840	272/840	27/840	216/840	41/840

order

$$\boxed{\text{nquad} = \begin{cases} \text{ceil}\left(\dfrac{\text{order}+1}{2}\right) & \text{Gauss} \\ (\text{order}+1) & \text{NC} \end{cases}}$$